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*Dedicated to Ray W. Ogden on the occasion of his 60th birthday*

# On travelling wave solutions of a generalized Davey–Stewartson system

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The generalized Davey–Stewartson (GDS) equations, as derived by Babaoglu & Erbay (2004, *Int. J. Non-Linear Mech.*, **39**, 941–949), is a system of three coupled equations in (2 + 1) dimensions modelling wave propagation in an infinite elastic medium. The physical parameters ($\gamma, m_1, m_2, \lambda$ and $n$) of the system allow one to classify the equations as elliptic–elliptic–elliptic (EEE), elliptic–elliptic–hyperbolic (EEH), elliptic–hyperbolic–hyperbolic (EHH), hyperbolic–elliptic–elliptic (HEE), hyperbolic–hyperbolic–hyperbolic (HHH) and hyperbolic–elliptic–hyperbolic (HEH) (Babaoglu *et al.*, 2004, preprint). In this note, we only consider the EEE and HEE cases and seek travelling wave solutions to GDS systems. By deriving Pohozaev-type identities we establish some necessary conditions on the parameters for the existence of travelling waves, when solutions satisfy some integrability conditions. Using the explicit solutions given in Babaoglu & Erbay (2004) we also show that the parameter constraints must be weaker in the absence of such integrability conditions.

*Keywords*: Davey–Stewartson equations; nonlinear Schrödinger equation; Pohozaev identity; radial solutions; travelling wave.

## 1. Introduction

In a recent study (Babaoglu & Erbay, 2004), a generalized Davey–Stewartson system involving three coupled nonlinear equations has been derived to study (2 + 1)-dimensional waves propagating in a bulk medium made of an elastic material with couple stresses (Truesdell & Noll, 1992). They are given by

$$\begin{aligned}
&\mathrm{i}A_T + pA_{XX} + rA_{YY} = q|A|^2A + \frac{k^2}{2\omega}(\gamma_3\phi_{1,X} + \gamma_1\phi_{2,Y})A,\\
&(c_g^2 - c_1^2)\phi_{1,XX} - c_2^2\phi_{1,YY} - (c_1^2 - c_2^2)\phi_{2,XY} = \gamma_3 k^2(|A|^2)_X,\\
&(c_g^2 - c_2^2)\phi_{2,XX} - c_1^2\phi_{2,YY} - (c_1^2 - c_2^2)\phi_{1,XY} = \gamma_1 k^2(|A|^2)_Y,
\end{aligned} \tag{1.1}$$

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where $c_g = c_2^2(k + 8m^2k^3)/\omega$ and $\omega = c_2k(1 + 4m^2k^2)^{\frac{1}{2}}$, and

$$\begin{aligned}
&c_1^2 = \frac{\tilde{\lambda} + 2\tilde{\mu}}{\rho_0}, \quad c_2^2 = \frac{\tilde{\mu}}{\rho_0}, \quad \gamma_1 = c_1^2 - 2c_2^2 + \frac{\mathcal{B}}{\rho_0}, \\
&\gamma_3 = c_1^2 + \frac{\mathcal{A} + 2\mathcal{B}}{2\rho_0}, \quad p = -\frac{1}{2\omega}(c_g^2 - c_2^2 - 24m^2c_2^2k^2), \\
&r = \frac{c_2^2}{2\omega}(1 + 8m^2k^2), \quad q = \frac{k^6\gamma_3^2}{\omega D_1(2k, 2\omega)}.
\end{aligned} \tag{1.2}$$

Here $X$, $Y$ and $T$ are slow spatial and time variables, respectively, $A$ is the complex amplitude of a short transverse wave mode propagating parallel to the $z$ direction, $\phi_1$ and $\phi_2$ are long longitudinal wave and long transverse wave modes propagating parallel to the $x$ and $y$ directions, respectively. Moreover, $k$ is the wave number and $\omega$ is the frequency, $m$, $\tilde{\lambda}$, $\tilde{\mu}$, $\mathcal{A}$ and $\mathcal{B}$ are material constants, $c_g$ is the group velocity of transverse waves whereas $c_1$ and $c_2$ are phase speeds of longitudinal and transverse waves, respectively. In terms of dimensionless variables, the generalized Davey–Stewartson (GDS) system takes the form

$$\begin{aligned}
&\mathrm{i}u_\tau + u_{\xi\xi} + \gamma u_{\eta\eta} = \chi|u|^2u + b(\varphi_{1,\xi} + \varphi_{2,\eta})u, \\
&\varphi_{1,\xi\xi} + m_2\varphi_{1,\eta\eta} + n\varphi_{2,\xi\eta} = (|u|^2)_\xi, \\
&\lambda\varphi_{2,\xi\xi} + m_1\varphi_{2,\eta\eta} + n\varphi_{1,\xi\eta} = (|u|^2)_\eta,
\end{aligned} \tag{1.3}$$

where $\tau$ is a non-dimensional time variable whereas $\xi$ and $\eta$ are non-dimensional spatial variables, $u$ is the complex amplitude of the short transverse wave mode and $\varphi_1$ and $\varphi_2$ are the real long longitudinal and long transverse wave modes, respectively. Here the non-dimensional coefficients that play the key role in the classification of the system are given by (for details, see Babaoglu & Erbay, 2004)

$$\begin{aligned}
&m_1 = \frac{c_1^2}{c_1^2 - c_g^2}\left(\frac{\gamma_3}{\gamma_1}\right)^2, \quad m_2 = \frac{c_2^2}{c_1^2 - c_g^2}\left(\frac{\gamma_3}{\gamma_1}\right)^2, \\
&\lambda = \frac{c_2^2 - c_g^2}{c_1^2 - c_g^2}, \quad n = \frac{c_1^2 - c_2^2}{c_1^2 - c_g^2}\left(\frac{\gamma_3}{\gamma_1}\right).
\end{aligned} \tag{1.4}$$

While the usual Davey–Stewartson (DS) equations (Davey & Stewartson, 1974; Djordjevic & Redekopp, 1977; Ablowitz & Segur, 1979; Ghidaglia & Saut, 1990) model the interaction between a short wave mode with a long wave (mean flow) in water waves and result in a coupled system of two equations, the system of equations (1.3), that may be called the GDS equations, model the interaction between a short transverse wave, a long transverse wave and a long longitudinal wave propagating in an infinite elastic medium. Due to the asymmetric nature of the interaction between the long longitudinal and long transverse waves, $\varphi_1$ and $\varphi_2$ satisfy a linear system of strongly coupled equations, i.e. they cannot be uncoupled using a linear change of variables and/or unknowns. The analysis of GDS equations becomes more challenging due to this asymmetry.

In Babaoglu & Erbay (2004), based on (1.4), the following constraints on the parameters were imposed:

$$(\lambda - 1)(m_2 - m_1) = n^2, \quad \lambda > 1 \text{ and } m_2 > m_1 \geqslant 1. \tag{1.5}$$

The GDS system can be classified according to the values of these parameters. In addition, in the classification of $(1.3)_1$ the $\tau$ derivative (time evolution) is not taken into account; as in the DS system, the first equation of the GDS system is an elliptic or a hyperbolic Schrödinger equation when the sign of $\gamma$ is positive or negative, respectively. Considering the last two coupled equations of system (1.3) involving the variables $\varphi_1$ and $\varphi_2$, one can rewrite the corresponding homogeneous problem as a first-order linear system

$$\mathbf{A}\mathbf{v}_\xi + \mathbf{B}\mathbf{v}_\eta = \mathbf{0}, \tag{1.6}$$

where

$$\mathbf{A} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & \lambda & 0 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & m_2 & n & 0 \\ n & 0 & 0 & m_1 \end{pmatrix}, \tag{1.7}$$

in which $\mathbf{v} = (\varphi_{1,\xi}, \varphi_{1,\eta}, \varphi_{2,\xi}, \varphi_{2,\eta})^T$. Using $(\lambda - 1)(m_2 - m_1) = n^2$, the eigenvalues of $\mathbf{A}^{-1}\mathbf{B}$ are then expressed as

$$r_1 = -(-\frac{m_2}{\lambda})^{\frac{1}{2}}, \quad r_2 = (-\frac{m_2}{\lambda})^{\frac{1}{2}},$$
$$r_3 = -(-m_1)^{\frac{1}{2}}, \quad r_4 = (-m_1)^{\frac{1}{2}}.$$

Therefore, system (1.3) can be classified as EEE, EHH, and EEH according to the respective sign of $(m_1, m_2, \lambda)$: $(+,+,+)$, $(-,-,+)$, and $(+,+,-)$ if $\gamma$ is positive, and HEE, HHH, and HEH according to the respective sign of $(m_1, m_2, \lambda)$: $(+,+,+)$, $(-,-,+)$, and $(+,+,-)$ if $\gamma$ is negative (see Babaoglu *et al.*, 2005). Here we restrict our attention to two different cases, namely the EEE case and the HEE case.

This paper is organized as follows. In the next section we establish two non-existence results for travelling wave solutions, that satisfy some integrability conditions, for GDS equations. In the EEE case without any further parameter restrictions there exist no non-trivial travelling wave solutions for GDS in this class. The key role in the argument is played by a Pohozaev-type identity (2.4). This is no accident since Pohozaev-type identities are commonly used in establishing similar results for nonlinear Schrödinger and DS equations (Pohozaev, 1965; Ghidaglia & Saut, 1996; Cazanave, 1996; Sulem & Sulem, 1999). On the other hand, in the HEE case the existence of travelling waves imposes some constraints on the parameters, namely either $0 \leqslant -\frac{b}{m_1} \leqslant \chi \leqslant -b$ or $-b \leqslant \chi \leqslant -\frac{b}{m_1} < 0\cdot$ This result is achieved by deriving and utilizing a second Pohozaev-type identity, namely (2.5), and expressing it in the transformed domain as (2.17). This section ends with a brief discussion of a necessary condition for the existence of radial travelling wave solutions. In the final section, we relax the conditions assumed on the class of solutions and show that in the absence of such integrability conditions, weaker constraints must be imposed on the parameters. In fact, following Babaoglu & Erbay (2004) explicit travelling wave solutions can be written even in the EEE case.

## 2. Pohozaev-type identities and non-existence of travelling wave solutions

In this section we consider localized wave solutions of (1.3) in the form

$$\begin{aligned} u(\xi,\eta,\tau) &= U(x,y)\exp[\mathrm{i}(\omega\tau + kx + ly)], \\ \varphi_1(\xi,\eta,\tau) &= \Phi_1(x,y), \\ \varphi_2(\xi,\eta,\tau) &= \Phi_2(x,y), \end{aligned} \tag{2.1}$$

where $k, l, \omega \in \mathbb{R}$, $x = \xi - c\tau$, $y = \eta - d\tau$, $c, d \in \mathbb{R}$, and $U \in H^1(\mathbb{R}^2)$ and $\nabla\varphi_i \in \mathbb{L}^2(\mathbb{R}^2)$ $(i = 1, 2)$.

Inserting solution (2.1) into GDS system (1.3) we obtain

$$\begin{aligned}
&(kc + ld - k^2 - \gamma l^2 - \omega)U + U_{xx} + \gamma U_{yy} = \chi U^3 + b(\Phi_{1,x} + \Phi_{2,y})U,\\
&(2k - c)U_x + (2\gamma l - d)U_y = 0,\\
&\Phi_{1,xx} + m_2\Phi_{1,yy} + n\Phi_{2,xy} = (U^2)_x,\\
&\lambda\Phi_{2,xx} + m_1\Phi_{2,yy} + n\Phi_{1,xy} = (U^2)_y.
\end{aligned} \tag{2.2}$$

By the second equation of (2.2) $U$ is constant along the characteristics

$$(2k - c)y - (2\gamma l - d)x = \text{const.}$$

For an $H^1$ function $U$, this could be accomplished if and only if

$$c = 2k, \quad d = 2\gamma l.$$

If $\omega = \dfrac{c^2}{4} + \dfrac{d^2}{4\gamma}$, then system (2.2) reduces to

$$\begin{aligned}
&U_{xx} + \gamma U_{yy} = \chi U^3 + b(\Phi_{1,x} + \Phi_{2,y})U,\\
&\Phi_{1,xx} + m_2\Phi_{1,yy} + n\Phi_{2,xy} = (U^2)_x,\\
&\lambda\Phi_{2,xx} + m_1\Phi_{2,yy} + n\Phi_{1,xy} = (U^2)_y.
\end{aligned} \tag{2.3}$$

Now, we establish Pohozaev-type identities to prove the non-existence of travelling wave solutions.

THEOREM 2.1 (Pohozaev-type identities) Let $u, \varphi_1, \varphi_2$ be a travelling wave solution for (1.3) in the form (2.1) with $u \in H^1(\mathbb{R}^2)$ and $\nabla\varphi_i \in \mathbb{L}^2(\mathbb{R}^2)$ $(i = 1, 2)$. Then $U$, $\Phi_1$ and $\Phi_2$ must satisfy

$$\int_{\mathbb{R}^2} [(U_x)^2 + \gamma(U_y)^2]\,\mathrm{d}x\,\mathrm{d}y = 0, \tag{2.4}$$

$$\int_{\mathbb{R}^2} \{\chi U^4 + b[(\Phi_{1,x})^2 + m_2(\Phi_{1,y})^2 + \lambda(\Phi_{2,x})^2 + m_1(\Phi_{2,y})^2 + 2n\Phi_{1,x}\Phi_{2,y}]\}\,\mathrm{d}x\,\mathrm{d}y = 0. \tag{2.5}$$

*Proof.* The identities (2.4) and (2.5) are obtained by direct computations. Multiplying $(2.3)_1$ by $xU_x$ and integrating over $\mathbb{R}^2$, after a number of integrations by parts, we obtain

$$\int_{\mathbb{R}^2} \Big[(U_x)^2 - \gamma(U_y)^2 - \frac{\chi}{2}U^4 + bx\Phi_{1,x}(U^2)_x + b\Phi_2(U^2)_y + bx\Phi_{2,x}(U^2)_y\Big]\,\mathrm{d}x\,\mathrm{d}y = 0. \tag{2.6}$$

If the second and third equations of system (2.3) are used in (2.6), we obtain the following identity after several integrations by parts:

$$\int_{\mathbb{R}^2} \Big\{(U_x)^2 - \gamma(U_y)^2 - \frac{\chi}{2}U^4 + \frac{b}{2}\big[-(\Phi_{1,x})^2 + m_2(\Phi_{1,y})^2 - 3\lambda(\Phi_{2,x})^2 - m_1(\Phi_{2,y})^2 - 2n\Phi_{1,x}\Phi_{2,y}\big]\Big\}\,\mathrm{d}x\,\mathrm{d}y = 0. \tag{2.7}$$

Similarly, if we multiply $(2.3)_1$ by $yU_y$ and integrate the resulting equation over $\mathbb{R}^2$, we obtain

$$\int_{\mathbb{R}^2}\Big[(U_x)^2-\gamma(U_y)^2+\frac{\chi}{2}U^4-b\Phi_1(U^2)_x-by\Phi_{1,y}(U^2)_x-by\Phi_{2,y}(U^2)_y\Big]\,\mathrm{d}x\,\mathrm{d}y=0. \tag{2.8}$$

In a similar fashion, if the second and third equations of system (2.3) are used in (2.8), we obtain

$$\int_{\mathbb{R}^2}\Big\{(U_x)^2-\gamma(U_y)^2+\frac{\chi}{2}U^4+\frac{b}{2}\big[(\Phi_{1,x})^2+3m_2(\Phi_{1,y})^2 \\ -\lambda(\Phi_{2,x})^2+m_1(\Phi_{2,y})^2+2n\Phi_{1,x}\Phi_{2,y}\big]\Big\}\,\mathrm{d}x\,\mathrm{d}y=0. \tag{2.9}$$

Finally, if we multiply $(2.3)_1$ by $U$, and use the second and third equations of system (2.3), we obtain

$$\int_{\mathbb{R}^2}\Big\{(U_x)^2+\gamma(U_y)^2+\chi U^4+b\big[(\Phi_{1,x})^2+m_2(\Phi_{1,y})^2+\lambda(\Phi_{2,x})^2 \\ +m_1(\Phi_{2,y})^2+2n\Phi_{1,x}\Phi_{2,y}\big]\Big\}\,\mathrm{d}x\,\mathrm{d}y=0. \tag{2.10}$$

Subtracting the sum of (2.7) and (2.10) from (2.9) yields

$$\int_{\mathbb{R}^2}[(U_x)^2+\gamma(U_y)^2]\,\mathrm{d}x\,\mathrm{d}y=0. \tag{2.11}$$

Substituting (2.11) into (2.10) we obtain (2.5):

$$\int_{\mathbb{R}^2}\big\{\chi U^4+b\big[(\Phi_{1,x})^2+m_2(\Phi_{1,y})^2+\lambda(\Phi_{2,x})^2 \\ +m_1(\Phi_{2,y})^2+2n\Phi_{1,x}\Phi_{2,y}\big]\big\}\,\mathrm{d}x\,\mathrm{d}y=0. \tag{2.12}$$

□

COROLLARY 2.2 (Non-existence of travelling wave solutions)

(a) Travelling wave solutions of the form (2.1) cannot exist in the EEE case.

(b) Travelling wave solutions of the form (2.1) can only exist in the HEE case if $0\leqslant-\frac{b}{m_1}\leqslant\chi\leqslant-b$ or $-b\leqslant\chi\leqslant-\frac{b}{m_1}<0$. In other words, if $b\leqslant0$ and $\chi+b>0$ or $b\leqslant0$ and $\chi+\frac{b}{m_1}<0$ then no travelling wave solutions can exist. Similarly, if $b>0$ and $\chi+b<0$ or $b>0$ and $\chi+\frac{b}{m_1}>0$ then no travelling wave solutions can exist.

(c) In the HEE case, if radial travelling wave solutions exist then

$$\chi+b\,\Gamma=0, \tag{2.13}$$

where

$$\Gamma=\frac{\sqrt{m_1}(1+m_1-2n)+m_2+\lambda m_1+\sqrt{m_2\lambda}(1+\sqrt{m_1})^2}{\sqrt{m_1}(1+\sqrt{m_1})(\sqrt{m_2}+\sqrt{\lambda})(\sqrt{m_2}+\sqrt{m_1\lambda})}.$$

*Proof.* (a) Let $\gamma$ be positive and the respective sign of $(m_1, m_2, \lambda)$ be $(+,+,+)$. This corresponds to the EEE case. In such a case, condition (2.11) is satisfied by an $H^1$ function if and only if $U = 0$. Thus (2.3) reduce to

$$\begin{aligned} \Phi_{1,xx} + m_2\Phi_{1,yy} + n\Phi_{2,xy} &= 0,\\ \lambda\Phi_{2,xx} + m_1\Phi_{2,yy} + n\Phi_{1,xy} &= 0. \end{aligned} \tag{2.14}$$

Note that $U = 0$ means absence of the short wave mode. We will now show that this implies the absence of localized disturbances. Now we look for a general solution of system (2.14) involving the variables $\Phi_1$ and $\Phi_2$, when $U = 0$. Taking the Fourier transform of (2.14) leads to a homogeneous linear system for the unknown functions $\hat{\Phi}_1$ and $\hat{\Phi}_2$:

$$\begin{aligned} (\xi_1^2 + m_2\xi_2^2)\hat{\Phi}_1 + n\xi_1\xi_2\hat{\Phi}_2 &= 0,\\ n\xi_1\xi_2\hat{\Phi}_1 + (\lambda\xi_1^2 + m_1\xi_2^2)\hat{\Phi}_2 &= 0, \end{aligned} \tag{2.15}$$

where $(\xi_1, \xi_2)$ is the dual variable of $(x, y)$, and $\hat{\Phi}_1$ and $\hat{\Phi}_2$ are the Fourier transforms of $\Phi_1$ and $\Phi_2$, respectively. The determinant of the coefficient matrix, given as

$$\begin{aligned} \Delta &= \lambda\xi_1^4 + (m_1 + \lambda m_2 - n^2)\xi_1^2\xi_2^2 + m_1m_2\xi_2^4\\ &= (\lambda\xi_1^2 + m_2\xi_2^2)(\xi_1^2 + m_1\xi_2^2), \end{aligned} \tag{2.16}$$

is positive since $m_1 + \lambda m_2 - n^2 = \lambda m_1 + m_2$, which follows from (1.5). Therefore, system (2.15) has only the trivial solution. This shows that no travelling wave solutions of the form (2.1) to the GDS system (1.3) exist in the EEE case.

(b) Let $\gamma$ be negative and the respective sign of $(m_1, m_2, \lambda)$ be $(+,+,+)$. This corresponds to the HEE case. Utilizing Plancherel's theorem, (2.12) is readily expressed as

$$\int (\chi + \alpha b)|\hat{f}|^2 \, \mathrm{d}\xi_1 \, \mathrm{d}\xi_2 = 0 \tag{2.17}$$

where $\hat{f}$ denotes the Fourier transform of $|U|^2$, and

$$\alpha = \alpha(\xi_1, \xi_2) = \frac{\lambda\xi_1^4 + (1 + m_1 - 2n)\xi_1^2\xi_2^2 + m_2\xi_2^4}{\lambda\xi_1^4 + (\lambda m_1 + m_2)\xi_1^2\xi_2^2 + m_1m_2\xi_2^4}. \tag{2.18}$$

Here $(1.3)_2$ and $(1.3)_3$ are used to express the Fourier transforms of $\Phi_1$ and $\Phi_2$ as

$$\begin{aligned} \hat{\Phi}_1 &= \frac{\mathrm{i}\xi_1}{\Delta}(n\xi_2^2 - \lambda\xi_1^2 - m_1\xi_2^2)\hat{f},\\ \hat{\Phi}_2 &= \frac{\mathrm{i}\xi_2}{\Delta}(n\xi_1^2 - \xi_1^2 - m_2\xi_2^2)\hat{f}, \end{aligned} \tag{2.19}$$

where $\Delta$ is the same quantity given in (2.16). In Babaoglu *et al.* (2005), it is shown that for $m_1 \geqslant 1$ one has $\frac{1}{m_1} \leqslant \alpha \leqslant 1$. Together with (2.17) this shows that $\chi$ and $b$ are of opposite signs. If $b \leqslant 0$ then $\chi \geqslant 0$ and consequently $\chi + b \leqslant 0$ and $\chi + \frac{b}{m_1} \geqslant 0$. On the other hand, if $b > 0$ then $\chi < 0$ and consequently $\chi + \frac{b}{m_1} \leqslant 0$ and $\chi + b \geqslant 0$. Therefore, if the system (1.3) has travelling wave solutions of the form (2.1) then $0 \leqslant -\frac{b}{m_1} \leqslant \chi \leqslant -b$ or $-b \leqslant \chi \leqslant -\frac{b}{m_1} < 0$ holds.

(c) In the HEE case, note that (2.17) can also be rewritten as follows:

$$\int (\chi + b - \beta(\xi_1, \xi_2)\, b)|\hat{f}|^2\, \mathrm{d}\xi_1\, \mathrm{d}\xi_2 = 0, \tag{2.20}$$

where

$$\beta(\xi_1, \xi_2) = \frac{1}{\Delta}[(\lambda m_1 + m_2 - 1 - m_1 + 2n)\xi_1^2\xi_2^2 + m_2(m_1 - 1)\xi_2^4].$$

If $U$ is a radial solution, i.e. $U = U(r)$, then (2.20) simplifies to

$$\int_0^{2\pi} [\chi + b - B(\theta)b]\, \mathrm{d}\theta = 0, \tag{2.21}$$

where

$$B(\theta) = \frac{(\lambda m_1 + m_2 - 1 - m_1 + 2n)\cos^2\theta \sin^2\theta + m_2(m_1 - 1)\sin^4\theta}{\lambda \cos^4\theta + (\lambda m_1 + m_2)\cos^2\theta \sin^2\theta + m_1 m_2 \sin^4\theta}.$$

Using Mathematica, the integral (2.21) is evaluated and the following condition is obtained for the existence of the radial solutions to system (1.3):

$$\chi + b\Gamma = 0, \tag{2.22}$$

where

$$\Gamma = \frac{\sqrt{m_1}(1 + m_1 - 2n) + m_2 + \lambda m_1 + \sqrt{m_2\lambda}(1 + \sqrt{m_1})^2}{\sqrt{m_1}(1 + \sqrt{m_1})(\sqrt{m_2} + \sqrt{\lambda})(\sqrt{m_2} + \sqrt{m_1\lambda})}.$$

This necessary condition for the existence of radial solutions to system (1.3) is rather complicated when compared to that of the DS system given in Ghidaglia & Saut (1996). ☐

REMARK 2.3 Pertaining to (c), let us note that since there exist no travelling wave solutions in the EEE case, the same applies to radial solutions.

REMARK 2.4 The conditions $\nabla\varphi_i \in \mathbb{L}^2(\mathbb{R}^2)$ $(i = 1, 2)$ are the natural ones for the EEE case and HEE case. When the second and third equations are both of hyperbolic type then, following Ablowitz & Segur (1979), boundary conditions should be imposed along characteristic lines at infinity (see also Ghidaglia & Saut, 1990). These boundary conditions do not allow us to perform integration by parts as in Ozawa (1992), Fokas *et al.* (2001) etc. hence we cannot derive Pohozaev-type identities.

## 3. A remark on a class of travelling wave solutions

In order to emphasize the dependence of the results in the previous section on the assumption that $u \in H^1(\mathbb{R}^2)$ and $\nabla\varphi_i \in \mathbb{L}^2(\mathbb{R}^2)$ $(i = 1, 2)$, we will now establish that in the absence of such conditions much weaker constraints are to be imposed on the parameters. To wit, in Babaoglu & Erbay (2004) travelling wave solutions to GDS system (1.3) were sought in the following form:

$$u = f(\zeta)\mathrm{e}^{\mathrm{i}\theta}, \quad \varphi_1 = g(\zeta), \quad \varphi_2 = h(\zeta), \tag{3.1}$$

where $\theta = l_1\xi + l_2\eta - \Omega\tau$, and the amplitudes of the short wave $f$, and the long travelling waves $g$ and $h$ are real functions of $\zeta = k_1\xi + k_2\eta - 2(l_1k_1 + \gamma l_2k_2)\tau$. Here we assume that $f$, $f'$, $g'$ and $h'$ tend to zero as $|\zeta| \to \infty$. Substituting solution (3.1) into system (1.3) leads to a set of coupled ordinary differential equations

$$\begin{aligned}
&(k_1^2 + \gamma k_2^2) f'' - (l_1^2 + \gamma l_2^2 - \Omega) f = \chi f^3 + b(k_1 g' + k_2 h') f,\\
&(k_1^2 + m_2 k_2^2) g'' + n k_1 k_2 h'' = 2k_1 f f',\\
&n k_1 k_2 g'' + (\lambda k_1^2 + m_1 k_2^2) h'' = 2k_2 f f',
\end{aligned} \tag{3.2}$$

where a prime denotes differentiation with respect to $\zeta$. Integrating equations $(3.2)_2$ and $(3.2)_3$ with respect to $\zeta$ once and solving the resulting algebraic equations for $g'$ and $h'$ in terms of $f^2$ gives

$$g' = \alpha_1 f^2, \qquad h' = \alpha_2 f^2, \tag{3.3}$$

where

$$\begin{aligned}
&\alpha_1 = \frac{k_1}{\Delta}(\lambda k_1^2 + m_1 k_2^2 - n k_2^2), \quad \alpha_2 = \frac{k_2}{\Delta}(k_1^2 + m_2 k_2^2 - n k_1^2),\\
&\Delta = (\lambda k_1^2 + m_2 k_2^2)(k_1^2 + m_1 k_2^2),
\end{aligned}$$

In such a case, the first equation of (3.2) takes the form

$$f'' = 2a_1 f^3 + a_2 f, \tag{3.4}$$

where the coefficients $a_1$ and $a_2$ are

$$\begin{aligned}
a_1 &= \frac{1}{2(k_1^2 + \gamma k_2^2)}[\chi + b(k_1\alpha_1 + k_2\alpha_2)],\\
a_2 &= \frac{1}{k_1^2 + \gamma k_2^2}(l_1^2 + \gamma l_2^2 - \Omega).
\end{aligned}$$

By integrating (3.4) with respect to $\zeta$ twice, in the case of $a_1 < 0$ and $a_2 > 0$, the following solitary wave solutions in terms of a hyperbolic secant function and hyperbolic tangent functions for the envelope function $f(\zeta)$ and the long waves $g(\zeta)$ and $h(\zeta)$ are obtained as

$$\begin{aligned}
f(\zeta) &= \pm\sqrt{\frac{a_2}{-a_1}}\ \operatorname{sech}(\sqrt{a_2}\ \zeta + \delta),\\
g(\zeta) &= -\frac{\alpha_1}{a_1}\sqrt{a_2}\ \tanh(\sqrt{a_2}\ \zeta + \delta),\\
h(\zeta) &= -\frac{\alpha_2}{a_1}\sqrt{a_2}\ \tanh(\sqrt{a_2}\ \zeta + \delta).
\end{aligned} \tag{3.5}$$

Note that for such solutions $u \notin H^1$ and $\nabla\varphi_i \notin \mathbb{L}^2$ since on the lines $\zeta =$ constant, $f$, $g'$, $h'$ are non-zero constants. So we can obtain travelling wave solutions for (1.3) if $a_1 < 0$ and $a_2 > 0$. But $\Omega$, $l_1$ and $l_2$ are arbitrary hence one can always guarantee the second constraint. Note that

$$a_1 = \frac{1}{2(k_1^2 + \gamma k_2^2)}[\chi + \alpha(k_1, k_2) b], \tag{3.6}$$

where $\alpha$ is given in (2.18), and $\alpha(k_1, k_2) \to 1$ for fixed $k_2$ if $k_1 \to \infty$ and $\alpha(k_1, k_2) \to \dfrac{1}{m_1}$ for fixed $k_1$ if $k_2 \to \infty$. In the following the EEE and HEE cases are considered and the constraints on the parameters are obtained for the existence of travelling wave solutions of the form (3.1).

(a) Let the parameter $\gamma$ be positive. This corresponds to the EEE case where $k_1^2 + \gamma k_2^2 > 0$. In such a case

  (i) if $b < 0$ then $\chi + b \leqslant \chi + \alpha b \leqslant \chi + \dfrac{b}{m_1}$. In addition, let $\chi + b < 0$. Thus, for fixed $k_2$ we can choose $k_1$ large enough so that $\chi + \alpha(k_1, k_2)b \to \chi + b$, guaranteeing $a_1 < 0$, i.e. travelling wave solutions to the GDS system can exist.

  (ii) If $b > 0$ then $\chi + \dfrac{b}{m_1} \leqslant \chi + \alpha b \leqslant \chi + b$, and if $\chi + \dfrac{b}{m_1} < 0$, for fixed $k_1$, we can choose $k_2$ large enough so that $\chi + \alpha(k_1, k_2)b \to \chi + \dfrac{b}{m_1}$, showing that $a_1 < 0$, i.e. travelling wave solutions can exist. This shows the existence of travelling waves even in the EEE case.

(b) On the other hand, in the HEE case where $\gamma < 0$, $k_1^2 + \gamma k_2^2$ may be of either sign depending on the values of $k_1$ and $k_2$. Thus,

  (i) if $b < 0$ and $\chi + b < 0$ then $\chi + b \leqslant \chi + \alpha b \leqslant \chi + \dfrac{b}{m_1}$. Hence, for fixed $k_2$ we can choose $k_1$ large enough so that $k_1^2 + \gamma k_2^2 > 0$ and $\chi + \alpha(k_1, k_2)b < 0$, guaranteeing $a_1 < 0$.

  (ii) If $b < 0$ and $\chi + b > 0$ then $\chi + \alpha(k_1, k_2)b > 0$. For fixed $k_1$, we can choose $k_2$ large enough so that $k_1^2 + \gamma k_2^2 < 0$, thus $a_1$ becomes negative. Therefore, the GDS equation (1.3) has travelling wave solutions of the form (3.1) without any constraint on the parameter $\chi$ if $b < 0$ in the HEE case.

  (iii) On the other hand, if $b > 0$ and $\chi + b < 0$ then $\chi + \dfrac{b}{m_1} \leqslant \chi + \alpha b \leqslant \chi + b$. Thus, $\chi + \alpha b < 0$. For fixed $k_2$, we can choose $k_1$ large enough such that $k_1^2 + \gamma k_2^2 > 0$, guaranteeing $a_1 < 0$.

  (iv) If $b > 0$ and $\chi + \dfrac{b}{m_1} > 0$, we can choose $k_2$ large enough so that $\chi + \alpha b > 0$ and $k_1^2 + \gamma k_2^2 < 0$, guaranteeing $a_1 < 0$.

Therefore, the GDS equation (1.3) has travelling wave solutions of the form (3.1) if $\chi < -b$ or $\chi > -\dfrac{b}{m_1}$ for $b > 0$. However, we do not know whether there exist travelling wave solutions of the GDS system when $-b \leqslant \chi \leqslant -\dfrac{b}{m_1}$ for $b > 0$ in the HEE case.

We summarize the above results in the following proposition.

PROPOSITION 3.1 If either of the conditions

$$\begin{aligned}
&\text{(a) EEE case: } \gamma > 0, \text{ and } b < 0 \text{ and } \chi + b < 0, \text{ or}\\
&\qquad \gamma > 0, \text{ and } b > 0 \text{ and } \chi + \frac{b}{m_1} < 0,\\
&\text{(b) HEE case: } \gamma < 0, \text{ and } b < 0 \text{ and } \chi + b \neq 0, \text{ or}\\
&\qquad \gamma < 0, \text{ and } b > 0 \text{ and } -\frac{b}{m_1} < \chi \text{ or } \chi < -b,
\end{aligned}$$

is met then there exist travelling wave solutions of the form (3.1).

REMARK 3.2 Solutions of purely hyperbolic tangent type are also established in Babaoglu & Erbay (2004) for the GDS system; these solutions, however, do not radically weaken the conditions imposed on the parameters.

REMARK 3.3 As discussed in Arkadiev *et al.* (1989), Ozawa (1992) and Fokas *et al.* (2001) various special types of solutions do exist for DS equations. In this paper, we have not tried to establish solutions of similar type for GDS equations; this will be pursued in a future work.

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