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Klein Geometries, Parabolic Geometries and Differential Equations of Finite Type

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Abstract. We define the infinitesimal and geometric orders of an effective Klein geometry G/H. Using these concepts, we prove (i) For any integer $m \geq 2$, there exists an effective Klein geometry G/H of infinitesimal order m such that G/H is a projective variety. (ii) An effective Klein geometry G/H of geometric order M defines a differential equation of order $M+1$ on G/H whose global solution space is G.
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1. Introduction

As stated in the review of the book [19], in current mathematics it is mainly the scores of Lie groups and Lie algebras that are extracted from the great symphony of Sophus Lie. Later, mainly due to the efforts of E. Cartan, W. Killing and H. Weyl, the classification of semisimple Lie algebras and their representations is achieved as one the most beautiful and complete theories in mathematics. Going back to the transformation groups of S. Lie, what do the root systems, Dynkin diagrams, Weyl groups... really correspond to in terms of the infinitesimal generators of a transitive and effective action of the Lie group? Conversely, if we start with a transitive and effective action of a Lie group, what do prolongations, differential invariants and other fundamental concepts arising in Lie's theory of transformation groups correspond to in terms of the Lie group and its Lie algebra? The purpose of this paper is to take a modest step towards the answers of these questions in the spirit of the framework proposed in [13]. The technical results stated in the above abstract appear as byproducts.

This paper is organized as follows. In Section 2 we work in the "universal envelope" of [13] and recall some well known facts about the infinite dimensional Lie algebra of formal vector fields $(J_\infty\mathfrak{X})_p$ with bracket $\{\ ,\ \}_\infty$ and k-jets of vector fields $(J_k\mathfrak{X})_p$ with the algebraic bracket $\{\ ,\ \}_k$ induced by $\{\ ,\ \}_\infty$, referring to [4], [8], [17], [14] for more details. $\{\ ,\ \}_k$ reduces the order by one (see (3)) and therefore does *not* endow $(J_k\mathfrak{X})_p$ with a Lie algebra structure.

In Section 3, we restrict our attention to geometries contained in the uni-

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versal filtration (1) and are defined by effective Klein geometries G/H. These geometries define filtration which stabilize at zero after a finite number of steps (see (9)). Using this fact, we define the infinitesimal order of an effective Klein geometry. This concept exists in the fundamental works [8], [17] and also in [4], pg.6. Now $\{\ ,\ \}_k$ restricts to the k-jets of the infinitesimal generators $J_k(\overline{\mathfrak{g}})_o \subset (J_k\mathfrak{X})_p$ of the action of G on G/H at some $o \in G/H$. On the other hand, we define another bracket $[\ ,\ ]_k$ (see (11)) which also reduces the order by one. The definition of this new bracket uses only the Lie algebras $\mathfrak{g}$, $\mathfrak{h}$ of G and H and seems to be unrelated to jets. Our main result (Proposition 3.4) shows that $\{\ ,\ \}_k$ and $[\ ,\ ]_k$ coincide and become the bracket of $\mathfrak{g}$ if G acts effectively on G/H and k is sufficiently large. This result allows us to detect jets inside G/H using only group theory.

In Section 4, we consider $|k|$-graded semisimple Lie algebras. This well known grading is induced by a choice of positive simple roots and seems to be totally unrelated to the grading in terms of jets used before. However, the main result of Section 4 (Proposition 4.2) shows that they indeed coincide. This fact implies the first result in the above abstract (Corollary 4.3), gives an affirmative answer to a question in [13] and also settles an open problem posed in [20] on pg. 325. It also opens the way to express many standard concepts in the theory of semisimple Lie groups in terms of jet theory by expressing them in terms of the coefficients of the Taylor expansions of the infinitesimal generators, but much remains to be done in this direction.

In Section 5, we introduce the concept of geometric order of an effective Klein geometry. This concept exists also in [20] and is implicit in [16]. We prove the analog of Proposition 3.4 on the group level (Proposition 5.5) and derive some consequences. The most notable is Corollary 5.7 which shows the existence of some canonical splittings.

In Section 6 we show that Corollary 5.8 implies the second result in the abstract. The mentioned differential equation is an ODE if $\dim G/H = 1$ and a system of PDE's if $\dim G/H \geq 2$. This differential equation reduces to Lie's First fundamental Theorem when geometric order is zero and also generalizes the well known Schwarzian differential equation for Mobius transformations.

In the Appendix we make some comments on the relation of our work to [8], [17], [6].

2. Formal vector fields

In this section we recall some well known facts in the form which will be needed in the next sections. We refer to [4], [8], [17], [14] for more details.

Let M be a differentiable manifold with $\dim M = n$ and $p \in M$. Let $\mathfrak{X}$ denote the Lie algebra of smooth vector fields on M and for $X \in \mathfrak{X}$, let $(j_\infty X)_p$ denote the ∞-jet of X at p. We define the vector space $(J_\infty\mathfrak{X})_p \doteq \{(j_\infty X)_p \mid X \in \mathfrak{X}\}$. For simplicity of notation, we denote $(J_\infty\mathfrak{X})_p$ by J_∞ in this section. Let $\widetilde{J}_k \subset J_\infty$ denote the subspace consisting of those $(j_\infty X)_p$ vanishing at all orders up to and including order $k \geq 0$. We set $\widetilde{J}_{-1} \doteq J_\infty$. Thus we obtain the following descending filtration of subspaces

$$\cdots \subset \widetilde{J}_2 \subset \widetilde{J}_1 \subset \widetilde{J}_0 \subset \widetilde{J}_{-1} = J_\infty \tag{1}$$

In the spirit of the framework proposed in [13], we call (1) the universal filtration at $p \in M$.

We now define the vector space $\widehat{J}_k \doteq \widetilde{J}_k / \ \widetilde{J}_{k+1}$, $-1 \leq k$. Note that $\widehat{J}_{-1} = T(M)_p =$ the tangent space of M at p. Thus we obtain

$$J_\infty = \widehat{J}_{-1} \oplus \widehat{J}_0 \oplus \widehat{J}_1 \oplus \widehat{J}_2 \oplus \cdots \tag{2}$$

We define a bracket $\{\ ,\ \}_\infty$ on J_∞ by $\{(j_\infty X)_p, (j_\infty Y)_p\}_\infty \doteq (j_\infty[X,Y])_p$. This gives a Lie algebra homomorphism $\mathfrak{X} \to J_\infty$ defined by $X \to (j_\infty X)_p$. The bracket $\{\ ,\ \}_\infty$ turns $\widehat{J}_0 \oplus \widehat{J}_1 \oplus \widehat{J}_2 \oplus \cdots$ into a graded Lie algebra: $\{\widehat{J}_i, \widehat{J}_j\} \subset \widehat{J}_{i+j}$, $0 \leq i,j$. We also have $[\widehat{J}_{-1}, \widehat{J}_i] \subset \widehat{J}_{i-1}$, $i \geq 0$ but $[x,y]$ is undefined for $x, y \in \widehat{J}_{-1}$ which can be checked using coordinates. It is standard to define $[\widehat{J}_{-1}, \widehat{J}_{-1}] = 0$ and turn J_∞ into a graded Lie algebra by setting $\widehat{J}_{-2} \doteq 0$. This definition turns out to be incompatible with the present framework (see the paragraph below (24)). For this reason, we leave $[\widehat{J}_{-1}, \widehat{J}_{-1}]$ undefined. Note that (1) is now a descending filtration of ideals inside J_0 but not inside $\widetilde{J}_{-1} = J_\infty$ since $[\widetilde{J}_{-1}, \widetilde{J}_k] \subset \widetilde{J}_{k-1}$.

We now truncate (2) at $k-1$ and define $J_k \doteq \widehat{J}_{-1} \oplus \widehat{J}_0 \oplus \cdots \oplus \widehat{J}_{k-1}$, $0 \leq k$, so that $J_0 = \widehat{J}_{-1}$. Clearly, $J_k = J_\infty / \widetilde{J}_k$. An element of J_k is called a k-jet of a vector field at p and is denoted by $(j_k X)_p$. Thus $(j_k X)_p = [(j_\infty X)_p]_k$ where $[(j_\infty X)_p]_k$ denotes the equivalence class of $(j_\infty X)_p$ in $J_\infty / \widetilde{J}_k$.

The bracket $\{\ ,\ \}_\infty$ gives the algebraic bracket

$$\{\ ,\ \}_k : \quad J_k \times J_k \to J_{k-1} \qquad 1 \leq k \tag{3}$$

defined as follows: For $(j_k X)_p$, $(j_k Y)_p \in J_k$ where $(j_k X)_p = [(j_\infty X)_p]_k$ and $(j_k Y)_p = [(j_\infty Y)_p]_k$, we define $\{(j_k X)_p, (j_k Y)_p\}_k \doteq [(j_\infty[X,Y])_p]_{k-1}$.

Let $J_{k,j}$ denote the kernel of the projection map $\pi_{k,j} : J_k \to J_j$, $0 \leq j \leq k-1$. Thus we have the exact sequence

$$0 \longrightarrow J_{k,j} \longrightarrow J_k \xrightarrow{\pi_{k,j}} J_j \longrightarrow 0 \tag{4}$$

Now $\{\ ,\ \}_k$ restricts to $J_{k,0}$ as

$$\{\ ,\ \}_k : \quad J_{k,0} \times J_{k,0} \to J_{k,0} \tag{5}$$

Thus $J_{k,0}$ is a Lie algebra with bracket $\{\ ,\ \}_k$ and is called the isotropy subalgebra. In fact, let $(\mathfrak{G}_k)^p_p$ be the Lie group of k-jets of local diffeomorphisms with source and target at p. Any choice of coordinates near p identifies $(\mathfrak{G}_k)^p_p$ with the k'th order jet group $GL_k(n)$ and $J_{k,0}$ is the Lie algebra of $(\mathfrak{G}_k)^p_p$.

Now let $(\mathfrak{G}_k)^p_q$ denote the set of all k-jets of local diffeomorphisms with source at $p \in M$ and target at $q \in M$. If $j_{k+1}(f)^p_q \in (\mathfrak{G}_{k+1})^p_q$, then $j_{k+1}(f)^p_q$ induces an isomorphism $\natural j_{k+1}(f)^p_q : (J_k\mathfrak{X})_p \to (J_k\mathfrak{X})_q$. In particular, we obtain a representation of $(\mathfrak{G}_{k+1})^p_p$ on J_k, that is, a homomorphism

$$\natural : (\mathfrak{G}_{k+1})^p_p \longrightarrow GL(J_k) \tag{6}$$

defined by $j_{k+1}(f)^p_p \to \natural j_{k+1}(f)^p_p$. The representation $\natural$ is faithful.

For an explicit formula for $\natural$ to be used in Section 5, let $(j_kX)_p \in J_k$ and $j_{k+1}(f)_p^p \in (\mathfrak{G}_{k+1})_p^p$. The diffeomorphism $f \circ e^{tX} \circ f^{-1}$ is the identity when $t = 0$ and defines curves starting at points near p. Differentiating these curves at $t = 0$, we obtain a vector field defined near p which we denote by $\frac{d}{dt}(f \circ e^{tX} \circ f^{-1})_{|t=0}$. We have

$$\natural j_{k+1}(f)_p^p((j_kX)_p) = j_k(\frac{d}{dt}(f \circ e^{tX} \circ f^{-1})_{|t=0})_p \tag{7}$$

The infinitesimal representation induced by (6) is

$$\flat : J_{k+1,0} \longrightarrow gl(J_k) \tag{8}$$

A subset $\mathfrak{T} \subset J_k$ is called transitive if $\pi_{k,0}(\mathfrak{T}) = J_{k,0} \;\; = T(M)_p$.

The proof the next lemma is a straightforward computation in local coordinates.

Lemma 2.1. *Let $\mathfrak{T} \subset J_k$ be transitive and $j_{k+1}(f)_p^p \in (\mathfrak{G}_{k+1})_p^p$. If $\natural j_{k+1}(f)_p^p(Y) = Y$ for all $Y \in \mathfrak{T}$, then $j_{k+1}(f)_p^p = id$.*

The local formulas for $\{\;,\;\}_\infty$ are obtained by differentiating the usual bracket formula $[X, Y] = X^a\partial_a Y^i - Y^a\partial_a X^i$ successively infinitely many times, evaluating at p and substituting jets. We denote the formula obtained by k-times differentiation by A_k, $k \geq 0$. For instance, A_0 is $\{(j_\infty X)_p, (j_\infty Y)_p\}^i = X^aY_a^i - Y^aX_a^i$. We make use of these formulas in the proof of Lemma 3.3.

It is also crucial to observe that if we replace $\mathfrak{X}$ in the above construction by the germs of vector fields at p, we get the same J_∞, since a partition of unity argument shows that any such germ comes from some $X \in \mathfrak{X}$. This is far from being true in the (complex) analytic category. In this case, if $(j_\infty X)_p = (j_\infty Y)_p$ for some $X, Y \in \mathfrak{X}$, then $X = Y$ on M and so J_∞ contains global information. We have a special case of this situation below where for some integer m, $(j_mX)_p$ uniquely determines X and (3) turns into an honest Lie bracket on some finite dimensional Lie algebra for $k = m + 1$.

3. Infinitesimal order

Let G be a Lie group (not necessarily connected) and H a Lie subgroup. G acts on G/H by $L_g(xH) = gxH$. Other than this action, there is another fundamental concept inherent in the definition of the homogeneous space G/H : the H-principal bundle $G \to G/H$. To emphasize our choice in this paper, we make the following

Definition 3.1. *A global Klein geometry consists of the following:*

i) A homogeneous space G/H

ii) The (left) action of G on G/H

In this paper, Klein geometry means global Klein geometry. We denote a Klein geometry by G/H and call G/H effective if G acts effectively. If K is the largest normal subgroup of G contained in H, then G/H is effective iff $K = \{e\}$.

We denote H by G_0 and the Lie algebras of G, G_0 by $\mathfrak{g}$, $\mathfrak{g}_0$. Now following [8], [17], we inductively define $\mathfrak{g}_{k+1} \doteq \{x \in \mathfrak{g}_k \mid [x, \mathfrak{g}] \subset \mathfrak{g}_k, \; 0 \leq k\}$. Then

$\mathfrak{g}_{k+1} \subset \mathfrak{g}_k$ is an ideal for $0 \leq k$. Since $\mathfrak{g}$ is finite dimensional, there exists an integer m such that $\mathfrak{g}_{m+i} = \mathfrak{g}_m$ for all $0 \leq i$. Since $\mathfrak{g}_m \subset \mathfrak{g}$ is an ideal, there exists a connected and normal subgroup $H \lhd G_{-1}$ with Lie algebra $\mathfrak{g}_m$. Now H is contained in the connected component of G_0 since $\mathfrak{g}_m \subset \mathfrak{g}_0$. Therefore $H = \{e\}$ if G acts effectively on G/G_0. So we conclude $\mathfrak{g}_m = 0$ in this case.

Definition 3.2. *The integer m such that $\mathfrak{g}_m = 0$ but $\mathfrak{g}_{m-1} \neq 0$ is called the infinitesimal order of the effective Klein geometry G/G_0.*

Until Section 5, order means infinitesimal order.

Therefore, an effective Klein geometry determines the descending filtration

$$\{0\} \subset \mathfrak{g}_{m-1} \subset \cdots \subset \mathfrak{g}_2 \subset \mathfrak{g}_1 \subset \mathfrak{g}_0 \subset \mathfrak{g} \tag{9}$$

Now, since $\mathfrak{g}_k \subset \mathfrak{g}_0$ is an ideal, we have the homomorphism $ad_k : \mathfrak{g}_0 \rightarrow \mathfrak{gl}(\mathfrak{g}/\mathfrak{g}_k)$ defined by $ad_k(x)(y + \mathfrak{g}_i) = [x, y] + \mathfrak{g}_k = ad_x(y) + \mathfrak{g}_k$ where $[\ ,\ ]$ is the bracket of $\mathfrak{g}$. We observe that $\ker(ad_k) = \mathfrak{g}_{k+1}$. This gives an alternative and more conceptual definition of the spaces in (9). In particular, we obtain the faithful representation

$$ad_k : \mathfrak{g}_0/\mathfrak{g}_{k+1} \longrightarrow \mathfrak{gl}(\mathfrak{g}/\mathfrak{g}_k) \tag{10}$$

(for simplicity of notation, we keep the same notation for maps when we pass to quotients or make identifications).

We define the bracket

$$[\ ,\ ]_{k+1} : \quad \mathfrak{g}/\mathfrak{g}_{k+1} \times \mathfrak{g}/\mathfrak{g}_{k+1} \rightarrow \mathfrak{g}/\mathfrak{g}_k \qquad 0 \leq k \tag{11}$$

by $[a + \mathfrak{g}_{k+1}, b + \mathfrak{g}_{k+1}]_{k+1} \doteq [a, b] + \mathfrak{g}_k$. Since $[\mathfrak{g}, \mathfrak{g}_{k+1}] \subset \mathfrak{g}_k$, $[\ ,\ ]_{k+1}$ is well defined. Note that $[\ ,\ ]_{m+1} = [\ ,\ ]$. We also have the projection map

$$\overline{\pi}_{k,j} : \mathfrak{g}/\mathfrak{g}_k \rightarrow \mathfrak{g}/\mathfrak{g}_j \qquad j + 1 \leq k \tag{12}$$

with kernel $\mathfrak{g}_j/\mathfrak{g}_k$ and the restricted bracket

$$[\ ,\ ]_k : \quad \mathfrak{g}_0/\mathfrak{g}_k \times \mathfrak{g}_0/\mathfrak{g}_k \rightarrow \mathfrak{g}_0/\mathfrak{g}_k \qquad 1 \leq k \tag{13}$$

which is well defined since $\mathfrak{g}_k \subset \mathfrak{g}_0$ is an ideal.

Now we also have the Lie algebra homomorphism $\mathfrak{g} \rightarrow \mathfrak{X}$ which maps $X \in \mathfrak{g}$ to its infinitesimal generator $\overline{X} \in \mathfrak{X}$, where $\mathfrak{X}$ is the Lie algebra of smooth vector fields on $M = G/G_0$. We denote the Lie subalgebra of infinitesimal generators by $\overline{\mathfrak{g}} \subset \mathfrak{X}$. Now $\overline{\mathfrak{g}}$ is isomorphic to $\mathfrak{g}$ since the action is both transitive and effective. We define $J_k(\overline{\mathfrak{g}})_o \doteq \{(j_k\overline{X})_o \mid \overline{X} \in \overline{\mathfrak{g}},\ 0 \leq k\} \subset (J_k\mathfrak{X})_o$. All the constructions of Section 2 can be done now with $\overline{\mathfrak{g}}$ at o and all the spaces obtained in this way imbed in the spaces in Section 2 together with their grading. In particular, we obtain a filtration contained in the universal filtration (1).

The restriction of (3) gives

$$\{\ ,\ \}_{k+1} : \quad J_{k+1}(\overline{\mathfrak{g}})_o \times J_{k+1}(\overline{\mathfrak{g}})_o \rightarrow J_k(\overline{\mathfrak{g}})_o \qquad 0 \leq k \tag{14}$$

(14) follows from [14] but can be checked also directly. Note that $J_0(\overline{\mathfrak{g}})_o = \mathfrak{g}/\mathfrak{g}_0 = T(G/G_0)_o$. The restriction of $\pi_{k,j}$ gives

$$\pi_{k,j} : J_k(\overline{\mathfrak{g}})_o \to J_j(\overline{\mathfrak{g}})_o \qquad j+1 \leq k \tag{15}$$

Clearly (15) commutes with (14). We have

$$\{\ ,\ \}_k : \quad J_{k,0}(\overline{\mathfrak{g}})_o \times J_{k,0}(\overline{\mathfrak{g}})_o \to J_{k,0}(\overline{\mathfrak{g}})_o \qquad 1 \leq k \tag{16}$$

We also have the faithful representation

$$\flat_{|(\mathfrak{g},\mathfrak{g}_0)} : J_{k+1,0}(\overline{\mathfrak{g}})_o \longrightarrow \mathfrak{gl}(J_k(\overline{\mathfrak{g}})_o) \tag{17}$$

The faithfulness of (17) follows from the infinitesimal analog of Lemma 2.1 and the fact that $J_k(\overline{\mathfrak{g}})_o \subset (J_k\mathfrak{X})_o$ is transitive.

To clarify the analogy between (11)-(14), (12)-(15), (13)-(16) and (10)-(17), we define the map

$$\begin{aligned} \theta_k &: \mathfrak{g} \to J_k(\overline{\mathfrak{g}})_o \\ \theta_k(x) &= j_k(\overline{x})_o \quad k \geq 0 \end{aligned} \tag{18}$$

where $\overline{x}$ is the infinitesimal generator of x. Now θ_k is clearly linear and is surjective by the definition of $J_k(\overline{\mathfrak{g}})_o$.

Lemma 3.3. *The kernel of θ_k is $\mathfrak{g}_k$.*

Proof. We prove by induction on k that, for $0 \leq k$, $(j_k\overline{X})_o = 0$ if and only if $X \in \mathfrak{g}_k$.

For $k = 0$, the claim is that $\overline{X}(o) = 0$ if and only if $X \in \mathfrak{g}_0$. Let $\pi : G \to G/G_0$ be the quotient map, inducing $\overline{\pi} = d\pi(e) : \mathfrak{g} \to \mathfrak{g}/\mathfrak{g}_0$. By definition, $\overline{X}(o) = \frac{d}{dt}_{|t=0}(e^{tX}(o)) = \overline{\pi}(\frac{d}{dt}_{|t=0}e^{tX}) = \overline{\pi}(X)$. Hence the claim follows.

Now $X \in \mathfrak{g}_{k+1}$

$$\begin{aligned} &\Longleftrightarrow\ X \in \mathfrak{g}_k \text{ and } [X,Y] \in \mathfrak{g}_k \text{ for all } Y \in \mathfrak{g} \quad (\text{definition of } \mathfrak{g}_{k+1}) \\ &\Longleftrightarrow\ (j_k\overline{X})_o = 0 \text{ and } j_k\overline{[X,Y]}_o = 0 \text{ for all } Y \in \mathfrak{g} \quad (\text{induction hypothesis}) \\ &\Longleftrightarrow\ (j_k\overline{X})_o = 0 \text{ and } j_k[\overline{X},\overline{Y}]_o = 0 \text{ for all } Y \in \mathfrak{g} \\ &\Longleftrightarrow\ (j_k\overline{X})_o = 0 \text{ and } \{(j_{k+1}\overline{X})_o, (j_{k+1}\overline{Y})_o\}_{k+1} = 0 \text{ for all } \overline{Y} \in J_0(\overline{\mathfrak{g}})_o \\ &\qquad\qquad (\text{definition of } \{\ ,\ \}_{k+1}) \end{aligned} \tag{19}$$

We now choose some coordinate system (x^i) around o and recall that the components of $\{(j_{k+1}\overline{X})_o, (j_{k+1}\overline{Y})_o\}_{k+1}$ are given by the formulas $A_0, A_1, \ldots, A_k$. Since $(j_k\overline{X})_o = 0$, all terms in the formulas $A_0, A_1, \ldots, A_k$ vanish except the last term in A_k which is $-\overline{Y}^a\overline{X}^i_{aj_k\cdots j_1}$. Therefore, the last formula in (19) is equivalent to

$$\overline{Y}^a\overline{X}^i_{aj_k\cdots j_1} = 0 \quad \text{for all } \overline{Y} \in J_0(\overline{\mathfrak{g}})_o \tag{20}$$

Since $J_k(\overline{\mathfrak{g}})_o$ is transitive, (20) is equivalent to $\overline{X}^i_{j_{k+1}j_k\cdots j_1} = 0$, that is, $(j_{k+1}\overline{X})_o = 0$. This completes the inductive step. ■

Lemma 3.3 gives the linear isomorphism

$$\theta_k : \mathfrak{g}/\mathfrak{g}_k \longrightarrow J_k(\overline{\mathfrak{g}})_o \tag{21}$$

and its proof shows that (9) coincides with the filtration described above inside the universal filtration (1).

Proposition 3.4. *The diagrams*

$$\begin{array}{ccc} \mathfrak{g}/\mathfrak{g}_k & \xrightarrow{\theta_k} & J_k(\overline{\mathfrak{g}})_o \\ \downarrow_{\overline{\pi}_{k,j}} & & \downarrow_{\pi_{k,j}} \\ \mathfrak{g}/\mathfrak{g}_j & \xrightarrow{\theta_j} & J_j(\overline{\mathfrak{g}})_o \end{array} \tag{22}$$

$$\begin{array}{ccc} \mathfrak{g}/\mathfrak{g}_{k+1} \times \mathfrak{g}/\mathfrak{g}_{k+1} & \xrightarrow{[\ ,\ ]_{k+1}} & \mathfrak{g}/\mathfrak{g}_k \\ \downarrow_{\theta_{k+1}\times\theta_{k+1}} & & \downarrow_{\theta_k} \\ J_{k+1}(\overline{\mathfrak{g}})_o \times J_{k+1}(\overline{\mathfrak{g}})_o & \xrightarrow{\{\ ,\ \}_{k+1}} & J_k(\overline{\mathfrak{g}})_o \end{array} \tag{23}$$

commute for $k \geq 0$.

Proof. The commutativity of the first diagram is straightforward. As for the second, for $X, Y \in \mathfrak{g}_{-1}$ we have

$$\begin{aligned} \{\theta_{k+1}(X+\mathfrak{g}_{k+1}), \theta_{k+1}(Y+\mathfrak{g}_{k+1})\}_{k+1} &= \{(j_{k+1}\overline{X})_o, (j_{k+1}\overline{Y})_o\}_{k+1} \\ &= [(j_\infty\{\overline{X},\overline{Y}\})_o]_k \\ &= [(j_\infty\overline{[X,Y]})_o]_k \\ &= (j_k\overline{[X,Y]})_o \end{aligned}$$

while

$$\theta_k([X+\mathfrak{g}_{k+1}, Y+\mathfrak{g}_{k+1}]_{k+1}) = \theta_k([X,Y]+\mathfrak{g}_k) = (j_k\overline{[X,Y]})_o$$

■

Thus we also obtain the linear isomorphisms $\theta_m : \mathfrak{g} \to J_m(\overline{\mathfrak{g}})_o$, $\theta_k : \mathfrak{g}_0/\mathfrak{g}_k \to J_{k,0}(\overline{\mathfrak{g}})_o \subset (J_{k,0}\mathfrak{X})_o$ and the commutative diagram

$$\begin{array}{rccc} ad_k : & \mathfrak{g}_0/\mathfrak{g}_{k+1} & \longrightarrow & \mathfrak{gl}(\mathfrak{g}/\mathfrak{g}_k) \\ & \parallel\ \theta_{k+1} & & \parallel\ \theta_k \\ \flat_{|(\mathfrak{g},\mathfrak{g}_0)} : & J_{k+1,0}(\overline{\mathfrak{g}})_o & \longrightarrow & \mathfrak{gl}(J_k(\overline{\mathfrak{g}})_o) \end{array} \tag{24}$$

Lemma 3.3 and Proposition 3.4 show that $J_m(\overline{\mathfrak{g}})_o \simeq J_{m+1}(\overline{\mathfrak{g}})_o$ and (11) gives the bracket $\{\ ,\ \}_{m+1} : J_m(\overline{\mathfrak{g}})_o \times J_m(\overline{\mathfrak{g}})_o \to J_m(\overline{\mathfrak{g}})_o$ which coincides with the bracket of $\mathfrak{g}$. Therefore θ_m is also a Lie algebra isomorphism. This statement holds also for $m = 0$, that is, when G_0 is discrete. For $m = 0$, note that the standard notation $\mathfrak{g}_{-1}$ for $\mathfrak{g}$ and the extension of the formula (11) to $k = -1$ is incorrect in the present framework if $\mathfrak{g}$ is not abelian.

Definition 3.5. *Let* G/G_0 *be a any Klein geometry whose descending filtration stabilizes at* $\{0\}$*. Then* G/G_0 *is called almost effective.*

If G/G_0 is almost effective, then K is discrete as we show in Section 5 (see the paragraph before Lemma 5.1). Therefore the effective Klein geometry $\frac{G/K}{G_0/K}$ defines the same filtration as G/G_0 and has order m. This fact is used in Proposition 4.2 and Corollary 4.3.

In this section we worked exclusively at the point $o \in G/G_o$. Using homogeneity, all the above constructions can be done at any point $p \in G/G_o$. This fact is needed only in Corollary 5.8.

4. Parabolic geometries

Let $\mathfrak{s}$ be a $|k|$-graded semisimple Lie algebra over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, $k \in \mathbb{Z}$. Thus

$$\mathfrak{s} = \mathfrak{s}_{-k} \oplus \mathfrak{s}_{-k+1} \oplus \cdots \oplus \mathfrak{s}_{-1} \oplus \mathfrak{s}_0 \oplus \mathfrak{s}_1 \oplus \cdots \oplus \mathfrak{s}_{k-1} \oplus \mathfrak{s}_k \tag{25}$$

To recall the grading in (25), we first assume that $\mathbb{F} = \mathbb{C}$, $\mathfrak{s}$ is a semisimple Lie algebra and $\mathfrak{p} \subset \mathfrak{s}$ is a parabolic subalgebra. We can find a Cartan subalgebra and a set of positive roots such that $\mathfrak{p}$ is standard with respect to a set Σ of simple roots. Now the Σ-height gives a $|k|$-grading on $\mathfrak{s}$, where k is the Σ-height of the maximal root of $\mathfrak{s}$ which gives the grading (25) with $\mathfrak{p} = \mathfrak{s}_0 \oplus \mathfrak{s}_1 \oplus \cdots \oplus \mathfrak{s}_{k-1} \oplus \mathfrak{s}_k$. For $\mathbb{F} = \mathbb{R}$, the complexification $\mathfrak{s}^{\mathbb{C}} = \mathfrak{g}$ is also $|k|$-graded and $\mathfrak{s}$ is a real form of the complex pair $(\mathfrak{g}, \mathfrak{p})$. For our purpose in this section, the relevant fact is that the origin of the grading in (25) seems to be totally unrelated to the gradings defined before in terms of jets.

Let S be a Lie group with Lie algebra $\mathfrak{s}$ and $P \subset S$ a Lie subgroup with Lie algebra $\mathfrak{p}$. To be consistent with our notation above, we should denote P by S_0 but we will not do this.

If $x \in \mathfrak{p}$ satisfies $[x, \mathfrak{s}] \subset \mathfrak{p}$, then the grading in (25) implies $x \in \mathfrak{s}_k$ and clearly $[\mathfrak{s}_k, \mathfrak{s}] \subset \mathfrak{p}$. Further, if $x \in \mathfrak{s}_k$ satisfies $[x, \mathfrak{s}] \subset \mathfrak{s}_k$, then $x = 0$. This fact implies the following

Proposition 4.1. *The Klein geometry S/P is almost effective with descending filtration*

$$\{0\} \subset \mathfrak{s}_k \subset \mathfrak{p} \subset \mathfrak{s} \tag{26}$$

We rewrite (25) as $\mathfrak{s} = \mathfrak{p}^- \oplus \mathfrak{s}_0 \oplus \mathfrak{p}^+$. Since $\mathfrak{p}^-$, $\mathfrak{p}^+$ are dual with respect to Killing form, they determine each other and (25) involves a redundancy. Our purpose is to increase the length of (26) by exploiting this redundancy.

To simplify things and also to be specific, we assume $S = SL(n, \mathbb{F})$, $n \geq 2$, $\mathfrak{s}_0$ is the Cartan subalgebra of diagonal matrices, and $\mathfrak{p}$ is the Borel subalgebra of upper triangular matrices so that $k = n - 1$ in (25). However, the main idea of our construction works in the broader context of (25).

We start by choosing an abelian subalgebra of $\mathfrak{s}_{-1}$ which we denote by $\mathfrak{a}_{-1}$. For instance, some 1-dimensional subspace of $\mathfrak{s}_{-1}$ will do. If $n = 6$, then the matrices $A(2,1)$, $A(4,3)$, $A(6,5)$ in the standard basis having 1's in the indicated entries and 0's elsewhere belong to $\mathfrak{s}_{-1}$ and they commute. Thus we can choose $\dim \mathfrak{a}_{-1} = 3$ in this case. It is easy to check that $\dim \mathfrak{a}_{-1}$ can be at most $[\frac{n}{2}]$.

Having fixed $\mathfrak{a}_{-1}$, we now define $\mathfrak{h} \doteq \mathfrak{a}_{-1} \oplus \mathfrak{p}$. Since any $x \in \mathfrak{a}_{-1}$ is a common eigenvector for all $y \in \mathfrak{s}_0$, it follows that $[\mathfrak{a}_{-1}, \mathfrak{s}_0] = \mathfrak{a}_{-1} \subset \mathfrak{h}$. Using the grading and the fact that $\mathfrak{a}_{-1}$ and $\mathfrak{p}$ are both subalgebras, we conclude that $\mathfrak{h} \subset \mathfrak{s}$ is a subalgebra. Note that $\mathfrak{h}$ is neither semisimple nor solvable (compare to pg. xvi of the Introduction of [11]). We now choose a Lie subgroup $H \subset S$ with Lie algebra $\mathfrak{h}$ and H/P defines the descending filtration

$$\begin{aligned} \mathfrak{h} &= \mathfrak{a}_{-1} \oplus \mathfrak{p} = \mathfrak{a}_{-1} \oplus \mathfrak{s}_0 \oplus \cdots \oplus \mathfrak{s}_{n-1} \\ \mathfrak{h}_0 &= \mathfrak{p} = \mathfrak{s}_0 \oplus \cdots \oplus \mathfrak{s}_{n-1} \\ \mathfrak{h}_1 &= \mathfrak{s}_1 \oplus \cdots \oplus \mathfrak{s}_{n-1} \\ \mathfrak{h}_2 &= \mathfrak{s}_2 \oplus \cdots \oplus \mathfrak{s}_{n-1} \\ &\vdots \\ \mathfrak{h}_{n-1} &= \mathfrak{s}_{n-1} \\ \mathfrak{h}_n &= 0 \end{aligned} \tag{27}$$

Lemma 3.3 and Proposition 3.4 show that (27) is consistent with the order of jets and (27) shows that H/P is almost effective. We define $G \doteq H/K$ and $G_0 \doteq P/K$ where K is discrete and obtain

Proposition 4.2. *The order of the effective Klein geometry G/G_0 is n.*

Note that $\mathfrak{h}_1/\mathfrak{h}_2 = \mathfrak{s}_1 =$ the span of positive simple roots. Now (22) gives

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{h}_1/\mathfrak{h}_2 & \longrightarrow & \mathfrak{h}/\mathfrak{h}_2 & \overset{\pi_{2,1}}{\longrightarrow} & \mathfrak{h}/\mathfrak{h}_1 & \longrightarrow & 0 \\ & & \downarrow_{\theta_2} & & \downarrow_{\theta_2} & & \downarrow_{\theta_1} & & \\ 0 & \longrightarrow & J_{2,1}(\overline{\mathfrak{g}})_o & \longrightarrow & J_2(\overline{\mathfrak{g}})_o & \overset{\pi_{2,1}}{\longrightarrow} & J_1(\overline{\mathfrak{g}})_o & \longrightarrow & 0 \end{array} \tag{28}$$

Thus $J_{2,1}(\overline{\mathfrak{g}})_o$ completely determines the Lie algebra structure of $\mathfrak{s}$. However, this is not so for $\mathfrak{h}$ due to the arbitrariness involved in the choice of $\mathfrak{a}_{-1}$.

For $\mathbb{F} = \mathbb{C}$, S/P is known to be a projective variety. Since H/P is closed in S/P and $G/G_0 = H/P$, we obtain

Corollary 4.3. *For every integer $n \geq 2$ and d, $1 \leq d \leq [\frac{n}{2}]$, there exists an effective Klein geometry G/G_0 of order n such that G/G_0 is a projective variety.*

We know nothing about the structure of the projective varieties given by Corollary 4.3 in general. Even the simplest case $\dim \mathfrak{a}_{-1} = 1$ raises some questions about these Riemann surfaces whose answers we do not know.

We conclude this section with the following remark: The tangent space $T(S/P)_o$ of S/P is $\mathfrak{s}/\mathfrak{p} = \mathfrak{s}_{-n+1} \oplus \mathfrak{s}_{-n+2} \oplus \cdots \oplus \mathfrak{s}_{-1}$. Therefore the filtration (27) (starting with $\mathfrak{h}_1$) determines a filtration in $T(S/P)_o$ which, to our knowledge, is observed and studied first in [2] (see also [3]) in the framework of general parabolic geometries defined using principal P-bundles.

5. Geometric order

In this section we resume our framework of Section 3, assume that G/G_0 is effective and G is connected. Following [16], we inductively define $G_{k+1} \doteq \{g \in G_k \mid Ad(g)x - x \in \widehat{\mathfrak{g}}_k \text{ for all } x \in \mathfrak{g},\ 0 \leq k\}$ where $\widehat{\mathfrak{g}}_k$ is the Lie algebra of G_k for $k \geq 1$. Now $G_{k+1} \lhd G_k$ is a normal subgroup for $k \geq 0$. Thus we obtain the filtration

$$\cdots \subset G_2 \subset G_1 \subset G_0 \subset G \tag{29}$$

(29) defines the filtration

$$\cdots \subset \widehat{\mathfrak{g}}_2 \subset \widehat{\mathfrak{g}}_1 \subset \mathfrak{g}_0 \subset \mathfrak{g} \tag{30}$$

Note that G_{k+1} is the kernel of $Ad_k : G_0 \rightarrow GL(\mathfrak{g}/\widehat{\mathfrak{g}}_k)$ defined by $Ad_k(x)(y + \widehat{\mathfrak{g}}_k) \doteq Ad_x(y) + \widehat{\mathfrak{g}}_k$ which gives the faithful representation

$$Ad_k : G_0/G_{k+1} \longrightarrow GL(\mathfrak{g}/\widehat{\mathfrak{g}}_k) \tag{31}$$

If $\rho : G \rightarrow R$ is *any* homomorphism of Lie groups with differential $d\rho : \mathfrak{g} \rightarrow \mathfrak{r}$, then $\ker(d\rho)$ is clearly the Lie algebra of $\ker(\rho)$. This fact together with the definitions of $\widehat{\mathfrak{g}}_k$ and $\mathfrak{g}_k$ shows $\widehat{\mathfrak{g}}_k = \mathfrak{g}_k$. In particular, $\widehat{\mathfrak{g}}_m = \{0\}$ and G_m is discrete. If G/G_0 is almost effective with $\mathfrak{g}_m = \{0\}$, then $K \subset G_m$ because (29) stabilizes at K which is therefore discrete.

Lemma 5.1. *For an effective Klein geometry G/G_0 with the descending filtration (29), we have either i) $G_m = \{e\}$ or ii) $G_m \neq \{e\}$ and $G_{m+1} = \{e\}$*

Proof. Suppose $G_m \neq \{e\}$. As G_{m+1} is the kernel of $Ad_m \colon G_0 \rightarrow GL(\mathfrak{g}/\widehat{\mathfrak{g}}_m) = GL(\mathfrak{g})$, we have $G_{m+1} \subset \ker(Ad : G \rightarrow GL(\mathfrak{g})) = Z(G) =$ center of G since G is connected. Therefore $G_{m+1} \subset Z(G) \cap G_0 = \{e\}$ since the action is effective. ■

Definition 5.2. *The integer M satisfying $G_M = \{e\}$, $G_{M-1} \neq \{e\}$ is called the geometric order of the effective Klein geometry G/G_0*

Lemma 5.1 shows that $M = m$ or $M = m + 1$ for an effective Klein geometry. How do we decide which one is the case? *ii*) below hints that the answer is related to the fundamental group of G/G_0. This is expected since fundamental group is the new issue when we pass from Lie algebra to Lie group.

Setting $k = M - 1$ in (31), we obtain

Proposition 5.3. *Let G_0 be any Lie group. Suppose there exists a connected Lie group G satisfying a) $G_0 \subset G$ is a Lie subgroup b) G acts effectively on G/G_0. Then G_0 is a matrix group.*

Now G_0 is the stabilizer of the point o. Therefore, $g \in G_0$ defines the isomorphism $\natural j_{k+1}(L_g)^o_o : (J_k\mathfrak{X})_o \rightarrow (J_k\mathfrak{X})_o$ and we obtain the representation

$$Ad_k : G_0 \longrightarrow GL((J_k\mathfrak{X})_o) \tag{32}$$

defined by $g \mapsto \natural j_{k+1}(L_g)^o_o$. In fact, for $g \in G_0$, $X \in \mathfrak{g}$, it follows from (7) that

$$\begin{aligned} \natural j_{k+1}(L_g)^o_o((j_k\overline{X})_o) &= j_k(\frac{d}{dt}(L_g \circ L_{e^{tX}} \circ L_{g^{-1}})_{t=0}))_o \\ &= j_k(\frac{d}{dt}(L_{ge^{tX}g^{-1}})_{t=0})_o \\ &= j_k(\overline{Ad(g)X})_o \end{aligned} \tag{33}$$

Note the fundamental difference between (7) and (33): (7) is local whereas (33) is global and involves analyticity.

(33) shows that (32) restricts as

$$Ad_k : G_0 \longrightarrow GL(J_k(\overline{\mathfrak{g}})_o) \tag{34}$$

Lemma 5.4. *The kernel of Ad_k is G_{k+1}.*

Proof. Let $g \in \ker(Ad_k)$ so that $\natural j_{k+1}(L_g)^o_o$ is identity on $J_k(\overline{\mathfrak{g}})_o$, that is, $\natural j_{k+1}(L_g)^o_o((j_k\overline{X})_o) = (j_k\overline{X})_o$ for all $X \in \mathfrak{g}_{-1}$. Now (33) gives $j_k(\overline{Ad(g)X}))_o = (j_k\overline{X})_o$ or equivalently $j_k(\overline{Ad(g)X - X})_o = 0$. By Lemma 3.3, this condition is equivalent to $Ad(g)X - X \in \mathfrak{g}_k = \widehat{\mathfrak{g}}_k$ and therefore to $g \in G_{k+1}$ by the definition of G_{k+1}. ■

Proposition 5.5. *There is a canonical injection of Lie groups $\Phi_k : G_0/G_k \to (\mathfrak{G}_k)^o_o \simeq GL_k(n)$, $n = \dim G_{-1}/G_0$.*

Proof. We define the homomorphism $G_0 \to (\mathfrak{G}_k)^o_o$ by $g \to j_k(L_g)^o_o$. In view of Lemma 2.1, $j_k(L_g)^o_o = id$ if and only if $\natural j_k(L_g)^o_o = id$ and the conclusion follows from Lemma 5.4. ■

Lemma 5.4 gives the faithful representation $Ad_k : G_0/G_{k+1} \to GL(J_k(\overline{\mathfrak{g}})_o)$, and Propositions 3.4, 5.5 give the commutative diagram

$$\begin{array}{rccc} Ad_k : & G_0/G_{k+1} & \longrightarrow & GL(\mathfrak{g}/\mathfrak{g}_k) \\ & \| \; \Phi_{k+1} & & \| \; \theta_k \\ \natural_{|(G,G_0)} : & \Phi_{k+1}(G_0/G_{k+1}) & \longrightarrow & GL(J_k(\overline{\mathfrak{g}})_o) \end{array} \tag{35}$$

which is the group analog of (24).

Setting $k = M$ in Proposition 5.5 gives

Corollary 5.6. *There is a canonical injection of Lie groups $\Phi_M : G_0 \to (\mathfrak{G}_M)^o_o$*

The following two special cases of Corollary 5.6 are of special interest.

i) Let $P \subset SL(n, \mathbb{C})$ be the Borel subgroup of upper triangular matrices. Then there exists a discrete and normal subgroup $K \subset P$ such that P/K injects canonically into $(\mathfrak{G}_1)^o_o \simeq GL_M(1)$, where $M = n$ or $n+1$. To prove this statement, we recall the construction of G/G_0 in Proposition 4.2 and choose $\dim \mathfrak{a}_{-1} = 1$.

Before we state *ii*), we recall the main result of [15]: A smooth manifold N is determined up to diffeomorphism by the abstract Lie algebra structure of $\mathfrak{X}$. This statement holds also in the analytic category and we refer to [5] for an extensive literature on the generalizations of this result. It turns out that certain subalgebras of $\mathfrak{X}$ also determine N. In the analytic category, all the information in $\mathfrak{X}$ is encoded at one point. The next statement shows how the fundamental group is encoded in first order jets in a special case.

ii) Let G be simply connected, $G_0 \subset G$ discrete and G/G_0 effective. Then we have the canonical injection $\pi_1(G/G_0, o) \to (\mathfrak{G}_1)^o_o$ and the faithful representation $\pi_1(G/G_0, o) \to GL(\mathfrak{g})$. For the proof, we note that $G_0 \simeq \pi_1(G/G_0, o)$ since G_0 is simply connected and $m = 0$ since G_0 is discrete. Assuming that $G_0 \neq \{e\}$, we conclude $M \neq 0$ and therefore $M = 1$ by Lemma 5.1. We now set $k = M - 1$ in (35).

The next corollary is an extension of Corollary 5.6.

Corollary 5.7. *For any integer $k \geq 1$, there is a canonical injection $\Phi_{M+k} : G_0 \to (\mathfrak{G}_{M+k})^o_o$ which makes the following diagram commute:*

$$\begin{array}{ccc} (\mathfrak{G}_{M+k})^o_o & \overset{\pi_{M+k,M}}{\longrightarrow} & (\mathfrak{G}_M)^o_o \\ & \nwarrow_{\Phi_{M+k}} & \uparrow_{\Phi_M} \\ & & G_0 \end{array} \tag{36}$$

Proof. We define Φ_{M+k} as in the proof of Proposition 14. ∎

In the same way as Levi-Civita connection is the object canonically determined by a Riemannian structure, the splitting given by (36) implies the existence of some objects canonically determined by some geometric structures. This problem will be studied elsewhere (see [2], [3] for the standard but very different approach to this problem).

We single out the next fact, which is essentially equivalent to Corollary 5.6, as a corollary for reference in Section 6.

Corollary 5.8. *Let $a, b \in G$ satisfy $L_a(p) = L_b(p) = q$, for some $p, q \in G/G_0$. If $j_M(L_a)^p_q = j_M(L_b)^p_q$, then $a = b$.*

Proof. First we assume $p = o$. Now $j_M(L_a)^o_q = j_M(L_b)^o_q$ if and only if $j_M(L_{a^{-1}})^q_o \circ j_M(L_b)^o_q = j_M(L_{a^{-1}b})^o_o = id$. Therefore $a^{-1}b \in G_M = \{e\}$ and $a = b$. The claim for arbitrary p follows from homogeneity. ∎

6. Differential equations of finite type

In this section, we derive an important consequence of Corollary 5.8.

Following Lie, we write the action of G on G/G_0 locally as

$$f^i(x^1, x^2, \ldots, x^r, y^1, y^2, \ldots, y^n) = z^i \quad 1 \leq i \leq n \tag{37}$$

where $\dim G/G_0 = n$ and (y^i), (z^i) are local parameters for G/G_0, $\dim G = r$ and (x^i) are local parameters for G. We write (37) shortly as $f(x, y) = z$ and fix some $\overline{x}, \overline{y}, \overline{z}$ with $f(\overline{x}, \overline{y}) = \overline{z}$. Thus $\overline{x} \in G$ determines the diffeomorphism $z = z(y)$ defined by $f(\overline{x}, y) = z$. The k-jet $j_k(\overline{x})^{\overline{y}}_{\overline{z}}$ of this diffeomorphism is given by

$$\frac{\partial^{|\mu|} f^i(\overline{x}, \overline{y})}{\partial y^\mu} = \frac{\partial^{|\mu|} z^i}{\partial y^\mu}(\overline{y}) \qquad 0 \leq |u| \leq k \tag{38}$$

Corollary 5.8 asserts that $\overline{x}$ is uniquely determined by (38) for $0 \leq |u| \leq M$. Thus we can solve $\overline{x}$ in terms of $\overline{y}$ and $\frac{\partial^{|\mu|} z}{\partial y^\mu}(\overline{y})$ from (38) for $0 \leq |u| \leq M$ and substitute the result into (38) for $0 \leq |u| \leq M+1$. The result is

$$\Phi^i_\mu(\overline{y}, \frac{\partial^{|\sigma|} z^i}{\partial y^\sigma}(\overline{y})) = \frac{\partial^{|u|} z^i}{\partial y^\mu}(\overline{y}) \quad 1 \leq |u| \leq M+1,\ 0 \leq |\sigma| \leq M \tag{39}$$

for some functions Φ^i_μ. The form of (39) does not depend on the choices $\overline{x}, \overline{y}$. Regarding $\overline{y}$ as a variable, (39) defines a system of PDE's of finite type. If

$\dim G/G_0 = 1$, then (39) is a system of ODE's. We can write (39) in a coordinate system such that we have also $0 \leq |\sigma| \leq |u|$. The number of equations in (39) is $\dim GL_{M+1}(n)$ and $\dim G_0$ of them are dependent.

To summarize, we have

Proposition 6.1. *An effective Klein geometry G/H of geometric order M defines a differential equation on G/H of order $M+1$. The global solution space of this differential equation is G.*

In order to solve the group parameters uniquely from (38), the number of unknowns should not exceed the number of equations. This gives an inequality relating M, $\dim G$ and $\dim H$, which is analogous to the inequalities in [12] (see pg.161-162).

Two instances of Proposition 6.1 are well known:

1) Note that $M = 0$ iff $G_0 = \{e\}$ so that a Klein geometry G/G_0 of geometric order zero is nothing but the Lie group G together with the left action of G on itself. In view of the simple computation in the proof of Theorem 5 in [14] on page 178, (39) becomes

$$\frac{\partial z^i}{\partial y^j} = \xi^i_a(z)\omega^a_j(y) \tag{40}$$

where the vector fields $\xi_j = \xi^a_j \partial_a$ are the infinitesimal generators of the action and the 1-forms $\omega^i = \omega^i_a dx^a$ on G are components of the Maurer-Cartan form. Therefore Proposition 6.1 reduces to Lie's First Fundamental Theorem.

2) Let $G = SL(2,\mathbb{C})$ and $H =$ the subgroup of upper triangular matrices and $K = \{\pm I\}$. Now $G \doteq G/K$ can identified with the group $\mathcal{M}$ of normalized Möbius transformations

$$\frac{az+b}{cz+d} = w, \quad ad - bc = 1, \tag{41}$$

$G/H = \frac{G/K}{H/K} \doteq G/G_0$ with the complex sphere $\mathcal{S}$ and the effective action of G on G/G_0 with the effective action of $\mathcal{M}$ on $\mathcal{S}$ as point transformations. In this case $M = 2$ and the process of deriving (39) from (37) amounts to differentiating (41) three times and eliminating the group parameters (see pg.21 of [14] for details). We have $\dim G_3(1) = 3$ equations in (39) and $\dim G_0 = 2$ of these equations are dependent. In fact, the first two equations reduce to identities and the third one is the well known Schwarzian ODE

$$w''' = \frac{3}{2}\frac{(w'')^2}{w'} \tag{42}$$

We refer to [14], [12] on the relation of (42) to differential invariants and to [9] for the use of (42) in defining projective structures on Riemann surfaces.

Note that the second statement of Proposition 6.1 is rather a definition than an assertion. We refer to [19] for a categorical approach to the global formulation of PDE's and their symmetries in terms of diffieties and to the classical book [11] for a systematic study of symmetries of differential equations.

We arrived at (39) starting from geometry. Conversely, we can also start with differential equations of finite type and study their geometrization as in [21],

[22]. It is therefore no coincidence that semisimple Lie groups, parabolic subgroups and projective imbeddings arise naturally also in [21], [22] as in this paper. This geometrization problem is studied for general exterior differential systems in the influential paper [18].

To conclude this section, it is standard to take the bundle $G \to G/H$ as the basis of a Klein geometry and generalize G to an auxiliary but extremely useful object on which a Lie group H acts freely (see the Foreword of [16] by S.S.Chern for a very concise formulation of this point of view). It is well known that this approach has been remarkably successful with far reaching results. However, we believe that the approach to geometry based on transitive actions of Lie groups has also much to offer.

7. Appendix

All ingredients of our constructions in Section 3 are contained in the fundamental papers [8], [17] in much more generality and we make here some comments on the relation of Section 3 to these works, emphasizing the novelty of our approach.

Let $\mathfrak{g} \subset \mathfrak{gl}(V)$ be a subalgebra. Following [8], [17], [7], we set $pr_0\mathfrak{g} \doteq \mathfrak{g}$ and define the k'th prolongation $pr_k\mathfrak{g}$ of $\mathfrak{g}$ inductively by the formula

$$pr_k\mathfrak{g} \doteq \{S \in \mathrm{Hom}(V, pr_{k-1}\mathfrak{g}) \mid S(v)w = S(w) \text{ for all } v, w \in V\}.$$

Definition 7.1. *$\mathfrak{g} \subset \mathfrak{gl}(V)$ is called finite type if $pr_m\mathfrak{g} = 0$ for some m. If $pr_{\widetilde{m}}\mathfrak{g} = 0$ and $pr_{\widetilde{m}-1}\mathfrak{g} \neq 0$, we call $\widetilde{m}$ the prolongation order of $\mathfrak{g} \subset \mathfrak{gl}(V)$.*

The best known matrix algebras which are finite type are the orthogonal, conformal, projective and affine algebras (see [8], [17] for details). As far as we know, for all first order G-structures studied so far in geometry, either the Lie algebra $\mathfrak{g}$ of G is of infinite type (as in symplectic or complex structures) or of finite type with $\widetilde{m} \leq 2$.

Now setting $k = 0$ in (24), we make

Definition 7.2. *$ad_0(\mathfrak{g}_0/\mathfrak{g}_1) \subset \mathfrak{gl}(\mathfrak{g}/\mathfrak{g}_0)$ is called the first order isotropy algebra of the effective Klein geometry G/G_0.*

Now combining the definitions and constructions in [7], [8] with those in this paper, one can easily prove the following

Proposition 7.3. *The first order isotropy algebra of an effective Klein geometry G/G_0 is of finite type and we have $\widetilde{m} = m$. Conversely, for any finite type $\mathfrak{g} \subset \mathfrak{gl}(V)$, there exists an effective Klein geometry G/G_0 with $\mathfrak{g}/\mathfrak{g}_0 = V$ and $ad_0(\mathfrak{g}_0/\mathfrak{g}_1) = \mathfrak{g}$.*

Proposition 7.3 implies that our approach is equivalent to that in [8], [17] in the special case of finite type structures. However, we produce abundance of finite type structures of any prescribed order using semisimple Lie algebras. We find it noteworthy that transitive actions of Lie groups do not play any role in [8], [17] even though "transitive" is used with the meaning "homogeneous" in [8] as stated in the introduction of [8].

Also, the concept of a rigid geometric structure is introduced in [6]. These structures are characterized by the fact that their infinitesimal automorphisms are determined by their jets of some fixed order. In view of our results in this paper, it is not surprising that this concept turns out to being equivalent to first order G-structures being of finite type, as shown in [1]. Therefore, we believe that there is also some conceptual overlap with [6], [1] and this paper.

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References

[1] Candel, A., and R. Quiroga-Barranco, *Gromov's centralizer theorem*, Geom. Dedicata **100** (2003), 123–135.

[2] Cap, A., and H. Schichl, *Parabolic geometries and canonical Cartan connections*, Hokkaido Math. J. **29** (2000), 453–505.

[3] Cap, A., J. Slovak, and V. Soucek, *Bernstein-Gelfand-Gelfand Sequences*, Ann. of Math. **154** (2001), 97–113.

[4] Fuks, D., "Cohomology of Infinite Dimensional Lie Algebras," Contemporary Soviet Mathematics, Consultants Bureau, New York and London, 1986.

[5] Grabowski, J., and N. Poncin, *Lie algebraic characterization of manifolds*, Cent. Eur. J. Math. **2** (2005), 811–825.

[6] Gromov, M., *Rigid transformation groups*, Géometrie differentielle, Paris, 1986, 65-139, in: Travaux en Cours **33**, Hermann, Paris, 1988.

[7] Guillemin, V. W., *The integrability problem for G-structures*, Trans. Amer. Math. Soc. **116** (1965), 544–560.

[8] Guillemin, V. W., and S. Sternberg, *An algebraic model of transitive differential geometry*, Bull. Amer. Math. Soc. **70** (1964), 16–47.

[9] Gunning, R. C., "Lectures on Riemann Surfaces," Princeton Mathematical Notes, Princeton University Press, Princeton, N.J. 1966.

[10] Krasilschik, I. S., V. V. Lychagin, and A. M. Vinogradov, "Geometry of Jet Spaces and Nonlinear Partial Differential Equations," Translated from the Russian by A. B. Sosinski, Advanced Studies in Contemporary Mathematics, 1; Gordon and Breach Science Publishers, New York, 1986.

[11] Olver, P. J., "Applications of Lie groups to Differential Equations," Graduate Texts in Mathematics, Springer-Verlag, New York, 1986.

[12] —, "Equivalence, Invariants, and Symmetry," Cambridge University Press, 1995.

[13] Ortaçgil, E., *The heritage of S. Lie and F. Klein: Geometry via transformation groups*, arXiv.org, math.DG/0604223, posted on April 2006.

[14] Pommaret, J. F., "Partial Differential Equations and Group Theory," New Perspectives for Applications **293**, Kluwer Academic Publishers, 1994.

[15] Shanks, M. E., and L. E. Pursell, *The Lie algebra of a smooth manifold*, Proc. Amer. Math. Soc. **5** (1954), 468–472.

[16] Sharpe, R. W., "Differential geometry. Cartan's generalization of Klein's Erlangen Program," Graduate Texts in Mathematics **166**, Springer-Verlag, New York, 1997.

[17] Singer, I. M., and S. Sternberg, *The infinite groups of Lie and Cartan, I. The transitive groups*, J. Analyse Math. **15** (1965), 1–114.

[18] Tanaka, N., *On the equivalence problems associated with simple graded Lie algebras*, Hokkaido Math. J. **8** (1979), 23–84.

[19] Vinogradov, A. M., "Cohomological Analysis of Partial Differential Equations and Secondary Calculus," translated from the Russian manuscript by Joseph Krasilshik, Translations of Mathematical Monographs **204**, Amer. Math. Soc., Providence, RI, 2001.

[20] Weingart, G, *Holonomic and semi-holonomic geometries*, Global Analysis and Harmonic Analysis, Marseille-Luminy, 1999, 307-328, in: Semin. Congr. **4**, Soc. Math. France, Paris, 2000.

[21] Yamaguchi, K., and T. Yatsui, *Geometry of higher order differential equations of finite type associated with symmetric spaces*, Advanced Studies in Pure Mathematics **37** (2000), 397–458.

[22] —, *Parabolic geometries associated with differential equations of finite type*, Preprint.

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