B U Department of Mathematics

Spring 2008 Math 162 - Discrete Mathematics,
Second Midterm, 9/5/2008, 17:00-19:00

Our aim is not only to solve the problems but also to tell the solutions clearly and rigorously. JUSTIFY your claims! Otherwise your proofs will not be accepted. **All the identities you need are given at the last page.**	Full Name : Student ID : over 60

1. [1pt] What is, more or less, the epitaph of Bernoulli, the father?

2. (a) [6pts] Ten distinct passengers got into an elevator on the ground floor of a building which has twenty more floors upstairs. What is the probability that they will all get off at different floors?

   (b) [10pts] There is a 30 percent chance that it rains on any particular day. What is the probability that there is at least one rainy day within a 7-day period? Given that it rained at least once in those seven days, what is the probability that there were at least two rainy days?

3. [10pts] Suppose that we roll a pair of dice until the sum of the numbers on the two dice is nine. What is the expected number of times we roll the dice?

4. [10pts] Use generating functions to determine the number of different ways ten identical balloons can be given to four children if each child receives at least two balloons. *Hint: First, let the number of balloons vary. What is the contribution of each child to the generating function?*

5. In the classroom, the professor gives a quiz, collects the papers at the end of the exam and distributes them back to the class at random. Sometimes by chance a student gets his/her paper back and sometimes not. In this question we are trying to find out the number of distributions which have no student getting his own paper back. For example if there are three students, Ali, Banu and Can, we do not want to give Ali's paper back to Ali. Meanwhile, Ali's paper to Banu, Banu's to Can, Can's to Ali is OK. Consider the set $S_n = \{1, 2, 3, \ldots, n\}$. Let a_n be the number of 1-1 onto maps from the set S_n to itself without any fixed points. We want to find a_n given n. That is, we want to count the 1-1 onto maps $f : S \to S$ such that for every $x \in S$, $f(x) \neq x$. For example, for $S_2 = \{1, 2\}$, there is just a single such map: it sends 1 to 2 and 2 to 1. Thus $a_2 = 1$.

   (a) [1pts] Find a_3.
   (b) [2pts] Find a_4.
   (c) [10pts] Show that $a_n = (n-1)(a_{n-1} + a_{n-2})$ for $n \geq 4$.

6. [10pts] *Catalan numbers* are the sequence of numbers $C_1, C_2, C_3, \cdots$ satisfying the recurrence relation

$$C_n = \sum_{k=0}^{n-1} C_k C_{n-1-k}$$

with the initial conditions $C_0 = C_1 = 1$. Show that if $G(x)$ is the generating function for the sequence of Catalan numbers then $x(G(x))^2 - G(x) + 1 = 0$.

Some identities: $(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$, $\frac{1}{1-x} = 1+x+x^2+\cdots$, $\frac{1}{(1-x)^2} = \sum_{k=0}^{\infty}(k+1)x^k$, $\frac{1}{(1-x)^n} = \sum_{k=0}^{\infty} \binom{n+k-1}{k} x^k$.