Fall 2004 Math 488 - Calculus on Manifolds
Homework 1 - Integration and topology basics
Due: 13/10/2004

1. Prove that an open subset of $\mathbb{R}^n$ with respect to product topology is also open in Euclidean metric topology and vice versa.

2. Construct a subset of $\mathbb{R}^2$ with boundary the square region $\{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq 1, 0 \leq y \leq 1\}$ in $\mathbb{R}^2$.

3. Prove that every compact subset of $\mathbb{R}^n$ is closed and bounded.

4. Let A be a compact subset of $\mathbb{R}^n$ and S be an infinite collection of points in A. Prove that S has a limit point.

5. Evaluate
$$\int_{D^2 \times T} (x + yz + t)\, dx\, dy\, dz\, dt$$
where $D^2 \subset \mathbb{R}^2$ is the unit disk and $T \subset \mathbb{R}^2$ is the triangular region with vertices at $(0, 0)$, $(1, 0)$ and $(0, 1)$.

6. (a) Let $\mathcal{C}$ be the curve given as the graph in $\mathbb{R}^2$ of a differentiable function $f : [a, b] \to \mathbb{R}$. Find the length of $\mathcal{C}$.

   (b) Let $\mathcal{S}$ be the surface given as the graph in $\mathbb{R}^3$ of a differentiable function $f : D^2 \to \mathbb{R}$. Find the area of $\mathcal{S}$.

   (c) Let Ω be the domain given as the graph in $\mathbb{R}^{n+1}$ of a differentiable function $f : D^n \to \mathbb{R}$ where $D^n = \overline{B(0, 1)} \subset \mathbb{R}^n$, the closed (solid) ball centered at the origin with radius 1. Try to invent a formula for the volume of Ω.

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