Fall 2004 Math 488 - Calculus on Manifolds
Homework 3 - Integration; measure and content zero sets
Due: 08/11/2004

1. (question 2.33 in Spivak) Show that the continuity of $D_1 f^i$ at point a may be eliminated from the hypothesis of Theorem 2.8.

2. Construct a map $f : M \to N$ between two topological spaces M and N such that f is one-to-one, onto its image, continuous but not a homeomorphism.

3. (question 3.7 in Spivak) Let $f : [0,1] \times [0,1] \to \mathbb{R}$ be defined by
$$f(x,y) = \begin{cases} 0, & x \text{ or } y \text{ irrational} \\ 1/q, & x \in \mathbb{Q},\ y = \frac{p}{q} \text{ in lowest terms} \end{cases}$$
Show that f is integrable and $\displaystyle\int_{[0,1]\times[0,1]} f = 0$.

4. (question 3.9 in Spivak)
(a) Show that an unbounded set cannot have content 0.
(b) Give an example of a closed set of measure 0 which does not have content 0.

5. (question 3.11 in Spivak) Let A be an open set that is the union of the open intervals (a_i, b_i) such that each rational number in (0,1) is contained in some (a_i, b_i). If $\sum_{i=1}^{\infty}(b_i - a_i) < 1$, show that the boundary of A does not have measure 0. Can you imagine such a strange set?

6. (question 3.22 in Spivak) If A is a Jordan-measurable set and $\varepsilon > 0$, show that there is a compact Jordan-measurable set $K \subset A$ such that $\displaystyle\int_A \chi_{A-K} < \varepsilon$.

7. (question 3.26 in Spivak) Let $f : [a,b] \to \mathbb{R}$ be integrable and non-negative and let $A_f = \{(x,y) : a \le x \le b \text{ and } 0 \le y \le f(x)\}$. Show that A_f is Jordan-measurable and has area $\int_a^b f$.

8. Let Φ be a partition of unity for A and let K be a compact subset of A. Show that all but finitely many $\phi \in \Phi$ are 0 on K.

Ferit Öztürk, Boğaziçi University 2004