Fall 2004 Math 488 - Calculus on Manifolds
Homework 5 - Differential forms
Due: 1/12/2004

1. Not all alternating k-tensors in $\mathbb{R}^n$ can be written as a product of k 1-tensors. Find such a tensor (Hint: Remember the solution of question 3d of homework 4).

2. For α and β two differential forms, show that $d(\alpha + \beta) = d\alpha + d\beta$.

3. (a) For $f : \mathbb{R}^n \to \mathbb{R}$, compute $df \wedge df$ in standard basis.
   (b) For any $\alpha \in \Omega^1(\mathbb{R}^n)$ show that $\alpha \wedge \alpha = 0$.
   (c) Find a 2-form $\omega^c \in \Omega^2(\mathbb{R}^4)$ such that $\omega^c \wedge \omega^c \neq 0$.
   (d) A 2-form $\omega \in \Omega^2(\mathbb{R}^n)$ is *nondegenerate* if for any $p \in \mathbb{R}^n$ and any vector $v_1 \in T_p\mathbb{R}^n$ at p, there is a vector v_2 at p such that $w_p(v_1, v_2) \neq 0$. Is ω^c above nondegenerate?
   (e) Is there a nondegenerate 2-form on $\mathbb{R}^3$?
   (f) Find a closed nondegenerate 2-form in $\mathbb{R}^{2n}$. Such a form is called a *symplectic form*.

4. A goal of using 'differential form' language is to get free from the choice of coordinates. Remember that a differential k-form ω in $\mathbb{R}^n$ is a differentiable map $\omega : \mathbb{R}^n \to \bigwedge^k(\mathbb{R}^n)$. Find formulas for the transformation of 1- and 2-forms in $\mathbb{R}^n$ under a coordinate change.

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