Fall 2004 Math 488 - Calculus on Manifolds
Midterm - 6/12/2004 - 2 hours

1. Consider $f : \mathbb{R}^4 \to \mathbb{R}^3, f(x,y,z,t) = (xy^2, x+z, t^3x+yz)$, the image expressed in coordinates x_1, x_2, x_3. Find $f^*\omega$ for (a) $\omega = dx_1 + x_3 dx_2$; (b) $\omega = dx_1 \wedge dx_3$.

2. Write down the definition, which we had in the class, of the exterior differential of a k-form $\omega = \sum_{I \in S_k} a_I dx_I$ in $\mathbb{R}^n$. Show that this definition is equivalent to the following:

   The exterior differential $d : \Omega^k(\mathbb{R}^n) \to \Omega^{k+1}(\mathbb{R}^n)$ is the linear operator defined at each $p \in \mathbb{R}^n$ for each $\omega \in \Omega^k(\mathbb{R}^n)$ as above by

   $$(d\omega)_p(\nu_1, \cdots, \nu_{k+1}) = \sum_{i=1}^{k+1} (-1)^{i-1} D_p\omega(\nu_i)(\nu_1, \cdots, \widehat{\nu_i} \cdots, \nu_{k+1})$$

   where $(\nu_1, \cdots, \widehat{\nu_i} \cdots, \nu_{k+1}) = (\nu_1, \cdots, \nu_{i-1}, \nu_{i+1}, \cdots, \nu_{k+1})$ and $D_p\omega$ is a multilinear map that is defined on each of the standard basis vectors $\{e_1, \cdots, e_n\}$ by

   $$D_p\omega(e_j) = \sum_I D_j a_I(p) dx_I, \quad (1 \leq j \leq n).$$

3. (a) Find a 1-form $\alpha \in \Omega^1(\mathbb{R}^3)$ such that $\omega = \alpha \wedge d\alpha$ is nowhere zero, that is, $(\alpha \wedge d\alpha)_p$ is a nonzero tensor for every $p \in \mathbb{R}^3$. Such a 1-form is called a *contact form.*

   (b) For the 1-form α you found in part (a), define

   $$\xi_p = \ker(\alpha_p) = \{\nu \in T_p(\mathbb{R}^3) | \alpha_p(\nu) = 0\},$$

   a linear subspace of $T_p(\mathbb{R}^3)$. Find ξ_p explicitly for your α (either purely algebraically or using the geometric approach). What is the dimension of ξ_p? Try to draw a picture of $\mathbb{R}^3$ which illustrates the distribution of ξ_p at as reasonably many points as possible to give an idea.

   (c) Show that the closed form $d\alpha$ is a *symplectic form* when restricted to ξ, i.e. for any $p \in \mathbb{R}^3$, $d\alpha_p$ is nondegenerate on ξ_p.

   (d) Similarly a contact form on $\mathbb{R}^{2n+1}$ is a 1-form $\alpha \in \Omega^1(\mathbb{R}^{2n+1})$ such that $(\alpha \wedge (d\alpha)^n)_p$ is a nonzero tensor for each $p \in \mathbb{R}^{2n+1}$. Find a contact form on $\mathbb{R}^{2n+1}$.

4. What we are going to observe in this question is that once we identify vectors and vector fields with tensors and forms on $\mathbb{R}^3$, tensor and form algebra becomes very much familiar to us.
First, for any $x \in \mathbb{R}^3$ define

$$\alpha_x : \mathbb{R}^3 \to \mathbb{R},\ u \mapsto x \cdot u$$

and

$$\omega_x : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R},\ (u, v) \mapsto x \cdot (u \times v).$$

(a) Show that $\alpha_x \in \bigwedge^1(\mathbb{R}^3)$ and $\omega_x \in \bigwedge^2(\mathbb{R}^3)$.

(b) For any $x, y \in \mathbb{R}^3$, show that $\alpha_x \wedge \alpha_y = \omega_{x \times y}$. You might want to use the identity $a \times (b \times c) = (a \cdot c)b - (a \cdot b)c$.

(c) Let X be a smooth vector field over $\mathbb{R}^3$. Define a differential 1-form α_X on $\mathbb{R}^3$ by $(\alpha_X)_p = \alpha_{(X_p)}$ for each $p \in \mathbb{R}^3$. Find a familiar compact expression for the 2-form $d\alpha_X \in \Omega^2(\mathbb{R}^3)$.

As a summary:

The correspondance in $\mathbb{R}^3$ between form algebra and vector calculus

1-form α_X	vector field X (and dot product)
2-form ω_Y	vector field Y (and triple product)
exterior product of 1-forms	cross product of vector fields
exterior differential	...?...

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