PS 8 Q 10

$f: A \longrightarrow B$, A: compact.

Let Γ_f denote the graph of f in $A \times B$.

$\Gamma_f = \{ (a,b) \in A \times B \mid b = f(a) \}$.

<u>Prove</u>: f is continuous $\iff$ Γ_f is compact.

($\Rightarrow$) It will be shown that Γ_f is compact, by showing that every sequence in Γ_f has a convergent subsequence.

The metric introduced in Question 3 of HW 3 will be used for the metric on $A \times B$. It was shown there that

$d_{A\times B}: A \times B \longrightarrow \mathbb{R}$ defined by

$$d_{A\times B}((a_1,b_1),(a_2,b_2)) = d_A(a_1,a_2) + d_B(b_1,b_2)$$

is indeed a metric; when d_A is a metric on A & d_B; metric on B.

One can use equivalent metrics, to reach the same conclusion.

Part (b) of the exercise, showed that if (A, d_A) and (B, d_B) are complete metric spaces, then $(A \times B, d_{A\times B})$ is also complete.

To shorten the writing, the subscripts for metrics will be omitted as long as it is clear from the context which one is meant.

($\Rightarrow$) Let $\{(a_n,b_n)\}_{n=1}^{\infty}$ be a sequence in Γ_f. Then $\{a_n\}_{n=1}^{\infty}$ is a sequence in the compact space A, hence this has a convergent subsequence $\{a_{n_k}\}_{k=1}^{\infty}$, converging to, say: $a \in (A, d_A)$

Since f is continuous, $f(a_{n_k}) \longrightarrow f(a)$ as $k \to \infty$.

Hence $\forall \varepsilon > 0$ $\exists K_1 \in \mathbb{Z}^+$ so that $\forall k \geq K_1$ $d(a_{n_k}, a) < \varepsilon/2$

$\exists K_2 \in \mathbb{Z}^+$ so that $\forall k \geq K_2$ $d(f(a_{n_k}), f(a)) < \varepsilon/2$

Then $\forall k \geq \operatorname{Max}\{K_1, K_2\}$

$$d((a_{n_k}, f(a_{n_k})), (a, f(a)))$$

$$= d(a_{n_k}, a) + d(f(a_{n_k}), f(a)) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Therefore $(a_{n_k}, f(a_{n_k})) \longrightarrow (a, f(a))$ as $k \to \infty$ in $\Gamma_f \subset A \times B$.

So every sequence in Γ_f has a convergent subsequence, consequently Γ_f is compact.

($\Leftarrow$) Suppose that Γ_f is compact.

Remark 1: Then it is not necessary to assume that A is compact, since this follows from Γ_f being compact.

If $\{U_\alpha\}$ is an open cover of A, then $\{U_\alpha \times B\}$ is an open cover of Γ_f: if $(a,b) \in \Gamma_f$, then $a \in A$. Since $\{U_\alpha\}$ is an open cover $\exists\, \alpha(a)$ so that $a \in U_{\alpha(a)}$. $b \in B$. so $(a,b) \in U_{\alpha(a)} \times B$. Since Γ_f is compact, there is a finite subcollection of $\{U_\alpha \times B\}$ that covers Γ_f. Call it: $\{U_i \times B\}_{i=1}^{k}$. $\forall\, a \in A$ $f(a)$ is well-defined and is in B, since f is a function from A into B. So $(a, f(a)) \in \Gamma_f$. So $\exists\, i_a \in \{1, \dots, k\}$ so that $(a, f(a)) \in U_{i_a} \times B$. Then $a \in U_{i_a}$. So $\{U_i\}_{i=1}^{k}$ is a finite subcollection of $\{U_\alpha\}$, that covers A. Hence A is compact.

Remark 2: $\mathrm{Im}(f) = f(A)$ is also compact by a similar argument: Let $\{V_\beta\}$ be an open cover of $f(A)$. Then $\{A \times V_\beta\}$ is an open cover of Γ_f: $\forall\, (a,b) \in \Gamma_f$, $b \in \mathrm{Im}(f) = f(A)$ since $b = f(a)$. Hence $\exists\, \beta(b)$, so that $b \in V_{\beta(b)}$. Then $(a,b) \in A \times V_{\beta(b)}$. Since Γ_f is compact, there is a finite subcover: $\{A \times V_j\}_{j=1}^{m}$. Then $\{V_j\}_{j=1}^{m}$ covers $f(A)$ as well: $\forall\, b \in f(A)$, $\exists\, a \in A$ so that $f(a) = b$. $\Rightarrow$ $(a,b) \in \Gamma_f$. Since $\{A \times V_j\}_{j=1}^{m}$ is a finite subcover, $\exists\, j_b \in \{1, \dots, m\}$ so that $(a,b) \in A \times V_{j_b}$, thus $b \in V_{j_b}$.
Hence $f(A) = \mathrm{Im} f$ is compact.

Back to the proof: ($\Leftarrow$)

It will be shown that f is sequentially continuous.
Let $\{a_n\}_{n=1}^{\infty}$ be a sequence in A that converges to $a \in A$. Then $\{(a_n, f(a_n))\}_{n=1}^{\infty}$ is a sequence in the compact subspace Γ_f of $A \times B$. Hence this has a convergent subsequence $(a_{n_k}, f(a_{n_k})) \longrightarrow (\tilde{a}, \tilde{b}) \in \Gamma_f$.

- Since $(\tilde{a}, \tilde{b}) \in \Gamma_f$, $\tilde{b} = f(\tilde{a})$.
- By reading convergence in ε-language:
  $\forall\, \varepsilon > 0$ $\exists\, K \in \mathbb{Z}^+$ so that $\forall\, k \geq K$

$$d(a_{n_k}, \tilde{a}) \leq d(a_{n_k}, \tilde{a}) + d(f(a_{n_k}), f(\tilde{a})) = d\big((a_{n_k}, f(a_{n_k})), (\tilde{a}, f(\tilde{a}))\big) < \varepsilon$$

So $\{a_{n_k}\}_{k=1}^{\infty}$ converges to $\tilde{a}$. But $a_n \longrightarrow a$ by assumption.
Since the limit is unique, $\tilde{a} = a$.

- So $f(a_{n_k}) \longrightarrow f(a)$ as $k \to \infty$.

Proof-by-contradiction:

Now suppose that $f(a_n) \not\to f(a)$. Then

$\exists \varepsilon > 0$ so that $\forall N \in \mathbb{Z}^+$ $\exists n \geq N$ such that $d(f(a_n), f(a)) \geq \varepsilon$

Take $N = 1$ first. Then $\exists n_1 \geq 1$ s.t. $d(f(a_{n_1}), f(a)) \geq \varepsilon$

Having found $n_1 < \dots < n_r$ s.t. $d(f(a_{n_i}), f(a)) \geq \varepsilon$
for $i = 1, \dots, r$

let $N = n_r + 1 > n_r$. Then $\exists n_{r+1} \geq N > n_r$ so that
$d(f(a_{n_{r+1}}), f(a)) \geq \varepsilon$.

So we find a subsequence $\{f(a_{n_r})\}_{r=1}^{\infty}$ of $\{f(a_n)\}_{n=1}^{\infty}$

such that $d(f(a_{n_r}), f(a)) \geq \varepsilon$ for all $r \in \mathbb{Z}^+$. $(*)$

Since $\{f(a_{n_r})\}_{r=1}^{\infty}$ is still a subsequence of $\{f(a_n)\}_{n=1}^{\infty}$ and by the same argument we've applied before:

- $\{(a_{n_r}, f(a_{n_r}))\}_{r=1}^{\infty}$ is a sequence in compact set Γ_f, so has a convergent subsequence $\{(a_{n_{r_s}}, f(a_{n_{r_s}}))\}_{s=1}^{\infty}$ converging to say $(a', b') \in \Gamma_f$.
- $b' = f(a')$.
- Since $(a_{n_{r_s}}, f(a_{n_{r_s}})) \longrightarrow (a', f(a'))$; $a_{n_{r_s}} \longrightarrow a'$ by applying ε-definition of sequence convergence.
- But $a_{n_{r_s}}$, being a subsequence of convergent sequence a_n, converges to $a = \lim_{n \to \infty} a_n$. So $a' = a$.
- But then $f(a_{n_{r_s}}) \longrightarrow f(a)$ contradicting $(*)$.

($\Leftarrow$) alternative proof.

$f: A \to B$. Γ_f : graph of f in $A \times B \doteq \{(a,b) \in A \times B : b = f(a)\}$.

Γ_f is compact $\Rightarrow$ f is continuous.

Remark: The following are assumed and used in the proof below:

(1) metric on the cartesian product of two metric spaces, and its properties. (such as characterization of open & closed sets under this metric.)

(2) that Γ_f being compact implies that A is compact and $f(A)$ is compact.

These will be demonstrated at the end, for completeness.

Proof:

- Let $a \in A$ and $\varepsilon > 0$.
- Let $V \doteq B(f(a), \varepsilon)$. $V^c = \{b \in B : d(b, f(a)) \geq \varepsilon\}$ is closed in B.
  Hence $A \times V^c$ is closed in $A \times B$. (see lemma 2 after proof.)
  Then $X \doteq (A \times V^c) \cap \Gamma_f = \{(\tilde{a}, \tilde{b}) \in A \times B : \tilde{b} = f(\tilde{a})$ and $d(\tilde{b}, f(a)) \geq \varepsilon\}$
  is compact.
- Let $U_n \doteq \{a' \in A ; d(a, a') > 1/n\}$. This is open (see lemma 3 after proof)
  $\{U_n \times B\}_{n=1}^{\infty}$ is an open cover for X: (See lemma 1 after proof as well.)
  If $(\tilde{a}, \tilde{b}) \in X$, then $d(f(\tilde{a}), f(a)) \geq \varepsilon > 0 \Rightarrow f(\tilde{a}) \neq f(a)$.
  Since f is a function, $\tilde{a} \neq a \Rightarrow d(\tilde{a}, a) > 0$.
  By archimedean property, $\exists N \in \mathbb{Z}^+$ so that $N > \frac{1}{d(\tilde{a}, a)}$.
  Then $d(\tilde{a}, a) > \frac{1}{N} \Rightarrow \tilde{a} \in U_N$.
  Since $f(\tilde{a}) \in B$, $(\tilde{a}, f(\tilde{a})) \in U_N \times B$.
  So $\{U_n \times B\}_{n=1}^{\infty}$ is indeed a cover of X.
- Since X is compact, there is a finite subcover:
  $\{U_{n_i} \times B\}_{i=1}^{\nu}$ for $\nu \in \mathbb{Z}^+$.
  Let $\tilde{N} = \max\{n_i : i = 1, \dots, \nu\}$. Then $\bigcup_{i=1}^{\nu} (U_{n_i} \times B) = U_{\tilde{N}} \times B$
  (see lemma 3b after proof.)
- Hence $X = (A \times V^c) \cap \Gamma_f \subset U_{\tilde{N}} \times B$.
  So if $p \in B(a, 1/\tilde{N})$, then $d(p, a) < \frac{1}{\tilde{N}} \Rightarrow (p, f(p)) \notin U_{\tilde{N}} \times B$.
  $\Rightarrow (p, f(p)) \notin X = (A \times V^c) \cap \Gamma_f$ (by above).
  $\Rightarrow (p, f(p)) \notin A \times V^c$ (since $(p, f(p)) \in \Gamma_f$.)
  $\Rightarrow f(p) \notin V^c$ (since $p \in A$.)
  $\Rightarrow f(p) \in (V^c)^c = V = B(f(a), \varepsilon) \Rightarrow d(f(p), f(a)) < \varepsilon$.
- Thus f is continuous at each $a \in A$. $\blacksquare$