Spring 2005 Math 477 - Projective Geometry
First Midterm Solutions - 18/04/2005

(1) True or false? Justify your answers in full detail.

a) (3pts) Two planes intersect along a line in a projective 3-space.

Here $\dim V = 4$. Let $P(U_1), P(U_2)$ be two planes. Then $\dim U_1 = \dim U_2 = 3$. Since

$$\dim V \geq \dim U_1 + \dim U_2 - \dim U_1 \cap U_2,$$

we have $3 \geq \dim U_1 \cap U_2 \geq 2$. If it is 3 then $U_1 = U_2$. Otherwise, $\dim P(U_1) \cap P(U_2) = 1$.

b) (3pts) Let $a = [1:0:1], b = [0:1:2] \in P(V)$ and let $f = [3:2:-1] \in P(V')$ where f is expressed in the dual basis. Then the line $l_f \subset P(V)$ dual to f intersects the line ab at the point $[0:1:-2]$.

The point $[0:1:-2]$ is not on the line ab! (In the question the point should have read $[0:1:+2]$. There is a typo here. Redo the solution with this correction.)

c) (3pts) Let $P(U)$ be an $(n-2)$-dimensional subspace of $PG(n,7)$. The number of hyperplanes through $P(U)$ is 8.

The dual of the statement is: "The number of points on a line in $PG(n,7)$ is 8" which is true by definition.

d) (3pts) Let $\mathbb{F} = \mathbb{Z}_2$. Any non-singular conic in $\mathbb{F}P^2$ has exactly 3 points.

The "diagonalisation theorem" does not apply here since $\text{char}\mathbb{Z}_2 = 2$. The only non-degenerate symmetric bilinear form on $\mathbb{F}^3$ is given by $h(x,y) = x_1y_1 + x_2y_2 + x_3y_3$ because if there were cross terms like x_1x_2, its coefficient must be divisible by 2 for symmetry. Division by 2 is not allowed in $\mathbb{Z}_2$. The quadratic form $q(x) = x_1^2 + x_2^2 + x_3^2$ defined by this h describes a conic on $\mathbb{F}P^2$ with exactly 3 points: $[1:1:0], [1:0:1], [0:1:1]$. As a final remark, keep in mind that playing with quadratic forms over $\mathbb{Z}_2$ is not safe!

(2) (8pts) Prove that the number of triangles that occur in $PG(2,q)$ equals

$$\frac{1}{6}(q^2+q+1)q^3(q+1).$$

To construct a triangle in $PG(2,q)$, ▶ choose a point: (q^2+q+1) choices; ▶ choose two lines in its pencil: $\binom{q+1}{2}$ choices; ▶ choose one point on each line: (q^2) choices. In this way, we count each triangle thrice. Hence,

$$\text{number of triangles} = \frac{1}{3}(q^2+q+1)\frac{q(q+1)}{2}q^2.$$

(3) a) (6pts) Let $p_1, \ldots, p_5 \in \mathbb{R}P^2$ be five points such that no three are **collinear**. Show that there exists a **unique non-singular** conic passing through all of the five points.

We must determine $a,b,c,d,e,f \in \mathbb{R}$ in $q(p) = ax_1^2 + bx_1x_2 + cx_1x_3 + dx_2^2 + ex_2x_3 + fx_3^2$. We have 5 equations $q(p_i) = 0, (i = 1, \ldots, 5)$ in 6 unknowns. For p_i's distinct, we have the 5 equations linearly independent so that the solution for this system is $\mathbb{R}$-many, which describe the same conic. One has to show that this unique conic is nonsingular in the case when not any three of the five points are collinear. This follows from:

lemma: If a conic $C \subset \mathbb{R}P^2$ containing at least 2 points is singular then it contains a line.

proof: If C is singular then the corresponding matrix Q is singular; i.e. $\ker Q \neq 0$. Let $x \in \ker Q \subset \mathbb{R}^3$, $x \neq 0$. Then $Qx = 0$. For any other $P = [p] \in C$, which exists by hypothesis, consider $ap + bx$:

$$(ap+bx)^TQ(ap+bx) = a^2p^TQp + b^2x^TQx + 2abp^TQx = 0.$$

Then the line $\{[ap+bx] : a,b \in \mathbb{R}\}$ is in C.

With the lemma proved, the conic in question has at least 5 points. Assume it is singular. Then it contains lines, one for each p_i (or one less if one of p_i's is in $\ker Q$).

If any 3 of these lines are distinct then any line in the plane intersects the conic at 3 distinct points which is impossible. So some of these lines must coincide, implying that at least 3 of the 5 points are collinear. Contradiction with the initial assumption.

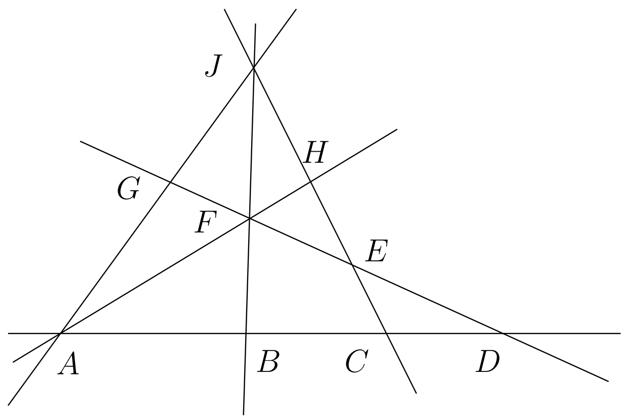
b) (2pts) Show that a projective transformation of $\mathbb{C}P^2$ which fixes a conic pointwise is identity.

On a nonsingular conic in $\mathbb{C}P^2$, there are infinitely many points. In particular, there are four points which are in general position in $\mathbb{C}P^2$ (Why? HW#5). If the transformation leaves the four points fixed then by the fundamental theorem of the projective geometry, it is the identity transformation.

(4) a) (4pts) Given four collinear points A, B, C, D in a projective plane, construct explicitly a projective transformation ϕ of the line $ABCD$ such that

$$\phi(A) = B, \phi(B) = A, \phi(C) = D, \phi(D) = C.$$

(*Hint: In the figure below, find a series of perspectivities so that their composition becomes the required* ϕ.*)*



Let $l_1 = ABCD, l_2 = GFED, l_3 = JHEC$. Observe in the figure:

$$\begin{array}{rl} \text{persp}_J: & l_1 \to l_2; A \mapsto G, B \mapsto F, C \mapsto E, D \mapsto D; \\ \text{persp}_A: & l_2 \to l_3; G \mapsto J, F \mapsto H, E \mapsto E, D \mapsto C; \\ \text{persp}_F: & l_3 \to l_1; J \mapsto B, H \mapsto A, E \mapsto D, C \mapsto C. \end{array}$$

Hence $\phi = \text{persp}_F \circ \text{persp}_A \circ \text{persp}_J$ is the required transformation.

b) (4pts) A projective transformation ψ of the plane is called an *involution* if $\psi \circ \psi$=id. Show that a projective transformation interchanging two points of the plane is an involution of the line determined by those two points. (*Hint: Start with two such points* $A, B \in P(V)$. *Take* $x \in AB$. *Use part (a) and fundamental theorem of projective geometry.)*

Let A, B be two such points; $l = AB$. Take $x \in l$. Let $\psi(x) = y$. Since a projective transformation sends lines to lines, y is on l. Then:

$$\psi : l \to l; A \mapsto B, B \mapsto A, x \mapsto y$$

and by part (a) there exists a map ϕ such that

$$\phi : l \to l; A \mapsto B, B \mapsto A, x \mapsto y, y \mapsto x.$$

Since A, B, x are in general position on l, by the fundamental theorem of projective geometry, $\psi = \phi$. Since x was arbitrary, ψ is an involution.

(5) (Bonus; 3pts) Why do we use the division signs ':' in the notation $[a : b : c]$ for a point in a projective plane?

Because we are interested in $a : b : c$.