PROOF OF THE WEINSTEIN CONJECTURE FOR OVERTWISTED CLOSED CONTACT 3-MANIFOLDS

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ABSTRACT

PROOF OF THE WEINSTEIN CONJECTURE FOR OVERTWISTED CLOSED CONTACT 3-MANIFOLDS

In this thesis, we study the proof of the Weinstein conjecture for 3-dimensional closed manifolds equipped with an overtwisted contact structure. The method of filling by pseudoholomorphic disks and the bubbling-off analysis are the main tools that are used in this proof.

ÖZET

3 BOYUTLU KAPALI ÇOKBÜKÜMLÜ KONTAKT ÇOKKATLILARDA WEINSTEIN SANISININ İSPATI

Bu tezde, çokbükümlü kontakt yapıya sahip üç boyutlu kapalı çokkatlılar için Weinstein sanısının ispatı incelenmiştir. Ḃu ispatta, temel araç olarak holomorfumsu disklerle doldurma yöntemi ve kabarcıklanma analizi kullanılmaktadır.

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LIST OF SYMBOLS/ABBREVIATIONS

$\square$	End of proof
$\wedge$	Wedge product of differential forms
$\hookrightarrow$	Embedding
$\oplus$	Direct sum of vector bundles
$\approx$	Used intergangably to denote a vector space isomorphism and a diffeomorphism
$\circ$	Composition of maps
$:=$	Definition
$\equiv$	Equal everywhere
$Aut(D)$	The set of biholomorphic transformations of the closed unit disk
$B(x,\epsilon)$	The open ball in $\mathbb{R}^n$ with center x and radius ϵ
$\mathcal{C}^0(\Omega,\mathbb{R}^n)$	The space of continuous functions $f:\Omega\to\mathbb{R}^n$
$\mathcal{C}^\infty(\Omega,\mathbb{R}^n)$	The space of infinitely differentiable functions
$\mathcal{C}_0^\infty(\Omega,\mathbb{R}^n)$	The space of infinitely differentiable functions compactly supported on Ω
$\mathcal{C}^k(\overline{\Omega},\mathbb{R}^n)$	The Banach space of all functions $\phi=(\phi_1,..,\phi_n)\in\mathcal{C}^k(\Omega,\mathbb{R}^n)$ for which $D^\alpha\phi_j$ is bounded and uniformly continuous on Ω for all j and $\lvert\alpha\rvert\le m$, i.e., it possesses a unique, bounded, continuous extension to $\overline{\Omega}$. $\mathcal{C}^k(\overline{\Omega},\mathbb{R}^n)$ is equipped with norm given by $$\lvert\lvert u\rvert\rvert_{\mathcal{C}^l(\overline{\Omega},\mathbb{R}^k)} := \max_{0\le\lvert\alpha\rvert\le m}\sup_{x\in\Omega}\lvert D^\alpha u(x)\rvert$$ where $\lvert D^\alpha u(x)\rvert$ length of the vector $D^\alpha u(x)$
$\mathcal{C}^k(\overline{\Omega})$	$\mathcal{C}^k(\overline{\Omega},\mathbb{R})$

$\mathcal{C}^k(\Omega, \mathbb{R}^n)$	The space all functions $\phi = (\phi_1, .., \phi_n)$ together with all partial derivatives $D^\alpha \phi_j$ of all orders $\lvert\alpha\rvert \leq m$, are continuous on Ω
$\mathcal{C}^k(\Omega)$	$\mathcal{C}^k(\Omega, \mathbb{R})$
d	Used interchangably to denote the exterior derivative of a differential form and the differtial of a map
$\mathring{D}$	Interior of D
$\partial\mathcal{D}$	Boundary of the manifold $\mathcal{D}$
D^α	$\dfrac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \dfrac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}}$, where $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\lvert\alpha\rvert = \alpha_1 + \cdots + \alpha_n$
$L^p(\Omega, \mathbb{R}^n)$	The Banach space of classes of measurable functions $u : \Omega \to \mathbb{R}^n$ such that $\int_\Omega \lvert u(x)\rvert^p < \infty$ if $1 \leq p < \infty$, or ess $\sup_\Omega \lvert u\rvert < \infty$ if $p = \infty$. $L^p(\Omega, \mathbb{R}^n)$ is equipped with the norm $$\lvert\lvert u\rvert\rvert_{L^p(\Omega,\mathbb{R}^n)} = \left(\int_\Omega \lvert u(x)\rvert^p dx\right)^{1/p}$$ if $p < \infty$ or $$\lvert\lvert u\rvert\rvert_{L^\infty(\Omega,\mathbb{R}^n)} = \text{ess } \sup_\Omega \lvert u\rvert$$ if $p = \infty$
$L^p(\Omega)$	$L^p(\Omega, \mathbb{R})$
$\mathcal{L}_Y$	Lie derivative along the vector field Y
$\partial_t u$	Partial derivative of u with respect to the variable t
u_t	Partial derivative of u with respect to the variable t
$\mathfrak{Re}(z)$	Real part of z
$\text{supp}\phi$	Support of ϕ
$\overline{W}$	Closure of W

$W^{k,p}(\Omega, \mathbb{R}^n)$	The Banach space of those functions $u : \Omega \to \mathbb{R}^k$, $u = (u_1, .., u_k)$ such that for all i, $u_i \in L^p(\Omega)$ and for any multi index $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{N}^n$ with $\vert\alpha\vert := \sum_{i=1}^n \alpha_i \leq m$, the weak derivative $D^\alpha u_i$ exists and lies in $L^p(\Omega)$ where $1 \leq p < \infty$. We set $D^\alpha u = \frac{\partial^{\vert\alpha\vert} u}{\partial^{\alpha_1} x_1 ... \partial^{\alpha_n} x_n} := (\frac{\partial^{\vert\alpha\vert} u_1}{\partial^{\alpha_1} x_1 ... \partial^{\alpha_n} x_n}, ..., \frac{\partial^{\vert\alpha\vert} u_k}{\partial^{\alpha_1} x_1 ... \partial^{\alpha_n} x_n})$ where $\frac{\partial^{\vert\alpha\vert} u_i}{\partial^{\alpha_1} x_1 ... \partial^{\alpha_n} x_n}$ stands for the weak derivative of u_i for each i. The norm on $W^{m,p}(\Omega, \mathbb{R}^k)$ is defined to be $\Vert u\Vert_{W^{m,p}(\Omega,\mathbb{R}^k)} := \Vert u\Vert_{m,p,\Omega} := \left(\sum_{0 \leq \vert\alpha\vert \leq m} \Vert D^\alpha u\Vert^p_{L^P(\Omega)} \right)^{1/p}$
$W^{k,p}(\Omega)$	$W^{k,p}(\Omega, \mathbb{R}^n)$
[x]	Equivalence class of x
$\vert x\vert$	Euclidean norm of x in$\mathbb{R}^n$
Δa	Laplacian of a
$\vert\nabla u\vert$	Gradient norm of a map $u : \mathbb{C} \to \mathbb{R}^n$, i.e., $\left(\left\vert \frac{\partial u}{\partial s} \right\vert^2 + \left\vert \frac{\partial u}{\partial t} \right\vert^2 \right)^{1/2}$ where $s + it \in \mathbb{C}$
$\phi^* \tau$	Pull back of τ by ϕ
$\phi_* v$	Push forward of v by ϕ
$\pi_1(X, x)$	The fundamental group of X with respect to the base point x

PDE	Partial differential equation

1. INTRODUCTION

The existence of a periodic orbit of a vector field that appears in a certain geometric setup or a vector field coming from a certain class of vector fields has been one of the important questions in geometry and dynamics.

In 1950, Seifert [1] raised the question whether a given non-singular vector field on 3-sphere admits a periodic orbit. In 1974, Schweizer [2] and in 1994, Kuperberg [3] gave $\mathcal{C}^1$ and $\mathcal{C}^\infty$ counterexamples respectively. These results showed that Seifert's question had to be modified considering the underlying dynamical system.

Indeed, Seifert's question got affirmative answers for some special dynamical systems. In the late 70's, Rabinowitz [4] and Weinstein [5] showed that a smooth compact regular energy surface for a Hamiltonian system in $\mathbb{R}^{2n}$ carries a periodic orbit of the Hamiltonian vector field if the energy surface bounds a star-shaped or convex domain respectively. After his analysis on Rabinowitz's work [4] and his own results [5], Weinstein posed his famous conjecture in [6], which has been called the general Weinstein conjecture.

Conjecture 1.0.1. *Assume that H is an autonomous Hamiltonian on a symplectic manifold (W, ω). Let $\Sigma = H^{-1}(E)$ be a closed regular level surface for H. Suppose there exists a 1-form λ on Σ such that $d\lambda = \omega|_\Sigma$ and $\lambda(z)(X_H(z)) \neq 0$ for all $z \in \Sigma$ where X_H is the Hamiltonian vector field. Then there exists a periodic orbit of X_H on Σ.*

The following breakthrough result of Viterbo gives a partial answer to the Weinstein conjecture [7].

Theorem 1.0.2. *Let $M = H^{-1}(E)$ be a compact smooth energy surface for some regular value E of the Hamiltonian H on the symplectic manifold $(\mathbb{R}^{2n}, \omega)$. If there exists a 1-*

form λ *on* M *such that* $d\lambda = \omega|_M$ *and* $\lambda(x)(X_H(x)) \neq 0$ *for all* $x \in M$ *where* X_H *is the Hamiltonian vector field. Then there exists a periodic orbit of* X_H *on* M*.*

This is the place where the contact geometry and the Reeb dynamics come into picture. We say a 1-form λ is a *contact form* on a $(2n+1)$-dimensional manifold M if $\lambda \wedge (d\lambda)^n$ is a volume form on M. With this data the Weinstein conjecture can be reformulated as the following:

Conjecture 1.0.3. *Let* (M, λ) *be a closed contact manifold and* X *be a non-singular vector field on* M *such that* $d\lambda(X, \cdot) = 0$ *and* $\lambda(X) > 0$*. Then* X *has a periodic orbit.*

We will see in the Chapter 2 that $\ker\lambda$ constitutes a $2n$-dimensional vector bundle on M and there is a unique vector field X_λ on M such that $d\lambda(X, \cdot) = 0$ and $\lambda(X) = 1$. We call $\xi = \ker\lambda$ the *contact structure* associated to λ and X_λ the associated *Reeb vector field*. We observe that for a vector field X as in Conjecture 1.0.3, there exists a smooth positive map f on M such that $X = fX_\lambda$. Moreover X_λ admits a periodic orbit if and only if $X = fX_\lambda$ admits a periodic orbit. With this observation the Conjecture 1.0.3 boils down to the claim that the Reeb vector field admits a periodic orbit.

Recently, Taubes proposed a proof of the Weinstein conjecture for 3-dimensional closed contact manifolds, which is based on the Seiberg–Witten equations. But in this study we will not deal with this proof. Instead, we will concentrate on the following partial proof of Hofer [9].

Theorem 1.0.4. *Assume that* X *is the Reeb vector field on a closed 3-dimensional contact manifold* (M, λ)*. Then* X *admits a periodic orbit if either* M *is diffeomorphic to* S^3 *or if* $\pi_2(M) \neq 0$ *or if the associated contact structure is overtwisted.*

The notion of an overtwisted contact structure is important in three-dimensional contact geometry. We say a contact form λ or a contact structure $\xi := \ker\lambda$ is *overtwisted* if there exists an embedded disk $\mathcal{D} \subset M$, which is called an *overtwisted disk*, such that

$$T(\partial\mathcal{D}) \subset \xi|_{\partial\mathcal{D}},$$
$$T_x\mathcal{D} \not\subseteq \xi_x \text{ for all } x \in \partial\mathcal{D}.$$

The main ingredient of the proof of Theorem 1.0.4 is the notion of a pseudoholomorphic curve. Pseudoholomorphic curves, which will be defined precisely in Chapter 3, are generalizations of holomorphic maps. They became basic tools of the symplectic geometry after the work of Gromov [10].

Now we will see the role of pseudoholomorphic curves in the proof of Theorem 1.0.4. The notions that will appear in what follows will be defined precisely in Chapter 2 and Chapter 3.

Let M be a closed 3-manifold and λ be a contact form. We denote by ξ and X_λ the associated contact structure and Reeb vector field, respectively. We pick a complex structure $J : \xi \to \xi$ compatible with $d\lambda$, i.e. $d\lambda(\cdot, J(\cdot)) > 0$ on $\xi \setminus \{0\}$. Then we define an almost complex structure $\tilde{J}$ on the "symplectization" $\mathbb{R} \times M$ in a certain way that will be explained later. We take a closed Riemannian surface (S, j) and a finite set $\Gamma \subset S$ and consider smooth maps $\tilde{u} : S \setminus \Gamma \to \mathbb{R} \times M$

$$d\tilde{u} \circ j = \tilde{J} \circ d\tilde{u}. \tag{1.1}$$

The equation above is called a *nonlinear Cauchy–Riemann equation* and $\tilde{u}$ is called a $(\tilde{J}, j)$*-holomorphic* or a *pseudoholomorphic curve.* In addition, we want $0 < E(\tilde{u}) < \infty$ where $E(\tilde{u})$ stands for the *energy* of $\tilde{u}$. Then we call $\tilde{u}$ a finite energy surface. We know that $E(\tilde{u}) = 0$ if and only if $\tilde{u}$ is constant. The set Γ stands for the set of "punctures" on S. Using the definition of energy and the Stokes' theorem one can show that if $\Gamma = \emptyset$ then

$\tilde{u}$ is constant [9].

Now we take $z_0 \in \Gamma$. The following observation, which is due to Hofer, on the behavior of $\tilde{u}$ near z_0 is the key fact that connects finite energy surfaces and the periodic orbits of X_λ. We parametrize S near z_0 by a holomorphic map $\sigma : [0, \infty) \times S^1 \to S$ and define

$$\tilde{v} := \tilde{u} \circ \sigma : [0, \infty) \times S^1 \to \mathbb{R} \times M.$$

Then we have the following fact [9].

Theorem 1.0.5. *Let $\tilde{v}$ be as above and put $\tilde{v} = (b, v)$. Then*

$$m(z_0) := \lim_{s \to \infty} \int_{S^1} v(s, \cdot)^* \lambda$$

exists. If $m(z_0) = 0$ then the map $\tilde{u}$ can be extended smoothly over z_0. If $m(z_0) \neq 0$ then any sequence $s_k \to \infty$ has a subsequence, also denoted by s_k such that

$$\lim_{k \to \infty} v(s_k, t) = x(m(z_0)t)$$

exists in $\mathcal{C}^\infty(S^1, M)$, where x is a $|m(z_0)|$-periodic orbit of X_λ.

Now, in order to prove Theorem 1.0.4 it is enough to show that there exists a non-constant finite energy plane

$$\tilde{u} : \mathbb{C} \to \mathbb{R} \times M,$$

$$d\tilde{u} \circ i = \tilde{J} \circ d\tilde{u},$$

$$0 < E(\tilde{u}) < \infty$$

if either M is diffeomorphic to S^3 or if $\pi_2(M) \neq 0$ or if λ is overtwisted. Indeed, $(\mathbb{C}, i)$ is a Riemann surface with one puncture only, namely $S^2 \backslash \{\infty\}$. By Theorem 1.0.5, either $\tilde{u}$ can be extended to $\{\infty\}$ or there exists a periodic orbit. But in the first case $\tilde{u}$ is necessarily constant.

In this thesis, strictly following [13] we will prove the following theorem in detail.

Theorem 1.0.6. *Assume that X is the Reeb vector field on a closed 3-dimensional contact manifold (M, λ). Then X admits a periodic orbit if λ is overtwisted.*

In Chapters 2, 3 and 4 we construct the machinery we need. In Chapter 2, we introduce basic notions of the contact geometry and give some details about surfaces in contact 3-manifolds, especially the overtwisted disk. In Chapter 3, first the general notion of a pseudoholomorphic curve is defined. Then we study special pseudoholomorphic disks in the symplectization $\mathbb{R} \times M$ of the 3-dimensional closed contact manifold M with an overtwisted disk $\mathcal{D}$. More precisely, we consider the smooth maps $\tilde{u} := (a, u) : D \to \mathbb{R} \times M$ satisfying

$$d\tilde{u} \circ i = \tilde{J} \circ d\tilde{u}, \tag{1.2}$$

$$u(\partial D) \subset \mathcal{D} \subset M \tag{1.3}$$

where D denotes the closed unit disk in $\mathbb{C}$. By pseudoconvexity of M in $\mathbb{R} \times M$, we make a crucial observation on the transversality of $u|_{\partial D}$ to the characteristic foliation of $\mathcal{D}$ (see 3.3.2). In Chapter 4, we establish Theorem 4.0.5, which is the regularity result for a certain set of maps $u : D \to W$ satisfying the nonlinear Cauchy–Riemann equation, where W is a 4-manifold with an almost complex structure. Using this result we prove Theorem 4.0.6, which is a compactness result saying that if a sequence of pseudoholomorphic disks $\tilde{u}_k : D \to \mathbb{R} \times M$ has a uniform gradient bound then it has a subsequence converging to a pseudoholomorphic disk in $\mathcal{C}^\infty$.

In Chapter 5, we show that the existence of a non-constant finite energy plane in a contact 3-manifold M implies the existence of a periodic orbit of the associated Reeb vector field (see Theorem 5.2.2). As noted above, what we prove is a special case of Theorem 1.0.5. In the proof we use the regularity result established in Chapter 4 and a "bubbling-off" analysis. However, constructing a non-constant finite energy plane requires much work. In Chapter 7, we see that if there exists a family of pseudoholomorphic disks

$$\tilde{u}_\tau = (a_\tau, u_\tau) : D \to \mathbb{R} \times M, \quad \tau \in (0,1)$$

with unbounded gradients then we can construct a non-constant finite energy plane (see Theorem 7.0.4). Therefore what remains for the proof of Theorem 1.0.6 is the existence of a family of pseudoholomorphic disks with unbounded gradients. To construct such a family we do the following.

First we pick an overtwisted disk $\mathcal{D}$ in M. By definition, there is no $x \in \partial\mathcal{D}$ such that $T_x\mathcal{D} = \xi_x$. We call such a point *singular*. Notice that there can be singular points in the interior of $\mathcal{D}$. Using Theorem 2.4.3, we assume there is only one singular point of $\mathcal{D}$. We call it e. Then in Section 6.2, Theorem 6.2.1, we construct a smooth embedding $\tilde{u} : (0,1) \times D \to \mathbb{R} \times M$ satisfying the following. Put $\tilde{u}_\tau := \tilde{u}(\tau, \cdot)$. Then the family $\tilde{u}_\tau$, which is called a *Bishop family*, consists of pseudoholomorphic disks with $\tilde{u}_\tau(\partial D) \subset \{0\} \times \mathcal{D} \setminus \{e\}$. In other words, each $\tilde{u}_\tau$ is a solution of the nonlinear Cauchy–Riemann equation with a boundary condition. In Section 6.3, we have an existence and uniqueness result for such solutions (see Theorem 6.3.1). This is a deep result that requires the Fredholm theory of the nonlinear Cauchy–Riemann operator corresponding to the nonlinear Cauchy–Riemann equation. Now we assume that each disk family in what follows has a uniform gradient bound. We choose a sequence $\tilde{u}_{\tau_k}$ ($\tau_k \to 1$) out of the Bishop family. By the compactness result from Chapter 4 we get a limit $\tilde{u}_1$ and using Theorem 6.3.1 we extend the Bishop family around $\tilde{u}_1$. We continue to extend the family in this way until one of $\tilde{u}_\tau(\partial D)$'s hits $\{0\} \times \partial\mathcal{D}$. Since this $\tilde{u}_\tau$ still satisfies $\tilde{u}_\tau(\partial D) \subset \{0\} \times \mathcal{D} \setminus \{e\}$, $\tilde{u}_\tau(\partial D)$ and $\{0\} \times \partial\mathcal{D}$ must intersect tangentially. But this is not possible in the view of Corollary 3.3.2. Therefore at some stage we must have a family of disks with unbounded gradients. This completes the proof. One can see the details of this argument in Remark 6.4.1.

To sum up, we can say that the proof of Theorem 1.0.6 follows in the following logical order: Theorem 2.4.3, Theorem 6.2.1, Remark 6.4.1, Theorem 7.0.4, Theorem 5.2.2.

2. PRELIMINARIES IN CONTACT GEOMETRY

2.1. Basic Definitions

Let M be $2n+1$ dimensional manifold. A 1-form λ is called a *contact form* and (M, λ) is called a *contact manifold* if the $(2n+1)$-form

$$\lambda \wedge (d\lambda)^n$$

is a volume form on M. A contact manifold (M, λ) is necessarily orientable since $\lambda \wedge (d\lambda)^n$ defines a natural orientation.

Given a contact manifold (M, λ) we define a vector bundle $\xi \to M$ by

$$\xi_p := ker(\lambda_p),\ p \in M.$$

We see that the linear map $\lambda_p : T_pM \to \mathbb{R}$ is nonzero for all $p \in M$ and ξ is a $2n$-dimensional vector bundle since $\lambda \wedge (d\lambda)^n$ is a volume form. We call ξ a *contact structure.* If τ is another 1-form on M such that $ker(\tau) = \xi$ then there exists a nowhere vanishing smooth map $f : M \to \mathbb{R}$ such that $\lambda = f\tau$ and

$$d\lambda|_{\xi\oplus\xi} = f d\tau|_{\xi\oplus\xi}.$$

A 2-form w on a $2n$-dimensional manifold W is called a *symplectic form* if it is closed, i.e., $dw = 0$ and non-degenerate, that is, for all $p \in W$, $w_p(u, v) = 0$ for all $u \in T_pW$ implies $v = 0$. In this case we say (W, w) is a *symplectic manifold* and we call w a *symplectic structure.* Similarly we say that an even dimensional vector bundle $\sigma \to M$

on an arbitrary dimensional manifold M is a *symplectic vector bundle* if there is a closed 2-form w on M such that $w|_{\sigma\oplus\sigma}$ is a symplectic form.

The following lemma relates contact structure to symplectic structure. For the proof see [15].

Lemma 2.1.1. *Let* (M, λ) *be a contact manifold and* ξ *be the corresponding contact structure. Then* $(\xi, d\lambda|_{\xi\oplus\xi})$ *is a symplectic vector bundle and there exists a unique vector field* X_λ *on* M *such that*

$$d\lambda(X_\lambda, \cdot) \equiv 0 \textit{ and } \lambda(X_\lambda) \equiv 1.$$

The unique vector field X_λ corresponding to the contact form λ is called the *Reeb vector field*. It is known that a contact structure ξ on (M, λ) defines a natural splitting of the tangent bundle as follows [12]:

$$TM = \mathbb{R}X_\lambda \oplus \xi.$$

The standard example of a contact manifold is $\mathbb{R}^{2n+1}$ with the contact form

$$\lambda = dz + \Sigma_{k=1}^{n} x_k dy_k$$

where $(x_1, .., x_n, y_1, ..y_n, z)$ are the coordinates of $\mathbb{R}^{2n+1}$. In this case we have

$$X_\lambda \equiv \frac{\partial}{\partial z}, \quad \text{and} \quad d\lambda = \Sigma_{k=1}^{n} dx_k \wedge dy_k.$$

Moreover a fiber ξ_p of the contact structure at $p = (x_1, ..., x_n, y_1, .., y_n, z)$ is spanned by the vectors $e_1, .., e_n, f_1, .., f_n$ where $\{e_k\}_{k=1}^{2n+1}$ is the standard basis of $\mathbb{R}^{2n+1}$ and

$$f_k = e_{n+k} - x_k e_{2n+1}.$$

We observe that

$$d\lambda(e_i, e_j) = 0, \quad d\lambda(f_i, f_j) = 0 \quad \text{and} \quad d\lambda(e_i, f_j) = \delta_{ij}.$$

Such a basis for ξ_p is called a *symplectic basis.*

The following theorem says that every contact manifold is locally $\mathbb{R}^{2n+1}$ with standard contact structure. For the proof see [13].

Theorem 2.1.2 (Darboux)**.** *Let (M, λ) and (N, τ) be two $2n + 1$-dimensional contact manifolds. Given $p \in M$ and $q \in N$ there exist a neighborhood U of p and a neighborhood V of q, and a diffeomorphism $\phi : U \to V$ such that*

$$\phi(p) = q \quad \textit{and} \quad \phi^*\tau = \lambda.$$

2.2. Contact Vector Fields

Let (M, λ) be a contact manifold. A vector field Y on M is called a *contact vector field* if its flow ϕ_t preserves the contact structure, that is

$$(\phi_t)_*\xi = \xi.$$

The following lemma, whose proof follows as in [13], gives different characterizations of a contact vector field.

Lemma 2.2.1. *Let Y be a vector field on (M, λ) with flow ϕ_t. Then the followings are equivalent:*

(i) $\mathcal{L}_Y \lambda = r\lambda$ *where* $r : M \to \mathbb{R}$ *is a smooth function,*

(ii) $(\phi_t)^* \lambda = f_t \lambda$ *for all* t*, where* $f_t : M \to (0, \infty)$ *is smooth,*

(iii) $(\phi_t)_* \xi = \xi$.

Given a function H on a contact 3-manifold we can find a contact vector field that has a decomposition into a component that lies on contact structure and a Reeb component which is determined by H.

Theorem 2.2.2. *Let $H : M \to \mathbb{R}$ be a smooth function. Then there is a unique vector field $\hat{V}_H \in ker\lambda$ such that*

$$V_H = HX_\lambda + \hat{V}_H$$

is a contact vector field.

Proof. We want to show that there exists a function $r : M \to \mathbb{R}$ such that $\mathcal{L}_{V_H} \lambda = r\lambda$. By Cartan's formula we have

$$\begin{aligned} r\lambda(\cdot) = \mathcal{L}_{V_H}\lambda(\cdot) &= d\lambda(V_H, \cdot) + d(\lambda(V_H))(\cdot) \\ &= d\lambda(HX_\lambda + \hat{V}_H, \cdot) + d(\lambda(HX_\lambda + \hat{V}_H))(\cdot) \\ &= d\lambda(\hat{V}_H, \cdot) + dH(\cdot). \end{aligned}$$

Then we get

$$r = r\lambda(X_\lambda) = d\lambda(HX_\lambda + \hat{V}_H, X_\lambda) + dH(X_\lambda) = dH(X_\lambda).$$

From above we have

$$d\lambda(\hat{V}_H, \cdot) = dH(X_\lambda)\lambda(\cdot) - dH(\cdot). \tag{2.1}$$

Let $p \in M$. Evaluating both sides on the basis vectors of ξ_p, we determine $\hat{V}_H(p)$. If we have another vector field $\hat{W}_H$ with the same properties, then

$$d\lambda(\hat{V}_H - \hat{W}_H, \cdot) \equiv 0.$$

By non-degeneracy of $d\lambda$ on ξ, we get $\hat{W}_H \equiv \hat{V}_H$. □

We call H a *contact Hamiltonian* and we say V_H is the *contact Hamiltonian vector field* associated to H.

Remark 2.2.3. Let $H : M \to \mathbb{R}$ be a smooth map and $F = H^{-1}(0)$ be a level surface of H. Let $x \in F$, then

$$\begin{aligned}
(\mathcal{L}_{V_H} H)(x) &= V_H[H](x) \\
&= dH(x)(V_H(x)) \\
&= dH(x)(H(x)X_\lambda(x) + \hat{V}_H(x)) \\
&= dH(x)(\hat{V}_H(x)) \\
&= dH(x)(X_\lambda(x))\lambda(x)(\hat{V}_H(x)) - d\lambda(x)(\hat{V}_H(x), \hat{V}_H(x)) \\
&= 0
\end{aligned}$$

Hence we have $(\mathcal{L}_{V_H} H)(x) = dH(x)(\hat{V}_H(x)) = 0$. This implies that H is invariant under the flow generated by V_H and

$$V_H(x) = \hat{V}_H(x) \in T_xF \cap \xi_x, \text{ for } x \in F.$$

2.3. Surfaces in Contact 3-Manifolds and the Characteristic Foliation

Let F be an orientable surface in a contact 3-manifold (M, ξ). A point $p \in F$ is called a *singular point* if $T_pF = \xi_p$; otherwise we call it *regular.* If we fix an orientation on F and a contact form λ inducing ξ, we say that a singular point p is positive if the orientation of T_pF and the orientation induced by $d\lambda$ coincide. Otherwise we say p is a negative singular point.

Let p be a singular point on the surface F. Let H be a smooth map defined on a neighborhood of p in M such that $F = H^{-1}(0)$ near p, with $dH(p) \neq 0$. Then for each $x \in F$ near p, we have $V_H(x) = \hat{V}_H(x) \in T_xM \cap \xi_x$. Moreover we have the following fact:

Theorem 2.3.1. *A point $p \in F$ is singular if and only if $V_H(p) = 0$.*

Proof. If p is a singular point then $T_pF = \xi_p$. For any $u \in T_pF = \ker dH(p) = \xi_p$, using (2.1) we have

$$d\lambda(V_H, u) = d\lambda(\hat{V}_H, u) = dH(X_\lambda)\lambda(u) - dH(u) = 0.$$

By non-degeneracy of $d\lambda$ on ξ_p we have $V_H(p) = 0$. Now assume $V_H(p) = 0$. For any $u \in T_pF$, by the equation above $dH(p)(u) = dH(p)(X_\lambda(p))\lambda(p)(u) = 0$. We had seen that $\mathcal{L}_{V_H}\lambda = dH(X_\lambda)\lambda$. So $dH(X_\lambda)$ never vanishes. Hence $\lambda(u) = 0$. This implies $T_pF \subset \xi_p$, therefore $T_pF = \xi_p$. $\square$

We say that a singular point p is *non-degenerate* if the the linearization of $\hat{V}_H$ at p has no purely imaginary eigeinvalues. The linearization of $\hat{V}_H$ at p is defined as

$$\hat{V}_H'(p)v := \frac{d}{dt}(d\phi_t^H(p))v\big|_{t=0},$$

where ϕ_t^H is the flow of $\hat{V}_H$ and $v \in T_pF$. In local coordinates (x_1, x_2) it is the 2×2 matrix of the form $\big((\frac{\partial \hat{V}_H^j}{\partial x_i}(p))_{ij}\big)$. One can see that non-degeneracy of p is independent of the choice of H. Indeed, if $F = K^{-1}(0)$ near p, for some function K then $\hat{V}_K = \rho \hat{V}_H$ for some non-vanishing function ρ and $\hat{V}_K'(p) = \rho(p)\hat{V}_H'(p)$, that is the linearization changes by multiplication with a nonzero real constant.

Let p be a non-degenerate singular point and λ_1, λ_2 be eigenvalues of $\hat{V}_H'(p)$. We have two cases. If $\lambda_1\lambda_2 > 0$ we say that p is an *elliptic singularity.* If $\lambda_1\lambda_2 < 0$ we say that p is a *hyperbolic singularity.* If $\lambda_1\lambda_2 > 0$ then either $\lambda_1, \lambda_2 \in \mathbb{R} \setminus \{0\}$ have the same sign or $\lambda_1 = \overline{\lambda}_2$ with nonzero real and imaginary parts. In the first case we call an elliptic singularity *nicely elliptic.*

The following theorem, gives a normal form for a surface around a singular point of it. For the proof see [13].

Theorem 2.3.2. *Let f be a smooth map on $\mathbb{R}^2$ with $f(0) = 0$ and $df(0) = 0$. Let F be the graph of f. Assume 0 is a non-degenerate singular point of F for the standard contact structure on $\mathbb{R}^3$. Moreover assume that there is no other singularity on $\{(x, y, f(x, y)) : (x, y) \in B(0, \epsilon_0)\}$ for some $\epsilon_0 > 0$. Then for a given $\epsilon \in (0, \epsilon_0)$ and $\delta > 0$ there exists a smooth map $\tilde{f}$ such that the associated surface $\tilde{F}$ has the same singularities as F. Moreover $f = \tilde{f}$ outside of $B(0, \epsilon)$, and $||f - \tilde{f}||_{C^0(B(0,\epsilon))} < \delta$. In addition we can choose*

$$\tilde{f}(x, y) = -\frac{1}{2}xy$$

near 0 if 0 is an elliptic singularity and we can choose

$$\tilde{f}(x, y) = \frac{1}{2}xy$$

near 0 if 0 is a hyperbolic singularity.

Now we will define the characteristic foliation of a compact, oriented surface in a contact 3-manifold. It is basically an equivalence class of vector fields on the surface which lie on the intersection of the tangent bundle of the surface and the contact distribution. We identify two vector fields if we can obtain one from the other by multiplication with a smooth strictly positive function on the surface.

Let $F \subset M$ be a compact, oriented surface and (M, λ) be a contact 3-manifold with the contact structure ξ. We pick a volume form w for F inducing the orientation. Let $i : F \to M$ be the inclusion. We define a vector field Z on F by the equation:

$$w(Z, \cdot) = i^* \lambda(\cdot). \tag{2.2}$$

Then we define the characteristic foliation for F by

$$[Z] = \{ fZ : f \in \mathcal{C}^\infty(F, (0, \infty)) \}.$$

It is known [13] that $[Z]$ is independent of the choice of w and the integral curves obtained on F is independent, up to reparametrization, of the choice of the vector field in $[Z]$. We call each integral curve of a given vector field in $[Z]$ a *leaf* of the characteristic foliation.

We observe that if $p \in F$ is a singular point, that is $T_pF = \xi_p$ then (2.2) yields $Z(p) = 0$. On the other hand, if T_pF is transversal to ξ_p then (2.2) implies that $Z(p) \in T_pF \cap \xi_p$ is nonzero.

2.4. The Overtwisted Disk

One of the main features that is used for differentiating contact 3-manifolds is the existence of a special embedded surface called an overtwisted disk. An *overtwisted disk* $\mathcal{D}$ in a contact 3-manifold (M, λ) is an embedded closed disk in M such that $T\partial\mathcal{D} \subset \xi|_{\mathcal{D}}$

and there is no singular point on $\partial\mathcal{D}$, that is, $\partial\mathcal{D}$ does not contain a point p such that $T_p\mathcal{D} = \xi_p$. If a contact 3-manifold contains an overtwisted disk we call the contact structure *overtwisted.* Otherwise we say it is *tight.* Similarly, a contact form is called *overtwisted (tight)* if it induces an overtwisted (tight) contact structure.

A standard example of an overtwisted contact form on $\mathbb{R}^3$ is

$$\tau = cos(r)dz + rsin(r)d\theta$$

where (r, θ, z) are cylindrical coordinates. For a sufficiently small ϵ we put

$$\mathcal{D} = \{\big(x, y, \epsilon(\pi^2 - (x^2 + y^2))\big) : x^2 + y^2 \leq 1\}.$$

We see that $\mathcal{D}$ is an overtwisted disk.

We call a diffeomorphism Ψ between contact manifolds (M, λ) and (N, τ) a *contactomorphism* if

$$\Psi^*\tau = \lambda.$$

If such a diffeomorphism exists then we say λ and τ are *isomorphic.* It is easy to see that a contactomorphism would preserve overtwisted disks.

The following theorem, whose proof follows as in [16], shows the existence of a tight contact structure on $\mathbb{R}^3$.

Theorem 2.4.1 (Bennequin)**.** *The standard contact structure on $\mathbb{R}^3$ is tight.*

Corollary 2.4.2. *On $\mathbb{R}^3$, there exist two non isomorphic contact structures.*

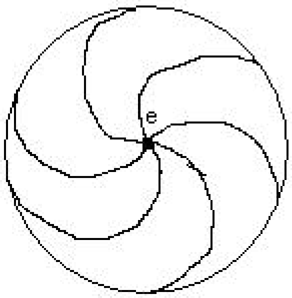

Figure 2.1. The characteristic foliation on an overtwisted disk with an elliptic singularity.

As we defined, above an overtwisted disk does not contain a singular point on its boundary. But in general, it may contain uncountably many singularities in the interior. Since boundary of an overtwisted disk is transverse to the contact structure, the boundary itself is a leaf of the characteristic foliation. The rest of the leaves may have a complicated geometry depending on the distribution of interior singularities and their types. However the work of Eliashberg, Fuch and Giroux on elimination of singularities gives the following important result:

Theorem 2.4.3. *Let M be a 3-manifold with an overtwisted contact form λ. There exists an overtwisted disk $\mathcal{D}$ so that $\mathcal{D}$ has exactly one singular point, an elliptic singularity e in the interior, and all leaves of the characteristic foliation connect e with $\partial\mathcal{D}$*

A detailed proof of this theorem can be found in [13].

3. PSEUDOHOLOMORPHIC CURVES, SYMPLECTIZATION, AND PSEUDOCONVEXITY

3.1. Pseudoholomorphic Curves

In this chapter we will introduce the notion of pseudoholomorphic curve in general. Then we will present our setup and study pseudoholomorphic curves there.

Pseudoholomorphic curves are generalizations of holomorphic maps from a domain $\Omega \subset \mathbb{C}$ into $\mathbb{C}^n$. We just replace Ω with a Riemann surface and $(\mathbb{C}^n, i)$ with an almost complex manifold (W, J).

An *almost complex structure* on a $2n$-dimensional manifold W is a smooth map $J : TW \to TW$ such that

$$J_x : T_xW \to T_xW, \text{ is linear and } J_x^2 = -id$$

for all $x \in W$. Then the pair (W, J) is called an *almost complex manifold.* We say that an almost complex structure J is *integrable* if there exists an atlas $\{\alpha, U_\alpha\}_\alpha$ on W such that for all $x \in W$,

$$d\alpha(x) \circ J_x = J_0 \circ d\alpha(x) : T_xW \to \mathbb{R}^{2n},$$

where J_0 is the matrix representing the standard complex multiplication on $\mathbb{R}^{2n} \approx \mathbb{C}^n$. If such an atlas exists then the transition maps are biholomorphic. We say that J is a *complex structure* if J is integrable. Integrability says that locally an almost complex structure is just the complex multiplication. A *Riemannian surface* (S, j) is a 2-manifold with a complex structure j.

Now we are ready to make the generalization. Let S be a Riemannian surface with a complex structure j and (W, J) be an almost complex manifold. A *pseudoholomorphic curve* (or a (j, J) *holomorphic curve*, or a J*-holomorphic curve* in short) is a smooth map

$$u : S \to W$$

which satisfies *the nonlinear Cauchy–Riemann equation*

$$du \circ j = J \circ du;$$

that is, the differential of u is complex linear with respect to j and J.

In order to see the local picture of such a map, we take an atlas $\{U_\alpha, \phi_\alpha\}$ on S such that

$$d\phi_\alpha \circ j = i \circ d\phi_\alpha$$

and transition maps $\phi_\alpha \circ \phi_\beta^{-1}$ are biholomorphic. Then u is pseudoholomorphic if and only if every local representation $u_\alpha = u \circ \phi_\alpha^{-1}$ of u is (i, J) holomorphic on $\phi_\alpha(U_\alpha)$. We see that in local coordinates $z = s + it$, the nonlinear Cauchy–Riemann equation looks like

$$\partial_s u_\alpha + J(u_\alpha)\partial_t u_\alpha = 0$$

or equivalently,

$$\partial_t u_\alpha - J(u_\alpha)\partial_s u_\alpha = 0$$

which are nonlinear first order partial differential equations.

3.2. Symplectization and Pseudoholomorphic Disks

Let M be a 3-dimensional, contact manifold with a an overtwisted contact structure ξ. Let λ be a one form inducing ξ and X_λ be the associated Reeb vector field. According to [13] we can choose a complex structure $J : \xi \to \xi$ compatible with $d\lambda|_{\xi\oplus\xi}$, that is

$$g_J(\cdot,\cdot) := d\lambda(\cdot, J\cdot)$$

is a Riemannian metric on ξ. Now consider the four manifold $\mathbb{R} \times M$. We have

$$T(\mathbb{R} \times M) = \mathbb{R}\partial_t \oplus \mathbb{R}X_\lambda \oplus \xi$$

where $\mathbb{R}\partial_t$ is the line bundle spanned by ∂_t and $\mathbb{R}X_\lambda$ is the line bundle spanned by the Reeb vector field X_λ. We define an almost complex structure $\tilde{J}$ on $\mathbb{R} \times M$ by

$$\tilde{J}(a\partial_t + bX_\lambda + v) := (-b\partial_t + aX_\lambda + Jv) \tag{3.1}$$

that is, $\tilde{J}$ acts on ξ as J and maps ∂_t to X_λ and X_λ to $-\partial_t$. Writing explicitly, we get

$$\tilde{J}(a,u)(h,k) := (-\lambda(u)k, J(u)\pi_\lambda k + hX_\lambda(u)) \tag{3.2}$$

where $(a,u) \in \mathbb{R} \times M$, $(h,k) \in T_{(a,u)}(\mathbb{R} \times M)$, and $\pi_\lambda : \mathbb{R}X_\lambda \oplus \xi \to \xi$ is the projection onto ξ.

Considering λ as a 1-form on $\mathbb{R} \times M$, we see that the 2-form $d(e^t\lambda)$ is a symplectic form on $\mathbb{R} \times M$. Therefore we call $(\mathbb{R} \times M, d(e^t\lambda))$ a symplectization of M. We observe that the almost complex structure $\tilde{J}$ is compatible with the symplectic form $d(e^t\lambda)$. Indeed,

$$d(e^t\lambda)\big(a_1\partial_t + b_1X_\lambda + v_1, \tilde{J}(a_2\partial_t + b_2X_\lambda + v_2)\big) = e^t\big(a_1a_2 + b_1b_2 + d\lambda(v_1, Jv_2)\big)$$

for $(a_i\partial_t + b_iX_\lambda + v_i) \in \mathbb{R}\partial_t \oplus \mathbb{R}X_\lambda \oplus \xi$ and it gives a Riemannian metric on $\mathbb{R} \times M$. It is also easy to see that for any smooth map $\varphi : \mathbb{R} \to R$ with $\varphi > 0$ and $\varphi' > 0$, $d(\varphi\lambda)$ is symplectic on $\mathbb{R} \times M$ and $\tilde{J}$ is compatible with $d(\varphi\lambda)$. For the sake of simplicity we define the Riemannian metric on $\mathbb{R} \times M$ by

$$g(\cdot,\cdot) := \frac{1}{e^t} d(e^t\lambda)(\cdot, \tilde{J}(\cdot)).$$

Writing explicitly we get

$$g_{(a,u)}((h_1,k_1),(h_2,k_2)) = h_1h_2 + \lambda(u)k_1\lambda(u)k_2 + g_J(\pi_\lambda k_1, \pi_\lambda k_2)$$

In this study we will mainly concentrate on pseudoholomorphic disks in the symplectization, i.e., the smooth maps

$$\tilde{u} : D \to \mathbb{R} \times M$$

defined on closed unit disk $D \subset \mathbb{C}$, which satisfy the following partial differential equation,

$$\partial_s \tilde{u} + \tilde{J}(\tilde{u}) \partial_t \tilde{u} = 0 \tag{3.3}$$

or equivalently,

$$\partial_t \tilde{u} - \tilde{J}(\tilde{u}) \partial_s \tilde{u} = 0 \tag{3.4}$$

on $\mathring{D}$, where $z = s + it$. If we put $\tilde{u} = (a, u)$, from (3.2) we get

$$\begin{aligned} 0 = (\partial_s a, \partial_s u) + \tilde{J}(a,u)(\partial_t a, \partial_t u) &= (\partial_s a, \partial_s u) + \big(- \lambda(u)\partial_t u, J(u)\pi_\lambda \partial_t u + \partial_t a X_\lambda(u)\big) \\ &= \big(\partial_s a - \lambda(u)\partial_t u, \partial_s u + J(u)\pi_\lambda \partial_t u + \partial_t a X_\lambda(u)\big) \end{aligned}$$

and

$$\begin{aligned} 0 = (\partial_t a, \partial_t u) - \tilde{J}(a,u)(\partial_s a, \partial_s u) &= (\partial_t a, \partial_t u) - \big(- \lambda(u)\partial_s u, J(u)\pi_\lambda \partial_s u + \partial_s a X_\lambda(u)\big) \\ &= \big(\partial_t a + \lambda(u)\partial_s u, \partial_t u - J(u)\pi_\lambda \partial_s u - \partial_s a X_\lambda(u)\big) \end{aligned}$$

Then we have

$$\partial a_s - \lambda(u)\partial_t u = 0, \text{ and } \partial a_t + \lambda(u)\partial_s u = 0, \tag{3.5}$$

which is clearly equivalent to

$$u^* \lambda = -da \circ i.$$

From (3.5), we deduce;

$$\begin{aligned}
0 = \partial_s u + J(u)\pi_\lambda \partial_t u + \partial_t a X_\lambda(u) &= \partial_s u + J(u)\pi_\lambda \partial_t u - \lambda(u)\partial u_s X_\lambda(u) \\
&= \lambda(u)\partial u_s X_\lambda(u) + \pi_\lambda \partial u_s + J(u)\pi_\lambda \partial_t u - \lambda(u)\partial u_s X_\lambda(u) \\
&= \pi_\lambda \partial u_s + J(u)\pi_\lambda \partial_t u
\end{aligned}$$

Hence the nonlinear Cauchy–Riemann equation yields,

$$\pi_\lambda \partial_s u + J(u)\pi_\lambda \partial_t u = 0, \quad u^*\lambda = -da \circ i. \tag{3.6}$$

Taking the exterior derivative of the both sides of $u^*\lambda = -da \circ i$ we get

$$u^* d\lambda = d(u^*\lambda) = d(-da \circ i) = (\Delta a) ds \wedge dt \tag{3.7}$$

On the other hand,

$$\begin{aligned}
(u^* d\lambda)\big((1,0),(0,1)\big) &= d\lambda(u_s, u_t) \\
&= d\lambda(\pi_\lambda u_s, \pi_\lambda u_t) \\
&= d\lambda(\pi_\lambda u_s, J(u)\pi_\lambda u_s) \\
&= g_J(\pi_\lambda u_s, \pi_\lambda u_s) \\
&= |\pi_\lambda u_s|_J^2
\end{aligned}$$

As a result we have

$$\Delta a = |\pi_\lambda u_s|_J^2 = |\pi_\lambda u_s|_J^2.$$

We define the *energy* of a pseudoholomorphic disk $\tilde{u}$ by

$$E(\tilde{u}) = \sup_{\varphi \in \Sigma} \int_D \tilde{u}^* d(\varphi\lambda),$$

where $\Sigma = \{\varphi : \mathbb{R} \to [0,1] : \varphi \text{ is smooth}, \ \varphi' \geq 0\}$. Here we see φ as a map $\varphi : \mathbb{R} \times M \to \mathbb{R}$ which depends only on the $\mathbb{R}$ component of $\mathbb{R} \times M$. We observe that

$$\begin{aligned}
\tilde{u}^* d(\varphi\lambda) &= d(\varphi \circ \tilde{u}) \wedge \tilde{u}^*\lambda + (\varphi \circ \tilde{u})\tilde{u}^*(d\lambda) \\
&= \varphi'(a) da \wedge u^*\lambda + \varphi(a) u^* d\lambda \\
&= \varphi'(a)\big[(a_s ds + a_t dt) \wedge (\lambda(u)u_s ds + \lambda(u)u_t dt)\big] + \varphi(a) u^* d\lambda \\
&= \bigg(\varphi'(a)\big[a_s\lambda(u)u_t - a_t\lambda(u)u_s\big] + \frac{1}{2}\varphi(a)\big[|\pi_\lambda u_s|_J^2 + |\pi_\lambda u_s|_J^2\big]\bigg) ds \wedge dt \\
&= \frac{1}{2}\bigg(\varphi'(a)\big[a_s^2 + a_t^2 + (\lambda(u)u_t)^2 + (\lambda(u)u_t)^2\big] + \varphi(a)\big[|\pi_\lambda u_s|_J^2 + |\pi_\lambda u_s|_J^2\big]\bigg) ds \wedge dt
\end{aligned}$$

where a_s and u_s stand for $\partial_s a$ and $\partial_s u$ respectively. So we conclude that a pseudoholomorphic disk has nonnegative energy. Moreover, (3.6) and (3.7) imply that $\tilde{u}$ is constant if and only if $E(\tilde{u}) = 0$. Indeed, $E(\tilde{u}) = 0$ implies,

$$a_s = a_t = \lambda(u)u_s = \lambda(u)u_t = 0, \text{ and } \pi_\lambda u_s = \pi_\lambda u_t = 0,$$

then

$$u_s = \lambda(u)(u_s)X_\lambda(u) + \pi_\lambda u_s = 0, \text{ and } u_s = \lambda(u)(u_t)X_\lambda(u) + \pi_\lambda u_t = 0.$$

3.3. Pseudoconvexity and Implications on Pseudoholomorphic Disks

In this section we define pseudoconvexity of a codimension 1 submanifold in an almost complex manifold and we derive some implications of this phenomenon on pseudoholomorphic disks in the symplectization of a contact 3-manifold.

Let (W, J) be an almost complex 4-manifold and M be an orientable codimension 1 submanifold of W. Assume we have a decomposition of W as follows: $W \setminus M = W^- \cup W^+$, where $W^- \cap W^+ = \emptyset$ and $\overline{W^-} \cap \overline{W^+} = M$. We say M is *pseudoconvex* with respect to W^- if there exists a smooth function $h : W \to \mathbb{R}$ such that

(i) $h^{-1}(0) = M$,
(ii) $dh(x) \neq 0$ for all $x \in M$,
(iii) $W^- = \{x \in W : h(x) < 0\}$,
(iv) $-d(dh \circ J)_x(v, J(x)v) > 0$ for all nonzero $v \in T_xW$, $x \in W$.

Each tangent space T_xM contains a unique complex line of J. Let us denote it by ξ_x. These complex lines give a plane distribution ξ in TM which is canonically oriented. Since TM is also oriented we have a trivial line bundle TM/ξ. Hence there is a vector field Y on M such that $TM = \xi \oplus \mathbb{R}Y$. If M is pseudoconvex, then ξ is the kernel of the one form $(dh \circ J)|_{TM}$. Moreover it is easy to see that $(dh \circ J) \wedge d(dh \circ J)$ is a nonvanishing top form on M. Hence the distribution of complex lines of J on TM is a contact structure induced by $dh \circ J$.

The following theorem shows the behavior of a pseudoholomorphic disk whose boundary lies on pseudoconvex submanifold.

Theorem 3.3.1. *Let $u : D \to W$ be a smooth solution of nonlinear Cauchy–Riemann equation with boundary condition $u(\partial D) \subset M$. Then either*

$$u(D) \subset M$$

or

$$u(\mathring{D}) \subset W^- \text{ and } u \text{ is transverse to } M \text{ at each } z \in \partial D, \text{ i.e.,}$$
$$du(z)T_zD \oplus T_{u(z)}M = T_{u(z)}W.$$

Proof. Let $h : W \to \mathbb{R}$ be as above and define $f = h \circ u : D \to \mathbb{R}$. Then

$$\begin{aligned} (-\Delta f) ds \wedge dt &= d(df \circ i) \\ &= d(dh \circ du \circ i) \\ &= d(dh \circ J \circ du) \\ &= d(u^*(dh \circ J)) \\ &= u^*(d(dh \circ J)) \\ &= d(dh \circ J)(u_s, u_t) ds \wedge dt \\ &= d(dh \circ J)(u_s, J(u)u_s) ds \wedge dt. \end{aligned}$$

Hence $\Delta f \geq 0$. We also have $f|_{\partial D} \equiv 0$. Applying the strong maximum principle [17], we get

$$f \equiv 0 \text{ on } D$$

or

$$f < 0 \text{ on } \mathring{D} \text{ and } \frac{\partial f}{\partial \nu} > 0 \text{ on } \partial D,$$

where $\frac{\partial f}{\partial \nu}$ is the outer normal derivative. We observe that first case is equivalent to

$u(D) \subset M$. In the second case we have $u(\mathring{D}) \subset W^-$. Moreover for fixed $z \in \partial D$,

$$\frac{\partial f}{\partial \nu}(z) = \frac{d}{dt} f(tz)\bigg|_{t=1} = \frac{d}{dt} h(u(tz))\bigg|_{t=1} > 0.$$

Then

$$d(h(u(z))) \circ du(z)(\mathbb{R}z) > 0.$$

Hence $d(u(z))\mathbb{R}z$ is not contained in $T_{u(z)}M$ since $dh(x)|_{T_xM} \equiv 0$. Therefore dimensions of $du(z)T_zD$ and $T_{u(z)}M$ add up to a number at least 4. $\square$

Returning to our case, we put $W = \mathbb{R} \times M$ and $J = \tilde{J}$ as before. Then we choose

$$h : W \longrightarrow \mathbb{R}, \quad (t, x) \mapsto e^t - 1$$

and $W^- = (-\infty, 0] \times M$, $W^+ = [0, +\infty) \times M$. We compute that

$$\begin{aligned} (dh \circ \tilde{J})_{(t,x)}(h, k) &= (e^t dt)_{(t,x)}(\tilde{J}(t, x)(h, k)) \\ &= (e^t dt)_{(t,x)}(-\lambda(x)k, J(x)\pi_\lambda k + hX_\lambda(x)) \\ &= -e^t \lambda(x)k \end{aligned}$$

and

$$\begin{aligned}
-d(dh \circ \tilde{J})_{(t,x)}((h,k), \tilde{J}(h,k)) &= \big(- d(-e^t\lambda)\big)((h,k), \tilde{J}(h,k)) \\
&= \big(e^t dt \wedge \lambda + e^t d\lambda\big)\big((h,k), (-\lambda(x)k, J(x)\pi_\lambda k + hX_\lambda(x))\big) \\
&= e^t\big(h^2 + (\lambda(x)k)^2 \\
&\quad +d\lambda(x)((h,k), (-\lambda(x)k, J(x)\pi_\lambda k + hX_\lambda(x))\big) \\
&= e^t\big(h^2 + (\lambda(x)k)^2 + d\lambda(x)(\pi_\lambda k, J(x)\pi_\lambda k)\big) \\
&= e^t\big(h^2 + (\lambda(x)k)^2 + |\pi_\lambda k|_J^2\big) \\
&> 0.
\end{aligned}$$

for all $(t,x) \in \mathbb{R} \times M$ and $(h,k) \in T_{(t,x)}\mathbb{R} \times M$, $(h,k) \neq 0$. Therefore we conclude that $M \approx \{0\} \times M$ is pseudoconvex with respect to $(-\infty, 0] \times M$.

Here we make an observation on pseudoholomorphic disks whose boundary lies specifically on an overtwisted disk in M.

Corollary 3.3.2. *Let M be a contact 3-manifold and $\mathcal{D} \subset M$ be an overtwisted disk with an interior singularity e. Let*

$$\tilde{u} = (a,u) : D \to \mathbb{R} \times M$$

be a $\tilde{J}$-holomorphic disk with

$$\tilde{u}(\partial D) \subset \mathcal{D}^* \approx \{0\} \times \mathcal{D}^* \subset \{0\} \times M.$$

where $\mathcal{D}^* = \mathcal{D} \setminus \{e\}$. Then either

$$\tilde{u} \equiv \quad \textit{constant}$$

or

$$\int_D u^* d\lambda > 0, \;\; a(int(D)) < 0, \;\; |\nabla a| \neq 0 \;\; on \;\partial D, \;\; and \; u|_{\partial D} : \partial D \to \mathcal{D}^*$$

is an immersion transverse to the one-dimensional distribution $\xi|_{\mathcal{D}^*} \cap T\mathcal{D}^* \to \mathcal{D}^*$.

Proof. Applying the previous theorem we have two cases. If $\tilde{u}(D) \subset M$ then $a \equiv 0$. By (3.6) and (3.7), we have $u^*\lambda = 0$ and $u^*d\lambda = 0$. Then we have

$$0 = u^*d\lambda = \frac{1}{2}(|\pi_\lambda u_s|_J^2 + |\pi_\lambda u_t|_J^2) ds \wedge dt, \text{ and } \lambda(u)u_s = \lambda(u)u_t = 0.$$

Hence $u_s = u_t = 0$ which implies that u is also constant.

If the second case holds, we have $\tilde{u}(\mathring{D}) \subset W^- = (-\infty, 0] \times M$, i.e. $a < 0$ on $\mathring{D}$. Moreover $\tilde{u}$ is transverse to M at each $z \in \partial D$. Then for $z \in \partial D$ we have

$$d\tilde{u}(z)T_zD + T_{\tilde{u}(z)}M = (da(z)T_zD, du(z)T_zD) + (0, T_{\tilde{u}(z)}M) = T_{\tilde{u}(z)}W.$$

Then $da(z)T_zD = \mathbb{R}$ for all $z \in \partial D$. Hence $|\nabla a| \neq 0$ on ∂D. We observe that

$$(-\Delta a) ds \wedge dt = d(da \circ i) = d(-u^*\lambda) = -u^*d\lambda.$$

Then $\int u^*d\lambda = 0$ implies $(-\Delta a) \equiv 0$. With boundary condition $a|_{\partial D} \equiv 0$, we get $a \equiv 0$ which contradicts to $|\nabla a| \neq 0$ on ∂D. Hence $\int u^*d\lambda > 0$.

Now we want to show that for all $z \in \partial D$

$$d(u|_{\partial D})(T_z\partial D) \cap \left(T_{u(z)}\mathcal{D}^* \cap \xi_{u(z)}\right) = \{0\}.$$

We already know that $d(u|_{\partial D})(T_z\partial D)$ lies on $T_{u(z)}\mathcal{D}^*$. Now we claim that $d(u|_{\partial D})(T_z\partial D) \nsubseteq \xi_{u(z)}$. Take $v \in T_z\partial D$. We have $T_zD = \mathbb{R}v + \mathbb{R}(iv)$ and $da(z)v = 0$ since $a \equiv 0$ on ∂D. On the other hand, $da(z)T_zD = \mathbb{R}$. Therefore $da(z)(iv) \neq 0$ which implies that $(u^*\lambda)v = -da(z)(iv) \neq 0$. Hence $\lambda(u(z))(u_*v) \neq 0$ i.e. $u_*v \notin \xi_{u(z)}$. $\square$

4. REGULARITY AND COMPACTNESS RESULTS FOR PSEUDOHOLOMORPHIC MAPS

Let J be an almost complex structure on $\mathbb{C}^n$ and $\Omega \subset \mathbb{C}$. We define the nonlinear Cauchy–Riemann operator

$$\overline{\partial}_J : \mathcal{C}^\infty(\Omega, \mathbb{C}^n) \to \mathcal{C}^\infty(\Omega, \mathbb{C}^n)$$

by

$$\overline{\partial}_J \phi := \phi_s + J(\phi)\phi_t,$$

where $z = s + it \in \Omega$. We see that pseudoholomorphic maps between Ω and $\mathbb{C}^n$ are zeros of the nonlinear Cauchy-Riemann operator.

In what follows, we will consider the Cauchy–Riemann operator corresponding to standard complex multiplication on $\mathbb{C}^n$ and state a theorem which gives estimates for this operator acting on compactly supported smooth maps from $\mathbb{C}$ into $\mathbb{C}^n$. Using this theorem we will establish some regularity results for pseudoholomorphic maps.

Theorem 4.0.3. *Given $1 < p < \infty$, $R > 0$, and $k \in \mathbb{N}$ there exists a constant $c > 1$, depending on p, R and k, such that for all $\phi \in \mathcal{C}_0^\infty(B_R, \mathbb{C}^n)$ we have*

$$||\phi||_{k,p,\mathbb{C}} \leq c||\overline{\partial}\phi||_{k-1,p,\mathbb{C}}.$$

Proof of the theorem is easy for $k = 1$ but it is much more complicated in general.

One can find the proof in [13]. Now we use this theorem and some other technical results to derive a regularity result for maps from closed unit disk into an almost complex manifold which satisfy nonlinear Cauchy–Riemann equation rather than a linear one.

Let W be a $2n$-dimensional almost complex manifold with an almost complex structure J. We embed W into $\mathbb{R}^N$ for some N using Withney's Embedding theorem and view u as a map from D into $\mathbb{R}^N$. We write

$$D^\alpha u = \left(\frac{\partial}{\partial s}\right)^{\alpha_1} \left(\frac{\partial}{\partial t}\right)^{\alpha_2} u,$$

where $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}^2$ is a multi-index. Let $K \subseteq W$ be a compact subset. We denote the closed unit disk by D. Let $\Gamma(c, K)$ denote the set of smooth solutions $u : D \to W$ of

$$\overline{\partial}_J u := u_s + J(u)u_t = 0$$

on D such that

$$\sup_{z \in D} |\nabla u(z)| \leq c$$

and $u(D) \subseteq K$, where

$$|\nabla u(z)|^2 = \left|\frac{\partial u}{\partial s}(z)\right|^2 + \left|\frac{\partial u}{\partial t}(z)\right|^2 .$$

Proposition 4.0.4. *Let $0 < p < \infty$, $\delta \in (0, 1)$, and $l \in \mathbb{N}$. Then there exists a constant $d > 0$ depending on p, δ, c, and K, such that*

$$\sup_{u \in \Gamma(c,K)} ||u||_{l,p,B(0,\delta)} \leq d.$$

Proof. We use induction for the proof. We first see that since $u(D) \subseteq K$ and K is compact in $\mathbb{R}^N$ the statement is true for $l = 0$. Moreover the existence of a uniform gradient bound makes the statement true for $l = 1$. We assume that the statement is true for $l \geq 1$. We see that if $\sup_{u \in \Gamma(c,K)} ||u||_{l+1,p,B(0,\delta)} = \infty$, then one can find a sequence $(u_k) \in \Gamma(c, K)$ such that $||u_k||_{l+1,p,B(0,\delta)} \to +\infty$. So it is enough to show that every sequence in $\Gamma(c, K)$ has a subsequence which is bounded in $W^{l+1,p}(B(0, \delta), \mathbb{R}^N)$. Fix $\delta' \in (\delta, 1)$ and take any sequence $(u_k) \in \Gamma(c, K)$. By induction hypothesis, there exists $c_{\delta'} > 0$ such that $||u_k||_{1,p,B(0,\delta)} \leq c_{\delta'}$. By the compact embedding

$$W^{1,p}(B(0, \delta), \mathbb{R}^N) \hookrightarrow \mathcal{C}^0(\overline{B(0, \delta')}, \mathbb{R}^N),$$

we may assume that $u_k \to u$ in $\mathcal{C}^0(\overline{B(0, \delta')})$ which also gives $u(\overline{B(0, \delta')}) \subseteq K$.

Now given $x_0 \in \overline{B(0, \delta')}$, there is neighborhood U in of $u(x_0)$ which is diffeomorphic to $\mathbb{R}^{2n} \simeq \mathbb{C}^n$ via ψ with $\psi(x_0) = 0$. Then there is an open set $\tilde{U}$ in $\mathbb{R}^N$ such that $W \cap \tilde{U} = U$ and there exists $R > 0$ such that $B(u(x_0), R) \subseteq \tilde{U}$. By continuity of u, there is $\epsilon_0 > 0$ such that $u(\overline{B(x_0, \epsilon_0)}) \subseteq B(u(x_0), R/4)$. Then by $\mathcal{C}^0$ convergence, $||u_k - u||_{\mathcal{C}^0} < R/4$ for large k. Hence for large k, $u_k(\overline{B(x_0, \epsilon_0)}) \subseteq B(u(x_0), R) \subseteq U$.

We define $\hat{J}$ on $\psi(U) = \mathbb{C}^n$ by

$$\hat{J} = d\psi \circ J \circ d\psi^{-1}.$$

After composing ψ with a linear map we may assume that $\hat{J}(0) = i$ and we define

$$v_k := \psi \circ u_k : \overline{B(x_0, \epsilon_0)} \to \mathbb{C}^n.$$

Now we are in a situation that

$$\overline{\partial} v_k = 0, \quad \text{on} \quad B(x_0, \epsilon_0),$$

and

$$v_k(x_0) \to 0, \quad v_k \to v \quad \text{in} \quad \mathcal{C}^0(\overline{B(x_0, \epsilon_0)}, \mathbb{C}^n).$$

Viewing ψ as a map on $\tilde{U}$ and making $\tilde{U}$ smaller before arguing as above, we may assume that ψ has bounded derivatives of all orders on $\tilde{U}$. We apply Theorem A.0.7 to u_k's and v_k's and conclude that (v_k) is bounded in $W^{l,p}(B(x_0, \epsilon_0), \mathbb{C}^n)$.

Now we claim that for some $\epsilon \in (0, \epsilon_0]$, (v_k) is bounded in $W^{l+1,p}(B(x_0, \epsilon), \mathbb{C}^n)$. Let $\epsilon \in (0, \frac{\epsilon_0}{2}]$. We define a smooth cut-off function $\beta_\epsilon : \mathbb{C} \to \mathbb{R}$ with

$$\text{supp}(\beta_\epsilon) \subset B(x_0, 2\epsilon) \subset B(x_0, \epsilon_0), \text{ and } \beta_\epsilon \equiv 1 \text{ on } B(x_0, \epsilon).$$

By construction, $\beta_\epsilon v_k$ is compactly supported so we can apply Theorem 4.0.3 to $\beta_\epsilon v_k$. With the constant coming from Theorem 4.0.3 we have,

$$\begin{aligned}
c^{-1}||v_k||_{l+1,p,B(x_0,\epsilon)} &\leq c^{-1}||\beta_\epsilon v_k||_{l+1,p,B(x_0,\epsilon_0)} \\
&\leq ||\overline{\partial}(\beta_\epsilon v_k)||_{l,p,B(x_0,\epsilon_0)} \\
&\leq ||(\beta_\epsilon v_k)_s + \hat{J}(v_k)(\beta_\epsilon v_k)_t||_{l,p,B(x_0,\epsilon_0)} \\
&\quad + ||(i - \hat{J}(v_k))(\beta_\epsilon v_k)_t||_{l,p,B(x_0,\epsilon_0)} \\
&\leq ||(\beta_\epsilon[(v_k)_s + \hat{J}(v_k)(v_k)_t]||_{l,p,B(x_0,\epsilon_0)} \\
&\quad + ||(\beta_\epsilon)_s v_k + (\beta_\epsilon)_t \hat{J}(v_k) v_k||_{l,p,B(x_0,\epsilon_0)} \\
&\quad + ||(i - \hat{J}(v_k))(\beta_\epsilon v_k)_t||_{l,p,B(x_0,\epsilon_0)} \\
&= ||(\beta_\epsilon)_s v_k + (\beta_\epsilon)_t \hat{J}(v_k) v_k||_{l,p,B(x_0,\epsilon_0)} + ||(i - \hat{J}(v_k))(\beta_\epsilon v_k)_t||_{l,p,B(x_0,\epsilon_0)}
\end{aligned}$$

where the last equality follows from the fact that v_k's are $\hat{J}$-holomorphic.

We consider the first expression at the last line. Since β_ϵ is smooth and

$$\mathrm{supp}(\beta_\epsilon) \subset B(x_0, 2\epsilon) \subset B(x_0, \epsilon_0),$$

all partial derivatives attains their maximum on $B(x_0, \epsilon_0)$. So we know that $||(\beta_\epsilon)||_{l,p,B(x_0,\epsilon_0)}$, $||(\beta_\epsilon)_s||_{l,p,B(x_0,\epsilon_0)}$ and $||(\beta_\epsilon)_t||_{l,p,B(x_0,\epsilon_0)}$ are finite and using Theorem A.0.5 we can make $||(\beta_\epsilon)_s v_k||_{l,p,B(x_0,\epsilon_0)}$ uniformly bounded. Now we view $\hat{J}$ as a smooth map from $\mathbb{R}^{2n}$ into $GL(\mathbb{R}^{2n}) \simeq \mathbb{R}^{4n^2}$. We remember that the parametrization ψ is a diffeomorphism between U and $\psi(U)$ and ψ can be chosen so that $\psi(U)$ is a bounded domain. Therefore choosing U smaller if necessary, we may assume that $\hat{J}$ has bounded derivatives of all orders on $\psi(U)$. We also have $v_k(B(x_0, \epsilon_0)) \subset \psi(U)$ for all k. Now we can apply Theorem A.0.7 to $\hat{J}(v_k)$ and we get

$$||\hat{J}(v_k) - \hat{J}(0)||_{l,p,B(x_0,\epsilon_0)} = ||\hat{J}(v_k) - i||_{l,p,B(x_0,\epsilon_0)} \leq c||v_k||_{l,p,B(x_0,\epsilon_0)}.$$

for some $c > 0$. By triangle inequality and boundedness of (v_k) in $W^{l,p}(B(x_0, \epsilon_0), \mathbb{C}^n)$, we conclude that $||\hat{J}(v_k)||_{l,p,B(x_0,\epsilon_0)}$ is uniformly bounded and therefore $||(\beta_\epsilon)_t \hat{J}(v_k)||_{l,p,B(0,\delta)}$ is uniformly bounded by Theorem A.0.5. Hence $||(\beta_\epsilon)_s v_k + (\beta_\epsilon)_t \hat{J}(v_k) v_k||_{l,p,B(0,\delta)}$ is bounded by some $C > 0$.

Now we consider the second expression at the last line. We see that

$$||(i - \hat{J}(v_k))(\beta_\epsilon v_k)_t||_{l,p,B(x_0,\epsilon_0)}$$

contains two type of integrals. First type consists of the followings:

$$\int |D^\beta (i - \hat{J}(v_k)) D^{\alpha-\beta} (\beta_\epsilon v_k)_t|^p dx$$

where $0 < |\beta| \leq |\alpha| = l$. In these expressions we have derivatives of v_k's up to degree l. By similar arguments as above, using induction hypothesis, Theorem A.0.7 and Theorem A.0.5 we conclude there is a constant $C' > 0$ such that these integrals are bounded independent of k.

In the second type, we have the integrals that contain derivatives of v_k's of degree $l + 1$, namely

$$\int |(i - \hat{J}(v_k)) D^\alpha (\beta_\epsilon v_k)_t|^p dx$$

where $|\alpha| = l$. Since $\text{supp}(\beta_\epsilon) \subset B(x_0, 2\epsilon)$, we have

$$\int_{B(x_0,\epsilon_0)} |(i - \hat{J}(v_k)) D^\alpha (\beta_\epsilon v_k)_t|^p dx \leq ||(i - \hat{J}(v_k))||^p_{L^\infty(x_0,2\epsilon)} \int_{B(x_0,\epsilon_0)} |D^\alpha (\beta_\epsilon v_k)_t|^p dx.$$

Hence

$$||(i-\hat{J}(v_k))D^{\alpha}(\beta_{\epsilon}v_k)_t||_{L^pB(x_0,\epsilon_0)} \leq ||(i-\hat{J}(v_k))||_{L^{\infty}(x_0,2\epsilon)}||\beta_{\epsilon}v_k||_{l+1,p,B(x_0,\epsilon_0)}.$$

For some constant $d>0$, counting number of α's with $|\alpha|=l$, we have

$$||(i-\hat{J}(v_k))(\beta_{\epsilon}v_k)_t||_{l,p,B(x_0,\epsilon_0)} \leq C' + d||(i-\hat{J}(v_k))||_{L^{\infty}(x_0,2\epsilon)}||\beta_{\epsilon}v_k||_{l+1,p,B(x_0,\epsilon_0)}$$

We combine all these results and get

$$c^{-1}||\beta_{\epsilon}v_k||_{l+1,p,B(0,\delta)} \leq C + C' + d||(i-\hat{J}(v_k))||_{L^{\infty}(x_0,2\epsilon)}||\beta_{\epsilon}v_k||_{l+1,p,B(x_0,\epsilon_0)}$$

Then we add and subtract

$$d||(i-\hat{J}(v_k))||_{L^{\infty}(x_0,2\epsilon)}$$

and deduce

$$\big(c^{-1}-\gamma(\epsilon)-\tau(k)\big)||\beta_{\epsilon}v_k||_{l+1,p,B(0,\delta)} \leq C + C',$$

where

$$\tau(k) := d\big(\ ||(i-\hat{J}(v_k))||_{L^{\infty}(x_0,2\epsilon)} - ||(i-\hat{J}(v))||_{L^{\infty}(x_0,2\epsilon)}\big)$$

and

$$\gamma(\epsilon) := d\big(||(i-\hat{J}(v))||_{L^{\infty}(x_0,2\epsilon)}\big).$$

We have $\gamma(\epsilon) \to 0$ as $\epsilon \to 0$ since $v(0) = 0$ and $J(0) = i$, and $\tau(k) \to 0$ as $k \to \infty$ since $v_k \to v$ uniformly. We choose ϵ small and k large so that we have

$$c^{-1} - \gamma(\epsilon) - \tau(k) > \frac{1}{2c}.$$

Then we have

$$||v_k||_{l+1,p,B(x_0,\epsilon)} \leq ||\beta_\epsilon v_k||_{l+1,p,B(x_0,\delta)} \leq \frac{C + C'}{c^{-1} - \gamma(\epsilon) - \tau(k)} \leq 2c(C + C')$$

Now for each $x_0 \in \overline{B(0,\delta)}$ there is $\epsilon \in (0, \epsilon_0]$ such that (v_k) is bounded in $W^{l+1,p}(B(x_0,\epsilon), \mathbb{R}^{2n})$. We compose each v_k with ψ^{-1}, apply Theorem A.0.7 and conclude that (u_k) is bounded in $W^{l+1,p}(B(x_0,\epsilon), \mathbb{R}^N)$. By compactness of $\overline{B(0,\delta)}$, we can cover $B(0,\delta)$ with finitely many $B(x_0,\epsilon)$. Hence (u_k) is bounded in $W^{l+1,p}(B(0,\delta), \mathbb{R}^N)$. $\square$

Since $B(0,\delta)$ has Lipschitz boundary for each $\delta \in (0,1)$ and $2 < p < \infty$, by Theorem A.0.6 we have a compact embedding

$$W^{l+1,p}(B(0,\delta), \mathbb{R}^N) \hookrightarrow \mathcal{C}^l(\overline{B(0,\delta)}, \mathbb{R}^N).$$

Therefore uniform Sobolev bounds give us uniform $\mathcal{C}^l$ bounds.

Theorem 4.0.5. *Let $\Gamma(c,K)$ be as above. Then given $\delta \in (0,1)$ and $l \in \mathbb{N}$, there is a constant $d > 0$, depending on l, c, δ and K such that*

$$||u||_{\mathcal{C}^l(\overline{B(0,\delta)})} \leq d$$

for every $u \in \Gamma(c,K)$

Now we will state a compactness theorem for pseudoholomorphic disks, which is one of the main ingredients for the proof of the Weinstein conjecture.

Theorem 4.0.6. *Let (M, λ) be 3-dimensional closed contact manifold and let $\mathcal{D}$ be an overtwisted disk in M with an exactly one singularity e. Let*

$$\tilde{u}_k = (a_k, u_k) : D \to \mathbb{R} \times M$$

be a sequence of $\tilde{J}$-holomorphic disks, where $\tilde{J}$ is an almost complex structure on $\mathbb{R} \times M$, such that

$$\tilde{u}_k(\partial D) \subset \{0\} \times \mathcal{D} \setminus \{e\}, \ \textit{and} \ \inf_k \textit{dist}(u_k(\partial D), e) > 0.$$

Assume also that

$$\sup_k ||\nabla \tilde{u}_k||_{\mathcal{C}^0(D)} < +\infty. \tag{4.1}$$

Then there is a subsequence of $(\tilde{u}_k)$ that converges to a $\tilde{J}$-holomorphic disk $\tilde{u} : D \to \mathbb{R} \times M$ with $\tilde{u}(\partial D) \subset \{0\} \times \mathcal{D} \setminus \{e\}$ in $\mathcal{C}^\infty(D)$.

Proof of this theorem relies heavily on the regularity result that we just proved. Using the fact that $a_k|_{\partial D} = 0$ for all k and the compactness of M we get a k-uniform $\mathcal{C}^0(D, \mathbb{R}^N)$ bound. Then we can apply Theorem 4.0.5 to the sequence $(\tilde{u}_k)$ since we have (4.1). Applying Theorem 4.0.5 and Arzela-Ascoli theorem successively and using a diagonal choice argument as we will see in the proof of Proposition 5.1.1 and Proposition 5.2.1, we construct a subsequence of $\tilde{u}_k$ that converges to a pseudoholomorphic disk $\tilde{u}$ in $\mathcal{C}^l(K, \mathbb{R}^N)$ for any $l \in \mathbb{N}$ and for any closed $K \subset D$ with $K \cap \partial D = \emptyset$. However this does not establish that this subsequence converges to $\tilde{u}$ in $\mathcal{C}^\infty(D, \mathbb{R}^N)$. To establish this we need a regularity result for pseudoholomorphic disks near the boundary of D. Using such a regularity one can establish the C^∞ convergence on D as in [9] and [13].

5. THE EXISTENCE OF A FINITE ENERGY PLANE IMPLIES THE EXISTENCE OF A PERIODIC ORBIT

In this chapter, we will show that the existence of a nontrivial finite energy plane in the symplectization $(\mathbb{R} \times M, \tilde{J})$ is a sufficient condition for the existence of a periodic orbit of the Reeb vector field correspoding to the overtwisted contact form λ on M. Recall that by a finite energy plane we mean a $\tilde{J}$-holomorphic map from $\mathbb{C}$ into $\mathbb{R} \times M$ which has finite energy.

5.1. Finite Energy Plane and the $d\lambda$ Energy

We start with proving that a nontrivial finite energy plane cannot have zero "$d\lambda$"-energy, where λ is defined in Chapter 3.

Proposition 5.1.1. *Let $\tilde{u} : \mathbb{C} \to \mathbb{R} \times M$ be a solution of the nonlinear Cauchy–Riemann equation*

$$\tilde{u}_s + \tilde{J}(\tilde{u})\tilde{u}_t = 0$$

with $\int_{\mathbb{C}} u^* d\lambda = 0$ and $E(\tilde{u}) < \infty$. Then $\tilde{u}$ is constant.

Proof. Recall from Chapter 3 that

$$u^* d\lambda = \frac{1}{2}(|\pi_\lambda u_t|_J^2 + |\pi_\lambda u_s|_J^2).$$

So, $\int_{\mathbb{C}} u^* d\lambda = 0$ implies $|\pi_\lambda u_t|_J^2 = |\pi_\lambda u_s|_J^2 = 0$. Remember also that we have the decomposition:

$$u_s = \lambda(u)(u_s)X_\lambda(u) + \pi_\lambda u_s, \text{ and } u_t = \lambda(u)(u_t)X_\lambda(u) + \pi_\lambda u_t. \tag{5.1}$$

Combining all these, we conclude that

$$u_s = \lambda(u)(u_s)X_\lambda(u), \quad u_t = \lambda(u)(u_t)X_\lambda(u).$$

Moreover, $u^*d\lambda = 0$ implies that $u^*\lambda$ is closed and therefore exact, hence there exist a smooth function f on $\mathbb{C}$ such that $df = u^*\lambda$. Recall also from Chapter 3 that $u^*\lambda = -da \circ i$. We define a map

$$\Phi := a + if : \mathbb{C} \to \mathbb{C}$$

and see that Φ is a holomorphic function since

$$df = u^*\lambda = -da \circ i.$$

Next, we will observe that since u_s and u_t have no contact plane components, we may omit two dimensions of the symplectization $\mathbb{R} \times M$, namely the contact planes, and instead of $\tilde{u}$ we may study Φ, which contains $\mathbb{R}$ and Reeb components of $\tilde{u}$, on $\mathbb{R} \times \mathbb{R}$.

Given $\varphi \in \Sigma = \{\varphi : \mathbb{R} \to [0,1] : \varphi \text{ is smooth}, \varphi' \geq 0\}$, we define a 2-form τ on $\mathbb{C}$ by $\tau := d(\phi(s)dt)$ where $z = s + it \in \mathbb{C}$. Since the energy of $\tilde{u}$ is finite, we have

$$\begin{aligned}
E(\tilde{u}) \geq \int_{\mathbb{C}} \tilde{u}^* d(\varphi\lambda) &= \int_{\mathbb{C}} \frac{1}{2}\Big(\varphi'(a)\big[a_s^2 + a_t^2 + (\lambda(u)u_t)^2 + (\lambda(u)u_t)^2\big] \\
&\quad + \varphi(a)\big[|\pi_\lambda u_s|_J^2 + |\pi_\lambda u_s|_J^2\big]\Big) ds \wedge dt \\
&= \int_{\mathbb{C}} \frac{1}{2}\Big(\varphi'(a)\big[a_s^2 + a_t^2 + (\lambda(u)u_t)^2 + (\lambda(u)u_t)^2\big]\Big) ds \wedge dt \\
&= \int_{\mathbb{C}} \varphi'(a)[a_s^2 + a_t^2] ds \wedge dt \\
&= \int_{\mathbb{C}} (a_s f_t - a_t f_s) ds \wedge dt \\
&= \int_{\mathbb{C}} \Phi^* \tau.
\end{aligned}$$

Observe that if Φ is constant, then a is constant therefore $da = 0$. This gives

$$0 = -da \circ i = u^* \lambda = \lambda(u)(u_s) ds + \lambda(u)(u_t) dt.$$

We already have $|\pi_\lambda u_t|_J^2 = |\pi_\lambda u_s|_J^2 = 0$ so by (5.1) we conclude that $u_s = u_t = 0$, which means that u is constant.

Now we want to show that Φ is constant. We first assume that Φ is nonconstant and has a bounded derivative. Then by Liouville's theorem we have $\Phi(z) = az + b$ for some $a, b \in \mathbb{C}$, with $a \neq 0$. Hence Φ is biholomorphic and

$$\int_{\mathbb{C}} \Phi^* \tau = \int_{\mathbb{C}} \tau = \int_{\mathbb{C}} \phi^{'}(s) ds \wedge dt$$

which diverges for a nonconstant $\phi \in \Sigma$ but $E(\tilde{u}) < \infty$.

We assume that Φ has unbounded derivatives and employ a "bubbling-off" argument.

We pick a sequence $(z_k) \subset \mathbb{C}$ such that

$$R_k := |\nabla\Phi(z_k)| := |\Phi'(z_k)| \to +\infty$$

We apply B.0.8 to the function $|\nabla\Phi(z)|$ and each z_k with $\epsilon_k = 1$ and get sequences (z_k') and (ϵ_k') with $\epsilon_k' \in (0,1]$ for all $k \in \mathbb{N}$ and $|z_k' - z_k| \leq 2$ such that

$$\epsilon_k'|\nabla\Phi(z_k')| \geq R_k,$$
$$|\nabla\Phi(z)| \leq 2|\nabla\Phi(z_k')| \text{ on } B(z_k', \epsilon_k') \subset \mathbb{C}.$$

Writing z_k and ϵ_k instead of z_k' and ϵ_k', we get $R_k = |\nabla\Phi(z_k)| \to +\infty$, $R_k\epsilon_k \to +\infty$ and

$$|\nabla\Phi(z)| \leq 2R_k \text{ on } B(z_k, \epsilon_k).$$

Using the biholomorphic transformation

$$z \mapsto z_k + \frac{z}{R_k}$$

between $B(z_k, \epsilon_k)$ and $B(0, R_k\epsilon_k)$, we define

$$\Phi_k : \mathbb{C} \to \mathbb{C}, \quad \Phi_k(z) := \Phi\left(z_k + \frac{z}{R_k}\right) - \Phi(z_k).$$

We see that each Φ_k satisfy the following: $|\nabla\Phi_k(0)| = 1$, $\Phi_k(0) = 0$. Moreover, $|\nabla\Phi_k(z)| \leq 2$ on $B(0, R_k\epsilon_k)$ since

$$|\nabla\Phi_k(z)| = \frac{1}{R_k}\left|\nabla\Phi\left(z_k + \frac{z}{R_k}\right)\right|.$$

Let us take a closed unit ball centered at the origin and call it B_1. Then for large k, $B_1 \subseteq B(0, R_k\epsilon_k)$. Since $\Phi_k(0) = 0$ and $|\nabla\Phi_k(z)| \leq 2$ on B_1, we get a k-uniform $\mathcal{C}^0(B_1, \mathbb{C})$ bound. Also using uniform gradient bound and Cauchy's integral formula we get k-uniform bounds for the higher derivatives on B_1. Then by the uniform C^1 bound on (Φ_k), using Arzela-Ascoli Theorem, we get a subsequence, say $(\Phi_{n_1(j)})_j$, that converges in $C^0(B_1, \mathbb{C})$. Now we have j-uniform C^2 bound, so by Arzela-Ascoli we get a subsequence, say $(\Phi_{n_{12}(j)})_j$, that converges in $C^1(B_1, \mathbb{C})$. Passing to a new subsequence for each derivative order, we get subsequences $(\Phi_{n_{12\ldots k}(j)})_j$ that converge in $C^{k-1}(B_1, \mathbb{C})$. Then we choose a subsequence $(\Phi_{\nu_1(j)})_j$ of (Φ_k) such that $\Phi_{\nu_1(j)} = \Phi_{n_{12\ldots j}(j)}$. Then $(\Phi_{\nu_1(j)})_j$ converges in $C^k(B_1, \mathbb{C})$ for all $k \in \mathbb{N}$. Now we take the closed ball with radius 2 centered at the origin, and call it B_2. We start with $(\Phi_{\nu_1(j)})_j$ and by the same argument as above we get a subsequence $(\Phi_{\nu_{12}(j)})_j$ that converges in $C^k(B_2, \mathbb{C})$ for all $k \in \mathbb{N}$. Passing to a new subsequence for each $B_n := \overline{B(0, n)}$, we get subsequences $(\Phi_{\nu_{12\ldots k}(j)})_j$ that converge in $C^k(B_n, \mathbb{C})$ for all $k \in \mathbb{N}$. Finally, we choose a subsequence (Φ_j) of (Φ_k) such that $\Phi_j = \Phi_{\nu_{12\ldots j}(j)}$. Then there exists a smooth map $\Psi : \mathbb{C} \to \mathbb{C}$ such that on any compact set $K \subset \mathbb{C}$, Φ_j converges to Ψ in $C^k(K, \mathbb{C})$ for all $k \in \mathbb{N}$, i.e.,

$$\Phi_j \to \Psi \text{ in } C^\infty_{loc}(\mathbb{C}).$$

Clearly, Ψ is holomorphic and satisfies $|\nabla\Psi(0)| = 1$ and $|\nabla\Psi(z)| \leq 2$ on $\mathbb{C}$. Again by Liouville's Theorem, Ψ is an affine nonconstant map.

Now let us check the "energy" of Ψ. We pick $\varphi \in \Sigma$, nonconstant, and put $\varphi_j(s) := \varphi(s - \mathfrak{Re}(\Phi(z_j)))$ and $\tau_\varphi := d(\varphi(s)dt)$. We had seen that $\Phi^*\tau_{\varphi_j} = \tilde{u}^* d(\varphi_j\lambda)$. Hence

$$\int_{B(0,R_j\epsilon_j)} \Phi_j^*\tau_\varphi = \int_{B(z_j,\epsilon_j)} \Phi^*\tau_{\varphi_j} \leq \int_{\mathbb{C}} \Phi^*\tau_{\varphi_j} = \int_{\mathbb{C}} \tilde{u}^* d(\varphi_j\lambda) \leq E(\tilde{u}) < +\infty.$$

Given $R > 0$, for large j, $R_j\epsilon_j > R$. We get

$$\int_{B(0,R)} \Phi_j^*\tau_\varphi \to \int_{B(0,R)} \Psi^*\tau_\varphi \qquad \text{as} \quad j \to +\infty.$$

Then for all $R > 0$, $E(\tilde{u}) \geq \int_{B(0,R)} \Psi^*\tau_\varphi$ and therefore $E(\tilde{u}) \geq \int_{\mathbb{C}} \Psi^*\tau_\varphi$. Since φ is nonconstant and Ψ is affine and nonconstant, the right hand side diverges. This is a contradiction. □

5.2. The Finite Energy Cylinder and Periodic Orbit

In order to prove the existence theorem for periodic orbits, it will be more convenient to use a pseudoholomorphic cylinder with finite energy rather than a finite energy plane. Therefore we change our setup as follows. We define a biholomorphic transformation

$$\phi : \mathbb{R} \times S^1 \to \mathbb{C} \setminus \{0\}, \quad \phi(s,t) := e^{2\pi(s+it)}$$

where $S^1 = \mathbb{R}/\mathbb{Z}$. Let $\tilde{u} = (a, u) : \mathbb{C} \to \mathbb{R} \times M$ be a finite energy plane with nontrivial energy. We define

$$\tilde{v} : \mathbb{R} \times S^1 \to \mathbb{R} \times M, \quad \tilde{v} = (b, v) := \tilde{u} \circ \phi.$$

Then we have $\partial_{\tilde{J}}\tilde{v} = 0$ on $\mathbb{R} \times S^1$ since ϕ is biholomorphic, and

$$\int_{\mathbb{C}} u^* d\lambda = \int_{\mathbb{R}\times S^1} v^* d\lambda > 0.$$

Moreover, for any $\varphi \in \Sigma$,

$$\int_{\mathbb{R}\times S^1} \tilde{v}^* d(\varphi\lambda) = \int_{\mathbb{R}\times S^1} (\tilde{u} \circ \phi)^* d(\varphi\lambda) = \int_{\mathbb{R}\times S^1} \phi^*\tilde{u}^* d(\varphi\lambda) = \int_{\mathbb{C}} \tilde{u}^* d(\varphi\lambda).$$

Hence

$$0 < E(\tilde{u}) = E(\tilde{v}) < +\infty$$

and $v(\mathbb{R} \times S^1) \subseteq M$. As in the previous chapters, we embed $\mathbb{R} \times M$ into $\mathbb{R}^N$ for sufficiently large N. Now we claim that gradient of such a finite energy cylinder is bounded.

Proposition 5.2.1. *Let $\tilde{v}$ be as above. Then there exists $c > 0$ such that*

$$|\nabla \tilde{v}(s,t)|^2 = |\partial_s \tilde{v}|^2 + |\partial_t \tilde{v}|^2 \leq c$$

for all $(s,t) \in \mathbb{R} \times S^1$.

Proof. We assume that the gradient of $\tilde{v}$ is not bounded and again we will employ a similar "bubbling-off" argument. We pick a sequence $(s_k, t_k) \subset \mathbb{R} \times S^1$ such that

$$R_k := |\nabla \tilde{v}(s_k, t_k)| \to +\infty.$$

Then we claim that $s_k \to +\infty$. Assume that (s_k) is bounded above by some $R > 0$. Clearly, the biholomorphic transformation ϕ above maps $(-\infty, R\,]$ onto a closed punctured disk say $B \setminus \{0\}$. We see that for all k,

$$R_k \leq \sup_{(-\infty,R\,]\times S^1} |\nabla \tilde{v}(s,t)| \leq \sup_{B} |\nabla \tilde{u}(z)| \cdot \sup_{(-\infty,R\,]\times S^1} |\nabla \phi(s,t)| < \infty$$

since B is compact. Hence $s_k \to +\infty$.

Now we view $\tilde{v} : \mathbb{R}^2 \approx \mathbb{C} \to \mathbb{R} \times M$ as a 1-periodic map in t, and we may assume $t_k \in [0,1]$. We apply Lemma B.0.8 to the function $|\nabla \tilde{v}(s,t)|$ for each (s_k, t_k) with $\epsilon_k = \frac{1}{4}$ and pass to the new sequences (s_k, t_k) and (ϵ_k) with *(i)* $|\nabla\tilde{u}(s,t)| \leq 2R_k$ for $(s,t) \in B((s_k,t_k),\epsilon_k)$,

(ii) $R_k\epsilon_k \to +\infty$,

(ii) $\epsilon_k \in (0, \frac{1}{4}]$.

We put $z_k := (s_k, t_k)$ and define

$$\tilde{v}_k : \mathbb{C} \to \mathbb{R} \times M, \quad \tilde{v}_k := (b_k(z), v_k(z)) := \left(b(z_k + \frac{z}{R_k}) - b(z_k), v(z + \frac{z}{R_k}) \right).$$

It is easy to see that $\overline{\partial}_{\tilde{J}}\tilde{v}_k = 0$ on $\mathbb{C}$, $|\nabla\tilde{v}_k(0)| = 1$, and $|\nabla\tilde{v}_k(z)| \leq 2$ on $B(0, R_k\epsilon_k)$ for all k. Now given $\varphi \in \Sigma$, we define $\varphi_k(s) = \varphi(s - b(z_k))$ and get

$$\int_{B(0,R_k\epsilon_k)} \tilde{v_k}^* d(\varphi\lambda) = \int_{B(z_k,\epsilon_k)} \tilde{v}^* d(\varphi_k\lambda) \leq \int_{\mathrm{R}\times[0,1]} \tilde{v}^* d(\varphi_k\lambda) \leq E(\tilde{v}) < +\infty \tag{5.2}$$

since $\tilde{v}$ is 1-periodic in t and $\epsilon_k \leq \frac{1}{4}$. We observe that

$$\int_{\mathrm{R}\times[0,1]} v^* d\lambda = \int_{\mathrm{R}\times[0,1]} \tilde{v}^* d(\varphi_0\lambda) \leq E(\tilde{v}) < +\infty. \tag{5.3}$$

where $\varphi_0 \equiv 1$. So given $R > 0$,

$$\int_{B(0,R)} v_k^* d\lambda = \int_{B(z_k,R/R_k)} v^* d\lambda$$

and by (5.3)

$$\int_{B(0,R/R_k)} v^* d\lambda \to 0 \text{ as } k \to +\infty$$

since $R/R_k \to 0$. Therefore

$$\int_{B(0,R)} v_k^* d\lambda \to 0 \text{ as } k \to +\infty. \tag{5.4}$$

Now we will show that there exists a subsequence of $(\tilde{v}_k)$ that converges in $\mathcal{C}^\infty_{loc}(\mathbb{C}, \mathbb{R}^N)$. For all $n \in \mathbb{N}$, we put $B_n := \overline{B(0,n)} \subset \mathbb{C}$. Similar to the proof of the previous proposition, we start with B_2. We have $b_k(0) = 0$ and $v_k(B_n) \subset M$ for all n where M is a compact subset of $\mathbb{R}^N$. So we get a k-uniform $\mathcal{C}^0(B_2, \mathbb{R}^N)$ bound on $(\tilde{v}_k)$. Observe that since we also have $|\nabla \tilde{v}_k(z)| \leq 2$ on B_1 for large k, after rescaling B_2 we can apply 4.0.5 to $(\tilde{v}_k)$ for $\delta = 1$ and we get k-uniform bounds on $(\tilde{v}_k)$ in $\mathcal{C}^l(B_1, \mathbb{R}^N)$ for all l. Notice that we cannot use Cauchy Integral Formula here, so we need 4.0.5 for bounds of higher order derivatives. Using $\mathcal{C}^1$ bound,by Arzela-Ascoli, we get a $\mathcal{C}^0$ convergent subsequence $(\tilde{v}_{n_1(j)})_j$. Using $\mathcal{C}^2$ bound on $(\tilde{v}_{n_1(j)})_j$ coming from 4.0.5, we get a $\mathcal{C}^1$ convergent subsequence $(\tilde{v}_{n_{12}(j)})_j$ of $(\tilde{v}_{n_1(j)})_j$. As in the proof of the previous proposition, we repeat this argument successively and we choose a diagonal subsequence that converges in $C^k(B_1)$ for all $k \in \mathbb{N}$. Then we repeat the whole argument for each B_n and finally get a subsequence, again say $(\tilde{v}_k)$, such that

$$\tilde{v}_k \to \tilde{w} \text{ in } C^\infty_{loc}(\mathbb{C}, \mathbb{R}^N)$$

where $\tilde{w} = (\beta, w) : \mathbb{C} \to \mathbb{R} \times M$. Then we have

$$\partial_{\tilde{J}} \tilde{w} = 0 \text{ on } \mathbb{C},$$
$$|\nabla \tilde{w}(0)| = 1,$$
$$|\nabla \tilde{w}(z)| \leq 2 \text{ for } z \in \mathbb{C}.$$

By (5.2) we also have $E(\tilde{w}) \leq E(\tilde{v}) < +\infty$ and by (5.4) we get $\int_{\mathbb{C}} w^* d\lambda = 0$. By Proposition 5.1.1, $\tilde{w}$ must be constant but $|\nabla \tilde{w}(0)| = 1$ so we have a contradiction. □

Now we are ready to state and prove the main result of this chapter, namely the existence theorem for periodic orbit.

Theorem 5.2.2. *Let* $\tilde{u} = (a, u) : \mathbb{C} \to \mathbb{R} \times M$ *be a solution of* $\overline{\partial}_{\tilde{J}} \tilde{u} = 0$ *satisfying*

$0 < E(\tilde{u}) < \infty$ *and let* $T := \int_{\mathbb{C}} u^* d\lambda$. *Then* $T > 0$ *and there exists a* T*-periodic solution of* $\dot{x}(t) = X_\lambda(x(t))$.

Proof. From Proposition 5.1.1 we know that $T > 0$. By previous discussions we may assume the existence of a map $\tilde{v} = (b, v) : \mathbb{R} \times S^1 \to \mathbb{R} \times M$ satisfying the following;

$$\begin{gathered} \partial_{\tilde{J}} \tilde{v} = 0 \text{ on } \mathbb{R} \times S^1, \\ \int_{\mathbb{R} \times S^1} v^* d\lambda = \int_{\mathbb{C}} u^* d\lambda > 0, \\ 0 < E(\tilde{v}) = E(\tilde{u}) < +\infty, \\ v(s,t) \to u(0) \text{ as } s \to -\infty. \end{gathered}$$

By Proposition 5.2.1, there exists $c > 0$ such that $|\nabla \tilde{v}(s,t)| \le c$ for all $(s,t) \in \mathbb{R} \times S^1$. We pick a sequence $s_k \to +\infty$ and define

$$\tilde{v}_k : \mathbb{R} \times S^1 \to \mathbb{R} \times M, \quad \tilde{v}_k := (b_k, v_k) = \big(b(s + s_k, t) - b(s_k, 0), v(s + s_k, t)\big).$$

Again we view $\tilde{v}_k$ as a map on $\mathbb{C}$ which is 1-periodic in the second argument. We see that $|\nabla \tilde{v}_k| \le c$ on $\mathbb{C}$, $b_k(0) = 0$ for all k, and we already have $\tilde{v}_k(\mathbb{C}) \subset M$. Using the same arguments as in the proof of Proposition 5.2.1, by Theorem 4.0.5 and the Arzela-Ascoli theorem we conclude that there exists a subsequence, say $(\tilde{v}_k)$, such that

$$\tilde{v}_k \to \tilde{w} \text{ in } \mathcal{C}^\infty_{loc}(\mathbb{R} \times S^1, \mathbb{R}^N)$$

where $\tilde{w} := (\beta, w) : \mathbb{R} \times S^1 \to \mathbb{R} \times M$ satisfies

$$\begin{gathered} \overline{\partial}_{\tilde{J}} \tilde{w} = 0 \text{ on } \mathbb{R} \times S^1, \\ |\nabla \tilde{w}| \le c \text{ on } \mathbb{R} \times S^1. \end{gathered}$$

We observe that

$$\sup_{\varphi\in\Sigma}\int_{\mathbb{R}\times S^1}\tilde{v_k}^{*}d(\varphi\lambda)=\sup_{\varphi\in\Sigma}\int_{\mathbb{R}\times S^1}\tilde{v}^{*}d(\varphi\lambda).$$

Therefore

$$E(\tilde{w})=E(\tilde{v})<+\infty. \tag{5.5}$$

Hence $\tilde{w}$ is also a finite energy cylinder. Now we fix $s_0\in\mathbb{R}$. Then

$$\begin{aligned}\int_{\{s_0\}\times S^1}v_k^{*}\lambda &= \int_{\{s_0+s_k\}\times S^1}v^{*}\lambda\\ &= \int_{\partial B(0,e^{s_0+s_k})}u^{*}\lambda\\ &= \int_{B(0,e^{s_0+s_k})}u^{*}d\lambda\\ &= \int_{(-\infty,s_k+s_0]\times S^1}v^{*}d\lambda.\end{aligned}$$

Therefore we get

$$\int_{\{s_0\}\times S^1}v_k^{*}\lambda\to\int_{\mathbb{R}\times S^1}v^{*}d\lambda=\int_{\mathbb{C}}u^{*}d\lambda=T\text{ as }k\to+\infty.$$

Taking $\varphi_0\equiv 1$ we get

$$\int_{\mathbb{R}\times S^1}\tilde{v}^{*}d(\varphi_0\lambda)=\int_{\mathbb{R}\times S^1}v^{*}d\lambda\le E(\tilde{v})<+\infty.$$

so for any $R > 0$,

$$\int_{[-R,R]\times S^1} v_k^* d\lambda = \int_{[-R+s_k,R+s_k]\times S^1} v^* d\lambda \to 0 \quad \text{as} \quad k \to +\infty.$$

This gives $\int_{[-R\times R]\times S^1} w^* d\lambda = 0$ and hence

$$\int_{\mathbb{R}\times S^1} w^* d\lambda = 0. \tag{5.6}$$

Again we view $\tilde{w} = (\beta, w) : \mathbb{C} \to \mathbb{R} \times M$ as a map 1-periodic in the second argument. We repeat exactly the same argument we sketched in the proof of Proposition 5.1.1. Then we have

$$w_s = \lambda(w)(w_s)X_\lambda, \text{ and } w_t = \lambda(w)(w_t)X_\lambda.$$

and we define a holomorphic map $\Phi := \beta + if$ on $\mathbb{C}$ where $df = w^*\lambda = -d\beta \circ i$. We observe that if Φ is constant then $0 = d\beta = (w^*\lambda) \circ i$ but $\displaystyle\int_{\{s_0\}\times S^1} w^*\lambda = T > 0$. So Φ is nonconstant. Moreover the gradient of Φ is bounded since

$$\sup_{\mathbb{C}} |\nabla\Phi|^2 = 2 \sup_{\mathbb{C}} |\nabla\beta|^2 \le 2 \sup_{\mathbb{C}} |\nabla\tilde{w}|^2 < +\infty.$$

Then Φ is an affine nonconstant map, that is $\Phi(z) = az + b$ for some $a, b \in \mathbb{C}$ with $a = a_1 + ia_2 \neq 0$ and $b = b_1 + ib_2$. Since β is 1-periodic in t we have $\beta(z) = \beta(s,t) = a_1 s - a_2 t + b_1 = a_1 s + b_1$. We compute

$$w_s = \lambda(w)(w_s)X_\lambda(w) = -\beta_t X_\lambda(w) = 0$$

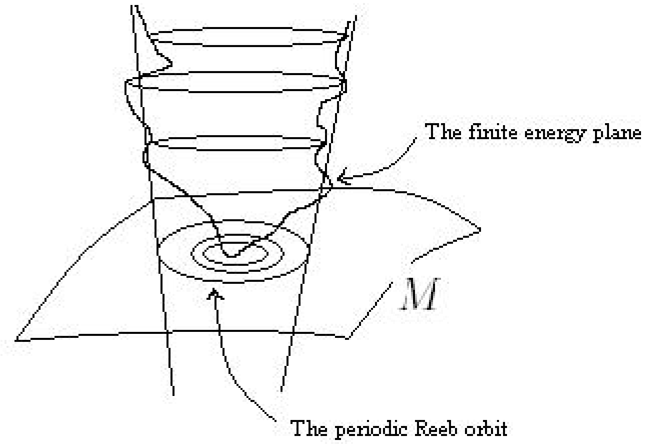

Figure 5.1. A finite energy plane

and

$$w_t = \lambda(w)(w_t)X_\lambda(w) = \beta_s X_\lambda(w) = a_1 X_\lambda(w).$$

We finally define $x(t) := w(s, \frac{t}{a_1})$, which is an integral curve of X_λ by previous line. Since w is 1-periodic, $x(t)$ is a_1-periodic. Furthermore

$$T = \int_{\{s_0\}\times S^1} v_k^* \lambda = \int_0^1 \lambda(w_t)dt = \int_0^1 a_1 dt = a_1.$$

$\square$

6. FAMILIES OF PSEUDOHOLOMORPHIC DISKS: EXISTENCE, UNIQUENESS, MAXIMALITY

In this chapter we will first define the Maslov index for a map from the unit disk to a four dimensional almost complex manifold, which satisfies a totally real boundary condition. Then we construct a family of pseudoholomorphic embedded disks in the symplectization of our overtwisted contact 3-manifold M, whose boundaries lie in an overtwisted disk, and which will have a certain Maslov index. This family will be called a Bishop family. After, we will state two important theorems that will be used in extending the Bishop family.

6.1. The Maslov Index

Let (W, J) be an almost complex 4-manifold and let F be a *totaly real* submanifold. This means that dim$F$=2 and $T_xF \oplus J(x)T_xF = T_xW$ for each $x \in F$. Let $u : D \to W$ be a smooth embedding such that

$$\overline{\partial}_J u := u_s + J(u)u_t = 0 \text{ on } \mathring{D},$$

and $u(\partial D) \subset F$. We know [13] that one can find a complex trivialization

$$\Phi : (u^*TW) \to (D \times \mathbb{C}^2, i)$$

of the the pullback bundle $u^*TW \to D$. Since u satisfies a totally real boundary condition, we can define a loop

$$\gamma_{u,F} : \partial D \to \daleth(2),$$

where $\daleth(2)$ is the manifold of totally real subspaces of $\mathbb{C}^2$ with respect to i, by

$$\gamma_{u,F}(z) := \mathrm{pr} \circ \Phi(\{z\} \times T_{u(z)}F)$$

where $pr : D \times \mathbb{C}^2 \to \mathbb{C}^2$ is the projection map. We pick a linear map χ on $\mathbb{C}^2$ such that $\chi(\gamma_{u,F}(1)) = \mathbb{R}^2$. Composing $\gamma_{u,F}$ with χ we get a representative of the homotopy class

$$[\chi \circ \gamma_{u,F}] \in \pi_1(\daleth(2), [\mathbb{R}^2]).$$

Now we know [13] that there is a distinguished group isomorphism

$$\mu_2 : \pi_1(\daleth(2), \mathbb{R}^2) \to \mathbb{Z}, \quad \mu_2([\alpha]) = 1,$$

where

$$\alpha : [0,1] \to \daleth(2), \quad \alpha(t) = e^{\pi i t}\mathbb{R} \oplus \mathbb{R}.$$

Using this isomorphism, we define the *Maslov index* of the map u by

$$\mu_2(u) := \mu_2([\chi \circ \gamma_{u,F}]).$$

One can show [13] that this definition is independent of χ and Φ.

6.2. The Bishop Family: A Family of Pseudoholomorphic Disks Coming Out From A Singular Point

After defining the Maslov index of an embedded pseudoholomorphic disk, we can now state the existence theorem for so-called the Bishop family.

Theorem 6.2.1. *Let (M, λ) be a 3-dimensional overtwisted contact manifold and let $\mathcal{D} \subseteq M$ be an overtwisted disk with exactly one elliptic singular point e which lies in the interior of $\mathcal{D}$. Then, after a $\mathcal{C}^0$ small perturbation of $\mathcal{D}$ near e (keeping e fixed without introducing any other singular points) and a suitable choice of a compatible complex structure J on $ker\lambda$ compatible with $d\lambda$, there exists a smooth family of embedded disks*

$$\tilde{u}_\tau : D \longrightarrow \mathbb{R} \times M; \quad \tau \in (0,1)$$

with

$$\overline{\partial}_{\tilde{J}}\tilde{u}_\tau := \partial_s \tilde{u}_\tau + \tilde{J}(\tilde{u}_\tau)\partial_t \tilde{u}_\tau = 0 \text{ on } D$$

where $\tilde{J}$ is the almost complex structure on $\mathbb{R} \times M$ associated to J and

- *(i)* $\tilde{u}_\tau(\partial D) \subseteq \{0\} \times \mathcal{D} \setminus \{e\}$,
- *(i)* $\tilde{u}_\tau(D) \cap \tilde{u}_\rho(D) = \emptyset, \quad \text{if} \quad \tau \neq \rho$,
- *(i)* $\{e\} \cup \bigcup_\tau \tilde{u}_\tau(\partial D)$ *is a neighborhood of e in $\mathcal{D}$,*
- *(i)* $\mu_2(\tilde{u}_\tau) = 2$ *for all* $\tau \in (0,1)$.

Proof. By 2.1.2, we assume that $M = \mathbb{R}^3$, $\lambda = dz + xdy$ and $e = (0,0,0)$. Near the origin, we write $\mathcal{D}$ as a graph of a function f on $\mathbb{R}^2$ with $f(0) = 0$ and $df(0) = 0$. Using Theorem

2.3.2, after a $\mathcal{C}^0$ small perturbation of $\mathcal{D}$ near e, we assume that

$$\mathcal{D} = \{(x, y, -\frac{1}{2}xy) : (x, y) \in B(0, \epsilon)\}$$

for a small $\epsilon > 0$. We see that $\ker\lambda = \{(1, 0, 0), (0, 1, -x)\}$ and define J on $\ker\lambda$ by

$$J(1, 0, 0) = (0, 1, -x) \quad \text{and} \quad J(0, 1, -x) = (-1, 0, 0).$$

Then the corresponding almost complex structure $\tilde{J}$ on the symplectisation $\mathbb{R} \times M$ maps $\frac{\partial}{\partial a}$ to $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial a}$ to $-\frac{\partial}{\partial a}$. We see that a map

$$\tilde{u}(s, t) = (a, x, y, z)$$

satisfies the nonlinear Cauchy-Riemann Equation corresponding to $\tilde{J}$ if and only if it satisfies

$$\begin{aligned}
\partial_s a - \partial_t z - x\partial_s y &= 0 \\
\partial_s z + \partial_t a - x\partial_t x &= 0 \\
\partial_s x - \partial_t y &= 0 \\
\partial_s y - \partial_t x &= 0.
\end{aligned}$$

where $z = s + it$. Moreover, the boundary conditions that should be satisfied are the following;

$$\begin{aligned}
z(e^{i\theta}) &= -\frac{1}{2}x(e^{i\theta})y(e^{i\theta}) \\
a(e^{i\theta}) &= 0.
\end{aligned}$$

One can easily verify that

$$\tilde{u}_\tau(s,t) = \left(\frac{\tau^2}{4}(s^2+t^2-1), \tau s, \tau t, -\frac{\tau^2}{2} st\right); \quad \tau > 0, \quad s^2+t^2 \leq 1$$

solves the system of PDE with the boundary conditions. One can also see that these solutions satisfy the first three property in the conclusion of the theorem. Actually, we observe that this family of maps form a smooth embedding

$$\tilde{u} : (0,1) \times D \to \mathbb{R} \times M, \quad \text{where} \quad \tilde{u}(\tau, \cdot) = \tilde{u}_\tau(\cdot).$$

Now, we claim that $\mu_2(\tilde{u}_\tau) = 2$ for all $\tau \in (0,1)$. It is easy to see that $\{0\} \times \mathcal{D} \setminus \{e\}$ is a totally real submanifold of $\mathbb{R} \times M$. So $\mu_2(\tilde{u}_\tau)$ is meaningful. We observe that

$$\tilde{J}(\tilde{u}_\tau(s,t))(h, k_1, k_2, k_3) = (-k_3 - \tau s k_2, -k_2, k_1, h - \tau s k_1)$$

for $(s,t) \in D$ and $(h, k_1, k_2, k_3) \in T_{\tilde{u}_\tau(s,t)}\mathbb{R} \times M$. We define a map

$$\Phi : \tilde{u}_\tau^* T(\mathbb{R} \times M) \to (D \times \mathbb{R}^4), \quad \Phi(s,t)(h, k_1, k_2, k_3) := (-\tau s k_2 - k_3, k_1, k_2, h).$$

If we identify $\mathbb{R}^4$ with $\mathbb{C}^2$ by $(h, k_1, k_2, k_3) \mapsto (h + ik_3, k_1 + ik_2)$, we get

$$\Phi(s,t)\tilde{J}(\tilde{u}_\tau(s,t))\Phi(s,t)^{-1} = i.$$

Hence Φ gives a bundle isomorphism between the complex vector bundles

$$(\tilde{u}_\tau^* T(\mathbb{R} \times \mathbb{R}^3), \tilde{J}(\tilde{u}_\tau)) \text{ and } (D \times \mathbb{R}^4, \tilde{J}(\tilde{u}_\tau)) \approx (D \times \mathbb{C}^2, i).$$

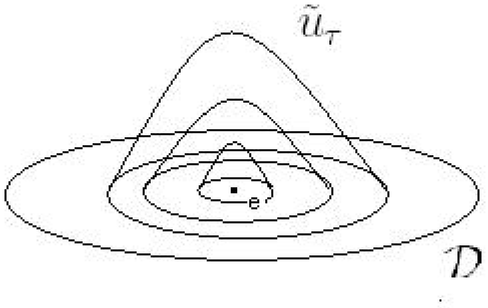

Figure 6.1. A Bishop family.

Therefore Φ is a complex trivialization of vector bundle $\tilde{u}_\tau^* T(\mathbb{R} \times \mathbb{R}^3)$. Now, for each $\tau \in (0,1)$ we should consider the loop $\gamma : [0, 2\pi] \to \urcorner(2)$ given by

$$\gamma(\theta) = \mathrm{pr} \circ \Phi\Big(e^{i\theta}, T_{\tilde{u}_\tau(e^{i\theta})}\big(\{0\} \times \mathcal{D}/\{0\}\big)\Big).$$

We pick a suitable linear map χ and see that $\chi \circ \gamma$ can be homotoped to the loop

$$\beta : [0, 2\pi] \to \urcorner(2)$$

$$\theta \mapsto e^{i\theta}\mathbb{R} \oplus \mathbb{R}$$

which has Maslov index 2. $\square$

6.3. The Implicit Function Theorem For The Nonlinear Cauchy-Riemann

Operator

In this section, we consider the nonlinear Cauchy-Riemann equation in the more general setting of an arbitrary almost complex 4-manifold. Without proving we will state an existence result for a family of solutions near an embedded solution and a global uniqueness result for such solutions . Later we will use these results to extend the Bishop family.

Let W be a 4-manifold equipped with an almost complex structure J. Moreover let $F \subset W$ be a totally real submanifold. We consider the nonlinear Cauchy–Riemann equation

$$\overline{\partial}_J u = u_s + J(u)u_t = 0 \text{ on } \mathring{D} \tag{6.1}$$

with boundary condition

$$u(\partial D) \subset F \tag{6.2}$$

where $D \subset \mathbb{C}$ is the closed unit disk.

Theorem 6.3.1. *Let u_0 be an embedded solution of (1) and (2) with Maslov index $\mu(u_0) = 2$. Then there exists a smooth embedding*

$$u : (-\epsilon, \epsilon) \times D \to W$$

such that

$$u(\tau)(\partial D) \subset F,$$
$$\overline{\partial}_J u(\tau) = 0 \textit{ for all } \tau \in (-\epsilon, \epsilon)$$
$$u(0)(z) = u_0(z) \textit{ for all } z \in D$$

where $u(\tau)(z) = u(\tau, z)$*. Moreover the associated disk family*

$$D_\tau = u(\tau)(D)$$

is unique up to biholomorphic parametrization of D *and smooth parametrization of* $(-\epsilon, \epsilon)$*.*

This result relies basically on the application of the Fredholm theory to the linearization of the nonlinear Cauchy–Riemann operator and it is out of the scope this study. However we will roughly summarize the main steps of the proof. A detailed proof can be seen in [13].

The first step is putting the setting into a normal form near the embedded solution u_0. It can be shown that a diffeomorphism from a neighborhood of $u_0(D)$ onto a neighborhood of $0 \in \mathbb{C}^2$ can be constructed such that, via this diffeomorphism we may take $W = \mathbb{C}^2$ locally and u_0 turns into $u_0(z) = (z, 0)$. Moreover, we may assume $J|_{D\times\{0\}} \equiv i$, and

$$F = \{(z, a)| \quad z \in \partial D; \quad a \in \mathbb{R}\}.$$

The second step is investigating the set $(\overline{\partial}_J)^{-1}(0)$ near the solution $u_0(z) = (z, 0)$. For this purpose, the nonlinear Cauchy–Riemann operator is viewed as a map between appropriate spaces. It can be shown that that for $2 < p < \infty$ and a sufficiently small neighborhood U of $D \times \{0\} = u_0(D)$

$$\overline{\partial}_J : \mathcal{B} \to L^p(D, \mathbb{C}^2)$$

is smooth where

$$\mathcal{B} = \{u \in W^{1,p}(D, \mathbb{C}^2) | \quad u(\partial D) \subset F; u(D) \subset U\}.$$

Here $\mathcal{B}$ is a Banach submanifold of $W^{1,p}(D, \mathbb{C}^2)$, which admits an atlas of one chart. A point in $\mathcal{B}$ can be parameterized as follows: one basically lets u_0 flow via the exponential map along "tangent functions" which constitute the space which is locally diffeomeorphic to $\mathcal{B}$. Next, the Fredholm theory can be utilized on the map

$$D\overline{\partial}_J(u_0) : T_{u_0}\mathcal{B} \to L^p(D, \mathbb{C}^2)$$

and it can proven that $\mu_2(u_0) = 2$ implies that $D\overline{\partial}_J(u_0)$ surjective and the kernel of $D\overline{\partial}_J(u_0)$ is a 4-dimensional subspace of $T_{u_0}\mathcal{B}$. Finally, by implicit function theorem one can show that $(\overline{\partial}_J)^{-1}(0)$ is a 4-dimensional submanifold of $\mathcal{B}$ which is diffeomorphic to a small ball in $\ker D\overline{\partial}_J(u_0)$. One can further show that near u_0, $(\overline{\partial}_J)^{-1}(0)$ is 1-dimensional and the remaining three dimensions correspond to the biholomorphic reparametrizations of the closed unit disk. This means that any element of $(\overline{\partial}_J)^{-1}(0)$, which is sufficiently close u_0 with respect to the norm of $W^{1,p}(D, \mathbb{C}^2)$, necessarily lies in the family given by the theorem (maybe after a reparametrization of D). This uniqueness result is much stronger then the statement of the theorem and the arguments in the next section will heavily lie on this uniqueness result.

6.4. The Maximal Family

Recall from the Chapter 5 that in order to show the existence of the periodic orbit it is enough to show that there exists a nontrivial finite energy plane in $\mathbb{R} \times M$. On the other hand, in the Chapter 7 we will see that the existence of a family of pseudoholomorphic disks with unbounded gradients gives rise to the nontrivial finite energy plane that we need. Now our aim is to establish the existence of a pseudoholomorphic disk family with unbounded gradients.

First we observe that gradients of pseudoholomorphic disks can be made arbitrarily large if the closed unit disk is suitably reparametrized. We know from complex analysis that conformal automorphisms of the closed unit disk D are of the form:

$$\phi(z) = e^{i\theta}\frac{z_0 - z}{1 - \overline{z_0}z}$$

where $\theta \in [0, 2\pi]$ and $z_0 \in \mathring{D}$. If we choose z_0 close to ∂D then we can make $||\nabla\phi||_{\mathcal{C}^0(D)}$ arbitrarily large. Since ϕ is biholomorphic, composing ϕ with a pseudoholomorphic map gives a pseudoholomorphic map with a gradient norm scaled by $||\nabla\phi||_{\mathcal{C}^0(D)}$. Hence by reparametrizing D we can construct a family of disks with unbounded gradients from a given family. But in Chapter 7, we will have to fix a special parametrization of D for each pseudoholomorphic disk of the family that will lead the existence of a finite energy plane. Therefore, we need a family of pseudoholomorphic disks whose gradient norms are unbounded independent of the parametrization we choose for D. We see that instead of the gradient norm the following quantity will be more convenient to use. We define

$$e(\tilde{u}_\tau) := \inf_{\phi\in \mathrm{Aut}(D)} ||\nabla(\tilde{u}_\tau \circ \phi)||_{\mathcal{C}^0(D)}$$

where $\mathrm{Aut}(D)$ stands for the set of all biholomorphic maps from D to itself. We know that for any $\tilde{u}_\tau$ we can find $\phi \in \mathrm{Aut}(D)$ such that

$$e(\tilde{u}_\tau) = ||\nabla(\tilde{u}_\tau \circ \phi)||_{\mathcal{C}^0(D)}$$

[9]. We say $\tilde{u}_\tau$ is *nicely parameterized* if $e(\tilde{u}_\tau) = ||\nabla\tilde{u}_\tau||_{\mathcal{C}^0(D)}$ already.

Remark 6.4.1. Now let us construct the disk family that we need. Let (M, λ) be a closed

3-manifold with an overtwisted contact structure kerλ. Let $\mathcal{D} \subset M$ be an overtwisted disk. By Theorem 2.4.3 we may assume that $\mathcal{D}$ has exactly one singularity $e \in \mathring{\mathcal{D}}$ which is elliptic. By Theorem 6.2.1 there exists an embedding

$$\tilde{u} : (0,1) \times D \to \mathbb{R} \times M, \quad \tilde{u}_\tau(\cdot) := \tilde{u}(\tau, \cdot)$$

with $\overline{\partial}_{\tilde{J}} \tilde{u}_\tau = 0$, $\tilde{u}_\tau(\partial D) \subset \{0\} \times \mathcal{D} \setminus \{e\}$ and $\mu_2(\tilde{u}_\tau) = 2$. We may assume that $(\tilde{u}_\tau)$ is nicely parameterized. We define

$$\delta := \inf_\tau \operatorname{dist}(\tilde{u}_\tau(\partial D), \partial \mathcal{D}).$$

Then we have two cases; either $\delta > 0$ or $\delta = 0$. First, we assume that $\delta > 0$. If $(e(\tilde{u}_\tau))$ is unbounded then we get the family we need. Otherwise, we pick a sequence $\tau_k \to 1$ and we have a sequence

$$\tilde{u}_{\tau_k} : D \to \mathbb{R} \times M$$

such that

$$\tilde{u}_{\tau_k}(\partial D) \subset \{0\} \times \mathcal{D} \setminus \{e\}, \text{ and } \inf_k \operatorname{dist}(\tilde{u}_{\tau_k}(\partial D), e) > 0,$$

since $(\tilde{u}_{\tau_k})$ comes from Theorem 6.2.1. Moreover we have

$$\sup_k ||\nabla \tilde{u}_{\tau_k}||_{\mathcal{C}^0(D)} < \infty.$$

Then by Theorem 4.0.6, we may assume that there exists a pseudoholomorphic map

$$\tilde{u}_1 : D \to \mathbb{R} \times M$$

such that $\tilde{u}_1(\partial D) \subset \{0\} \times \mathcal{D} \setminus \{e\}$ and

$$\tilde{u}_{\tau_k} \to \tilde{u}_1 \text{ in } \mathcal{C}^\infty(D).$$

Using the hypothesis on the winding number of $\tilde{u}_{\tau_k}|_{\partial D}$, and Corollary 3.3.2, we can show that $\tilde{u}_{\tau_k}$ and $\tilde{u}_1$ are embedding near the boundary. Then following [14], we define the selfintersection numbers $\mathrm{Int}(\tilde{u}_1),\ \mathrm{Int}(\tilde{u}_{\tau_k}) \in \mathbb{Z}$. We know that $\mathrm{Int}(\tilde{u}_1) = 0$ if and only if $\tilde{u}_1$ is an embedding. Hence we already have $\mathrm{Int}(\tilde{u}_{\tau_k}) = 0$. Then one can show that $\mathrm{Int}(\tilde{u}_1) = 0$ since $\tilde{u}_{\tau_k} \to \tilde{u}_1$ in $\mathcal{C}^\infty$. For the details see [9], [14].

Now, $\tilde{u}_1$ is an embedding with $\mu_2(\tilde{u}_1) = 2$. By Theorem 6.3.1, there exists a smooth embedding

$$\tilde{v} : (1-\epsilon, 1+\epsilon) \times D \to \mathbb{R} \times M, \quad \tilde{v}_\nu(\cdot) := \tilde{v}(\nu)(\cdot)$$

such that $\tilde{v}(1)(\cdot) = \tilde{u}_1(\cdot)$, $\overline{\partial}_{\tilde{J}} \tilde{v}_\nu = 0$ and $\tilde{v}_\nu(\partial D) \subset \{0\} \times \mathcal{D} \setminus \{e\}$ for all $\nu \in (1-\epsilon, 1+\epsilon)$. Now, since $\tilde{u}_{\tau_k} \to \tilde{u}_1$ in $\mathcal{C}^\infty(D, \mathbb{R}^N)$, by the arguments in the proof of Theorem 6.3.1 that are mentioned above we conclude that for a large k, $\tilde{u}_{\tau_k}$ lies in the 1-dimensional branch of solutions $(\tilde{v}_\nu)$, maybe after a reparameterization of its domain. But the solution family is also unique around $\tilde{u}_{\tau_k}$ therefore the families $\tilde{u}_\tau$ and $\tilde{v}_\nu$ fit together. Moreover $\tilde{v}$ extends $\tilde{u}$ and δ decreases since $\tilde{u}$ and $\tilde{v}$ are embedding.

This argument can be repeated unless we meet a family with unbounded gradients. So, if no family with unbounded gradients appears then δ can be improved until it is zero. Now assume that we reach a family $(\tilde{u}_\tau)_{\tau \in (0,\tau_0)}$ with

$$\inf_\tau \mathrm{dist}(\tilde{u}_\tau(\partial D), \partial\mathcal{D}) = 0.$$

Now if $(e(\tilde{u}_\tau))$ is still bounded, by an argument similar to the above we improve the family beyond $\tilde{u}_{\tau_0}$ which implies the existence of a pseudoholomorphic disk, say $\tilde{u}_{\tau_0}$, such that

$$\operatorname{dist}(\tilde{u}_{\tau_0}(\partial D), \partial\mathcal{D}) = 0$$

Since $\tilde{u}_{\tau_0}(\partial D)$ and $\partial\mathcal{D}$ are compact they must intersect and since $\tilde{u}_{\tau_0}(\partial D) \subset \mathcal{D}$ they must intersect tangentially. But this cannot happen since $\partial\mathcal{D}$ is a leaf of a characteristic foliation and by Corollary 3.3.2 we know that $\tilde{u}_{\tau_0}|_{\partial D}$ must be transverse to the leafs of the characteristic foliation. Therefore at some stage this chain of arguments break down so we conclude that there exists a family $\tilde{u}_\tau$ such that

$$\begin{gathered}
\sup_\tau e(\tilde{u}_\tau) = +\infty, \\
\tilde{u}_\tau(\partial D) \subset \{0\} \times \big(\mathcal{D} \setminus \{e\}\big), \\
\operatorname{dist}(\tilde{u}_\tau(\partial D), e) > 0.
\end{gathered}$$

7. UNBOUNDED GRADIENTS IMPLIES THE EXISTENCE OF A FINITE ENERGY PLANE

In this chapter we consider the case where gradients of a family of filling disks blow up. We first establish some energy estimates for a given family then we state and prove the main theorem of this chapter which says that unbounded gradients give rise to a non trivial finite energy plane.

Let (M, λ) be a 3-dimensional closed contact manifold and $\mathcal{D}$ be an overtwisted disk with an elliptic singularity $e \in \mathring{\mathcal{D}}$. In the previous chapter we chose a complex structure J on $\xi = \ker\lambda$ and we extended J to an almost complex structure $\tilde{J}$ on $\mathbb{R} \times M$ as in Chapter 3. Then we established the existence of a family of embedded $\tilde{J}$-holomorphic disks

$$\tilde{u}_\tau = (a_\tau, u_\tau) : D \to \mathbb{R} \times M, \quad \tau \in (0,1) \tag{7.1}$$

satisfying

$$\overline{\partial}_{\tilde{J}} \tilde{u}_\tau = 0 \text{ on } \mathring{D}, \tag{7.2}$$

$$u_\tau(\partial D) \subset \mathcal{D} \setminus \{e\}, \tag{7.3}$$

$$\inf\nolimits_\tau \operatorname{dist}(u_\tau(\partial D), e) > 0, \tag{7.4}$$

$$\sup\nolimits_\tau e(\tilde{u}_\tau) = +\infty. \tag{7.5}$$

Since $\tilde{u}_\tau$'s are embedding, we also know that the loop $u_\tau|_{\partial D}$ winds around e exactly once for each τ. Now we will first show that the energy of this family is bounded by a constant which depends only on $\mathcal{D}$ and λ.

Lemma 7.0.2. *Let $\tilde{u}_\tau$ be as above. Then there exists a constant $C > 0$, depending only on λ and $\mathcal{D}$, such that*

$$E(\tilde{u}_\tau) \leq C.$$

Proof. We know that if $\tilde{u}$ is nonconstant then by Corollary 3.3.2, $(u|_{\partial D})^*\lambda$ never vanishes. We also know that $u(\partial D) \subset \mathcal{D}$ is embedded and by the assumption on winding number it bounds a disk, say $\mathcal{D}_1$. Let $\varphi \in \Sigma$, by Stokes' Theorem we have

$$\begin{aligned} 0 \leq \int_D \tilde{u}^* d(\varphi\lambda) &= \int_{\partial D} \tilde{u}^*(\varphi\lambda) \\ &= \varphi(0) \int_{\partial D} u^*\lambda \\ &\leq \int_{\partial D} u^*\lambda \\ &= \int_{u|\partial D} \lambda \\ &\leq \int_{\mathcal{D}_1} |d\lambda| \\ &\leq \int_{\mathcal{D}} |d\lambda|. \end{aligned}$$

Hence

$$E(\tilde{u}) \leq \int_{\mathcal{D}} |d\lambda|.$$

□

Before we state and prove the main theorem of this chapter, which gives the existence of a finite energy plane, we have to overcome a technical problem. In the proof of the main theorem we will employ a bubbling–off argument again and we get the finite energy plane as a limit of a sequence of conformaly rescaled maps. The following proposition, which we will not prove, will establish the sufficient condition for the limit map to be defined on the whole plane.

Let $(\tilde{u}_\tau)_{\tau\in(0,1)}$ be a family of embedded pseudoholomorphic disks that satisfies the conditions (7.2) to (7.5) and remember that each $u_\tau|_{\partial D}$ winds exactly once around the elliptic singularity e. We parameterize the leaves of the characteristic foliation of $\mathcal{D}$ by $(l_\alpha)_{\alpha\in S^1}$. Now for each τ, we reparametrize D such that

$$u_\tau(1) \in l_1, \quad u_\tau(i) \in l_i, \quad u_\tau(-1) \in l_{-1}. \tag{7.6}$$

Then we have the following proposition, whose proof follows as in [13]:

Proposition 7.0.3. *Let $(\tilde{u}_\tau)_{\tau\in(0,1)}$ be a family of pseudoholomorphic disks as above. Assume also that they satisfy (7.6). If $\tau_k \to 1$ and $(z_k) \subset D$ are sequences with $R_k := |\nabla \tilde{u}_{\tau_k}(z_k)| \to \infty$, then $R_k dist(z_k, \partial D)$ is unbounded.*

Now we are ready to prove the main theorem of this chapter.

Theorem 7.0.4. *Assume that there exists a family of pseudoholomorphic disks $(\tilde{u}_\tau)_{\tau\in(0,1)}$ satisfying the conditions from (7.2) to (7.6). Then $\mathbb{R}\times M$ contains a nonconstant finite energy plane.*

Proof. We pick a sequence $(\tilde{u}_k)$ out of $(\tilde{u}_\tau)$ with $e(\tilde{u}_k) \to +\infty$. We already know that there exists a positive constants C such that

$$E(\tilde{u}_k) \leq C.$$

We pick a sequence $(z_k) \subset D$ with $R_k := |\nabla \tilde{u}_k(z_k)| \to +\infty$. Since D is compact we assume $z_k \to z_0$. We choose a sequence $\epsilon_k \searrow 0$ with $R_k\epsilon_k \to +\infty$ and $B(z_k, \epsilon_k) \subset D$. We may assume that

$$|\nabla \tilde{u}_k(z)| \leq 2R_k \text{ for } z \in B(z_k, \epsilon_k).$$

On $B(-R_k z_k, R_k)$, we again define rescaled maps

$$\tilde{v}_k(z) = (b_k, v_k) := \Big(a_k(\frac{z}{R_k} + z_k) - a_k(z_k), u_k(\frac{z}{R_k} + z_k)\Big).$$

Then we have $|\nabla \tilde{v}_k(0)| = 1$, $b_k(0) = 0$, and $|\nabla \tilde{v}_k(z)| \leq 2$ for $z \in \Omega_k := B(0, R_k\epsilon_k) \cap B(-R_k z_k, R_k)$. By the previous proposition we know that $R_k dist(z_k, \partial D) \to +\infty$ so we have $\bigcup_k \Omega_k = \mathbb{C}$.

As in the previous proofs, we can find a subsequence of $(\tilde{v}_k)$ that converges in $\mathcal{C}^\infty$ to a pseudoholomorphic plane

$$\tilde{v} : \mathbb{C} \to \mathbb{R} \times M.$$

$\tilde{v}$ is not constant since $|\nabla \tilde{v}(0)| = 1$ and given a compact set $K \subset \mathbb{C}$, for large k we have

$$sup_{\varphi \in \Sigma} \int_K \tilde{v}_k^* d(\varphi\lambda) \leq \sup_{\varphi \in \Sigma} \int_{B(-R_k z_k, R_k)} \tilde{v}_k^* d(\varphi\lambda) = \sup_{\varphi \in \Sigma} \int_D \tilde{u}_k^* d(\varphi\lambda) = E(\tilde{u}_k) \leq C.$$

We let $k \to +\infty$, then take supremum over all such K's and deduce that $\tilde{v}$ has finite energy. □

Notice that using $e(\tilde{u}_\tau)$ is necessary for the previous proof. Remember that we had to choose a specific parametrization of D for each $\tilde{u}_\tau$ in order to establish the condition (7.6). Therefore we needed a family whose gradients are unbounded independent of parametrization we choose for D and

$$\sup_{\tau} e(\tilde{u}_{\tau}) = \infty$$

is a sufficient condition for that.

APPENDIX A: Sobolev Spaces of Vector Valued Maps

Let $\Omega \subset \mathbb{R}^n$ be a connected open set, $m \in \mathbb{N}$ and $1 \leq p < \infty$. The definition and the basic properties of the Sobolev space $W^{m,p}(\Omega)$ are well known (see [17], [18]). Here we will give a generalization of $W^{m,p}(\Omega)$ for vector valued maps, namely the Sobolev space $W^{m,p}(\Omega, \mathbb{R}^k)$.

We define $W^{m,p}(\Omega, \mathbb{R}^k)$ to be the set of those $u : \Omega \to \mathbb{R}^k$, $u = (u_1, .., u_k)$ such that for all $i = 1, .., k$, $u_i \in L^p(\Omega)$ and for any multi index $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{N}^n$ with $|\alpha| := \sum_{i=1}^n \alpha_i \leq m$, the weak derivative $D^\alpha u_i$ exists and lies in $L^p(\Omega)$. We set

$$D^\alpha u = \frac{\partial^{|\alpha|} u}{\partial^{\alpha_1} x_1 ... \partial^{\alpha_n} x_n} := \big(\frac{\partial^{|\alpha|} u_1}{\partial^{\alpha_1} x_1 ... \partial^{\alpha_n} x_n}, ..., \frac{\partial^{|\alpha|} u_k}{\partial^{\alpha_1} x_1 ... \partial^{\alpha_n} x_n}\big) : \Omega \to \mathbb{R}^k$$

where $\frac{\partial^{|\alpha|} u_i}{\partial^{\alpha_1} x_1 ... \partial^{\alpha_n} x_n}$ stands for the weak derivative of u_i for each i. The norm on $W^{m,p}(\Omega, \mathbb{R}^k)$ is defined to be

$$||u||_{W^{m,p}(\Omega,\mathbb{R}^k)} := ||u||_{m,p,\Omega} := \left(\sum_{0 \leq |\alpha| \leq m} ||D^\alpha u||_{L^P(\Omega)} \right)^{1/p}$$

where

$$||D^\alpha u||_{L^P(\Omega,\mathbb{R}^k)} := \big(\int_\Omega |D^\alpha u(x)|^p dx \big)^{1/p}.$$

Here $|D^\alpha u(x)|$ stands for the Euclidean norm of the vector $D^\alpha u(x)$. It is easy to see that for any $u \in W^{m,p}(\Omega, \mathbb{R}^k)$, we have

$$||u_i||_{W^{m,p}(\Omega)} \leq ||u||_{W^{m,p}(\Omega,\mathbb{R}^k)} \tag{A.1}$$

for all i. Given $u \in W^{m,p}(\Omega, \mathbb{R}^k)$ and $|\alpha| \leq m$, there exists a constant $c > 0$, depending only on p and k, such that

$$|D^\alpha u(x)| = \left(\sum_{i=1}^{k} |D^\alpha u_i(x)|^2 \right)^{1/2} \leq c \left(\sum_{i=1}^{k} |D^\alpha u_i(x)|^p \right)^{1/p}$$

since there are equivalent norms of $\mathbb{R}^k$ on the both sides of the equation. Then we have

$$|D^\alpha u(x)|^p \leq c^p \sum_{i=1}^{k} |D^\alpha u_i(x)|^p$$

and therefore

$$\int_\Omega |D^\alpha u(x)|^p dx \leq c^p \sum_{i=1}^{k} \int_\Omega |D^\alpha u_i(x)|^p dx.$$

Summing up on α, we get

$$\sum_{0 \leq |\alpha| \leq m} \int_\Omega |D^\alpha u(x)|^p dx \leq c^p \sum_{i=1}^{k} \sum_{0 \leq |\alpha| \leq m} \int_\Omega |D^\alpha u_i(x)|^p dx,$$

that is,

$$||u||^p_{W^{m,p}(\Omega,\mathbb{R}^k)} \leq c^p \sum_{i=1}^{k} ||u||^p_{W^{m,p}(\Omega)}.$$

By Minkowski's inequality, we get

$$||u||_{W^{m,p}(\Omega,\mathbb{R}^k)} \leq c \sum_{i=1}^{k} ||u_i||_{W^{m,p}(\Omega)} \tag{A.2}$$

for a constant c depending only on p and k. Now using (A.1) and (A.2), one can generalize many properties of $W^{m,p}(\Omega)$ to $W^{m,p}(\Omega, \mathbb{R}^k)$. Here we will give the ones that we need . The first property is a generalization of the Banach algebra property of the Sobolev spaces.

Theorem A.0.5. *Let* $\Omega \subset \mathbb{R}^2$ *be a bounded domain,* $p > 2$*, and* $l \geq 2$*. Let* $A \in W^{l,p}(\Omega, \mathbb{R}^{k^2})$ *and* $u \in W^{l,p}(\Omega, \mathbb{R}^k)$*. We define* $Au : \Omega \to \mathbb{R}^k$ *by* $(Au)_i := \sum a_{ij} u_j$ *where* $A(x) = (a_{ij}(x)) \in GL(\mathbb{R}^k) \approx \mathbb{R}^{k^2}$ *for all* $x \in \Omega$*. Then there exists a constant* $c > 0$*, independent of* A *and* u*, such that*

$$||Au||_{W^{l,p}(\Omega,\mathbb{R}^k)} \leq c||A||_{W^{l,p}(\Omega,\mathbb{R}^{k^2})}||u||_{W^{l,p}(\Omega,\mathbb{R}^k)}.$$

This theorem easily follows from the Banach algebra property of $W^{l,p}(\Omega)$ applied to the maps a_{ij} and u_j and the observations (A.1) and (A.2). One can see from [18] that the dimension of Ω and the conditions on l and p are given so that the Banach algebra property can be applied.

Now we will generalize an imbedding theorem for our purposes.

Theorem A.0.6. *Let* $\Omega \subset \mathbb{R}^2$ *be a bounded domain with Lipschitz boundary,* $2 < p < \infty$*. Then the imbedding*

$$W^{l+1,p}(\Omega, \mathbb{R}^k) \hookrightarrow \mathcal{C}^l(\overline{\Omega}, \mathbb{R}^k)$$

is compact.

Here $\mathcal{C}^l(\overline{\Omega}, \mathbb{R}^k)$ stands for the Banach space of all functions $u = (u_1, .., u_k)$ for which $D^\alpha u_j$ is bounded and uniformly continuous on Ω for all j and $|\alpha| \leq m$, i.e., it possesses a unique, bounded, continuous extension to $\overline{\Omega}$. $\mathcal{C}^k(\overline{\Omega}, \mathbb{R}^n)$ is equipped with norm given by

$$||u||_{\mathcal{C}^l(\overline{\Omega},\mathbb{R}^k)} := \max_{0\leq|\alpha|\leq m} \sup_{x\in\Omega} |D^\alpha u(x)|$$

where $|D^\alpha u(x)|$ length of the vector $D^\alpha u(x)$. We easily deduce that for any $u \in \mathcal{C}^l(\overline{\Omega}, \mathbb{R}^k)$,

$$||u_i||_{\mathcal{C}^l(\overline{\Omega})} \leq ||u||_{\mathcal{C}^l(\overline{\Omega},\mathbb{R}^k)} \text{ for all } i, \tag{A.3}$$

and

$$||u||_{\mathcal{C}^l(\overline{\Omega},\mathbb{R}^k)} \leq \sum_{i=1}^{k} ||u_i||_{\mathcal{C}^l(\overline{\Omega})} \tag{A.4}$$

Then $u \in \mathcal{C}^l(\overline{\Omega}, \mathbb{R}^k)$ if and only if $u_i \in \mathcal{C}^l(\overline{\Omega})$ for all i.

Using (A.3) and (A.4), one can easily prove Theorem A.0.6 by applying the Rellich-Kondrachov theorem [18] to the components of u successively.

The last result that we need is a technical result about the regularity of the composition of two maps.

Theorem A.0.7. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain with Lipschitz boundary. Let $u \in W^{m,p}(\Omega, \mathbb{R}^l)$ with $mp > n$, and $f \in \mathcal{C}^\infty(\mathbb{R}^l, \mathbb{R}^k)$ with bounded derivatives up to order m. Then $f \circ u \in W^{m,p}(\Omega, \mathbb{R}^k)$ and there is a constant $c > 0$ such that*

$$||f \circ u - f(0)||_{m,p,\Omega} \leq c||u||_{m,p,\Omega}$$

for all $u \in W^{m,p}(\Omega, \mathbb{R}^l)$.

A detailed proof this theorem can be found in [13]. If we get into the proof this theorem we observe that if the derivatives of f are bounded on a set containing the image of u then the theorem still works. We should keep this fact in mind since we will apply this theorem in such cases.

APPENDIX B: A Technical Result on Metric Spaces

Here we state a technical lemma whose proof follows as in [13]. This lemma plays a crucial role in some "bubbling off" arguments that appear in the Chapters 5 and 7.

Lemma B.0.8. *Let* (X, d) *be a complete metric space. Let* $f : X \to [0, +\infty)$. *Then given* $x_0 \in X$ *and* $\epsilon_0 > 0$ *there exist* $\epsilon \in (0, \epsilon_0]$ *and* $x \in X$ *with*

(i) $\epsilon f(x) \geq \epsilon_0 f(x_0)$,

(ii) $d(x, x_0) \leq 2\epsilon_0$,

(iii) $f(y) \leq 2f(x)$ *for all* $y \in B(x, \epsilon)$.

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