FIRST STEPS INTO HEEGARD FLOER HOMOLOGY

by

Semih Özlem

B.S., Mathematics, Boğaziçi University, 2006

Submitted to the Institute for Graduate Studies in
Science and Engineering in partial fulfillment of
the requirements for the degree of
Master of Science

Graduate Program in Mathematics
Boğaziçi University
2008

FIRST STEPS INTO HEEGARD FLOER HOMOLOGY

APPROVED BY:

Asst. Prof. Ferit Öztürk (Thesis Supervisor)	....
Asst. Prof. Burak Gürel	....
Assoc. Prof. A. Muhammed Uludağ	....

DATE OF APPROVAL: Day.10.2008

ABSTRACT

FIRST STEPS INTO HEEGARD FLOER HOMOLOGY

Heegard Floer homology is a topological invariant for closed 3-manifolds equipped with a Spin^c-structure. Construction of Heegard Floer homology in the case when first Betti number is 0 is explained. The tools required in the construction are pseudo holomorphic disks, symmetric product space, Chern class, Maslov index, and spin^c-structures. These tools are studied.

TABLE OF CONTENTS

1. INTRODUCTION

Heegard Floer homology has been developed by Zoltán Szabó, and Peter Ozsváth in [1] as a topological invariant for closed oriented 3-manifolds . It is built on Gromov's theory of pseudo-holomorphic disks, and is inspired by Lagrangian Floer homology. Lagrangian Floer homology is defined for a symplectic manifold and a pair of Lagrangian submanifolds. It is generated by intersection points of the Lagrangian submanifolds. Its differential counts pseudoholomorphic disks. Heegard Floer homology follows this construction. The construction of the spaces playing the roles of the symplectic manifold and the pair of Lagrangian submanifolds is based on a Heegard decomposition of the 3-manifold. It turns out that the count of pseudoholomorphic disks in the symplectic manifold reveals the topology of the 3-manifold.

Heegard Floer homology can be placed in the larger context by noting that it comes after Donaldson polynomials which count the solutions of certains PDEs, and Seiberg-Witten invariants which are related to solutions of Seiberg-Witten equations. Heegard Floer homology can be seen as another step in attempts to relate count of solutions of certain differential equations with some boundary conditions to topological characteristics of some underlying space.

Having roughly placed Heegard Floer homology in a historical context, we now explicitly give the construction of Heegard Floer homology for a closed orientable 3-manifold equipped with a spin^c-structure.

Let Y be a connected, closed, oriented 3-manifold, equipped with a spin^c-structure s. (See Chapter 4 for definition of spin^c-structures.) Assume that the first Betti number $b_1(Y) = 0$, for simplicity. Let the union of sets $U_0 \cup_\Sigma U_1$ denote a Heegard decomposition of Y into diffeomorphic handlebodies U_0 and U_1 glued along their boundary Σ. Let the number g be the genus of Σ. Let the 4-tuple $(\Sigma, \alpha, \beta, z)$ denote a *pointed Heegard diagram* representing this Heegard decomposition. Here $\alpha = (\alpha_1, \ldots, \alpha_g)$ and $\beta = (\beta_1, \ldots, \beta_g)$ are g-tuples of closed curves in Σ. These curves are chosen so that the α

curves intersect the β curves transversally. The point z is any point on the surface Σ away from α and β curves.

From a Morse theoretic standpoint, α curves can be considered as the intersection of Σ with the ascending submanifolds of critical points of index 1, and β curves can be considered as the intersection of Σ with the descending submanifolds of critical points of index 2. The number of critical points of index 1 and the number of critical points of index 2 both equal the genus number g. Apart from these there is one critical point of index 0, and one critical point of index 3.

Let $\mathrm{Sym}^g(\Sigma)$ be the symmetric g-fold product of Σ, which is naturally equipped with a complex structure. Let $\mathbb{T}_\alpha = \alpha_1 \times \ldots \times \alpha_g$ and $\mathbb{T}_\beta = \beta_1 \times \ldots \times \beta_g$ be two g-tori in symmetric space $\mathrm{Sym}^g(\Sigma)$. In construction of Heegard Floer homology, the symmetric space $\mathrm{Sym}^g(\Sigma)$ plays the role of the symplectic manifold in Lagrangian Floer theory. The totally real subspaces $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$ play the role of Lagrangian submanifolds of Lagrangian Floer theory.

Intersection points $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ determine g trajectories in Y, by considering the gradient flow of a Morse function on Y that is compatible with the Heegard decomposition. Let $\gamma_{\mathbf{x}}$ denote the corresponding 1-chain in Y. Each point $z \in \Sigma$ that lies outside α and β curves also determines a trajectory connecting the critical point of index 0 to critical point of index 3. Let γ_z denote this trajectory.

Let $\mathbb{D}$ be the unit disk in $\mathbb{C}$. Let e_1 denote the *left arc* of the boundary of the disk, and e_2 denote the *right arc*. For points $\mathbf{x}, \mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, let $\pi_2(\mathbf{x}, \mathbf{y})$ denote the homotopy classes of maps $u : \mathbb{D} \to \mathrm{Sym}^g(\Sigma)$ satisfying the boundary conditions

$$u(-i) = \mathbf{x}, \quad u(i) = \mathbf{y};$$
$$u(e_1) \subset \mathbb{T}_\alpha, \quad u(e_2) \subset \mathbb{T}_\beta.$$

Let $\mathbf{x}, \mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$. If there exists a pseudo-holomorphic disc connecting the point

$\mathbf{x}$ to the point $\mathbf{y}$, then the 1-cycle $\gamma_x - \gamma_y$ is trivial in $H_1(Y, \mathbb{Z})$. This is a consequence of the identification $H_1(\mathrm{Sym}^g(\Sigma)) \cong H_1(\Sigma)$. (see Proposition 2.4.)

On the other hand, spin^c structures also come into play. Spin^c structures can be thought as equivalence classes of unit vector fields on Y, where two vector fields are equivalent if they agree outside a 3-ball in Y. Given a point $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, outside the trajectories $\gamma_{\mathbf{x}}$ and γ_z, gradient of a Morse function determines a spin^c-structure, which will be called $s_z(\mathbf{x})$. It can be shown that if there is a pseudo holomorphic disc u connecting $\mathbf{x}$ and $\mathbf{y}$, then $s_z(\mathbf{x}) = s_z(\mathbf{y})$. (see Proposition 4.2.)

As can be seen from the discussion above, existence of pseudo holomorphic discs between two points in the intersection of the totally real submanifolds $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$ are related both to the topology of Y and spin^c-structure on Y. This observation is one of the key aspects lying at the foundation of Heegard Floer homology.

A detailed analysis of pseudo holomorphic disks involves Fredholm theory, as in Gromov theory. This analysis provides the link mentioned before to investigating number of solutions of a partial differential equation. Holomorphic representatives of homotopy classes in $\pi_2(\mathbf{x}, \mathbf{y})$ are actually solutions of a generalized Cauchy Riemann equation. Fredholm theory is used to study the space of such solutions. Note that writing a differential equation to impose being holomorphic requires choice of a complex structure on Σ, which induces a complex structure on $\mathrm{Sym}^g(\Sigma)$.

Concerning pseudo holomorphic disks, two quantities are needed to define Heegard Floer homology. One of them is the Maslov index μ. (See Chapter 3). It measures the dimension of the moduli space of holomorphic representatives of a pseudo holomorphic disc. The other is the algebraic intersection number n_z of a pseudo holomorphic disc with the submanifold $z \times \mathrm{Sym}^{g-1}(\Sigma)$ of $\mathrm{Sym}^g(\Sigma)$.

Let ϕ be an element of $\pi_2(\mathbf{x}, \mathbf{y})$, whose Maslov index equals 1. Let $\mathcal{M}(\phi)$ denote the set of the holomorphic representatives of the pseudo holomorphic disc ϕ. Identify the infinite strip $\{z : \mathrm{Re}(z) \in [0, 1]\}$ with the unit disc $\mathbb{D}$. Translation by imaginary numbers in the infinite strip allows one to define an $\mathbb{R}$-action on $\mathcal{M}(\phi)$. Let $\hat{\mathcal{M}}(\phi)$ denote the quotient space $\mathcal{M}(\phi)/\mathbb{R}$. This space is called the moduli space of unparametrized holomorphic discs. Using Fredholm theory it can be shown that this space is a compact, zero-dimensional manifold [1].

It can also be shown that there is only one pseudo holomorphic disc $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$ satisfying the conditions $\mu(\phi) = 1$ and $n_z(\phi) = 0$.

Let $k(\mathbf{x}, \mathbf{y})$ denote the number of points in the space $\hat{\mathcal{M}}(\phi)$ when the set $pi_2(\mathbf{x}, \mathbf{y})$ is nonempty.

The chain complex used in defining Heegard Floer homology is constructed as follows: Let s be a spin^c-structure on Y. Let $\widehat{CF}(\alpha, \beta, s)$ be the abelian group generated by

$$\Upsilon = \{\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta : s_z(\mathbf{x}) = s\}.$$

A relative grading is introduced on this abelian group by

$$\mathrm{gr}(\mathbf{x}, \mathbf{y}) = \mu(\phi) - 2n_z(\phi),$$

where ϕ is any element of $\pi_2(\mathbf{x}, \mathbf{y})$. This is well-defined because of the effect of attaching a topological sphere to the Maslov index.

The boundary map for defining Heegard Floer homology is defined by

$$\partial \mathbf{x} = \sum_{\{\mathbf{y} \in \Upsilon : \mathrm{gr}(\mathbf{x}, \mathbf{y}) = 1\}} k(\mathbf{x}, \mathbf{y}) \mathbf{y}.$$

Note that this forces us to choose a pseudo holomorphic disc $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$ satisfying $n_z(\phi) = 0$.

In counting the number of points of the moduli space orientation must also be taken into account.

The abelian group $\widehat{CF}(\alpha, \beta, s)$, together with the boundary map above creates a chain complex. The homology group arising from this complex, denoted by $\widehat{HF}(\alpha, \beta, s)$ is called the Heegard Floer homology.

In order to show that this is a topological invariant for 3-manifolds equipped with a spin^c-structure, one must show that the arising group is independent of the Heegard decomposition, complex structure chosen on Σ. This is carried out in [1]. Two heegard decomposition of the same manifold can be obtained from each other through Heegard moves, in a manner similar in taste to Reidemeister moves in knot theory. It is shown in [1] that resulting Heegard Floer homologies remain invariant under these Heegard moves.

We will now look at Heegard Floer homology for a simple but nontrivial example outlined in [2]. Because the issue of orientation has been left out, we will use $\mathbb{Z}_2$-coeffiecients. Consider the genus 1 splitting of S^3 into two solid tori. A Heegard diagram for this splitting is given by a genus 1 surface $\Sigma_1 = S^1 \times S^1$, and the curves $\alpha = x \times S^1$ and $\beta = S^1 \times x$. In this case the symmetric space $\mathrm{Sym}^1(\Sigma)$ is nothing other than Σ_1. The totally real tori are $\mathbb{T}_\alpha = \alpha$ and $\mathbb{T}_\beta = \beta$. They intersect at a single point $\mathbf{x} = (x, x) \in S^1 \times S^1$. Therefore the complex $\widehat{CF}(\Sigma_1, \alpha, \beta)$ has a single generator, $\mathbf{x}$. Hence there are no differentials.

Some key remarks that could indicate the importance of Heegard Floer homology are the following. Heegard Floer homology is conjecturally equivalent to Seiberg-Witten-Floer homology [3]. Secondly, using Heegard Floer homology, one can find certain invariants associated with a contact structure on a 3-manifold, which give some information on whether the contact structure is tight or overtwisted.

In this thesis we have explored some topological issues, concepts and tools that go into the construction of Heegard Floer homology. We have not looked at Fredholm theory, and analytical aspects of pseudo holomorphic discs. We have not explored why Heegard Floer homology remains invariant under Heegard moves. This thesis is aimed as a list of some advanced exercises in low dimensional topology.

As already outlined in the construction of Heegard Floer homology above, the space that we are working on is the symmetric space $\mathrm{Sym}^g(\Sigma)$. We have explored the topology of this space in Chapter 2. In particular we showed that its first homology group is isomorphic to its fundamental group and also to the first homology group of Σ in proposition 2.4. Using this result, we were able to make the observation that existence of a pseudo holomorphic disc between two intersection points of the tori $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$, implied that the 1-cycle determined by the trajectories of the intersection points was trivial in the 3-manifold Y. Proposition 2.4 was also needed in the computation of the second homotopy group of $\mathrm{Sym}^g(\Sigma)$, which is done in Section 2.2.

Proposition 2.6, which calculates the homotopy group $\pi^2(\mathrm{Sym}^g(\Sigma))$, required understanding cohomology ring of $\mathrm{Sym}^g(\Sigma)$. Structure of this cohomology ring was also needed for computation of Chern class. Cohomology ring of $\mathrm{Sym}^g(\Sigma)$ is studied in detail in Section 2.3.

On the other hand, to understand pseudo holomorphic discs, one had to understand what went on when a topological sphere was attached to a pseudo holomorphic disc. In particular to show that the relative grading defined on $\widehat{CF}$ is well-defined, one had to prove a property of Maslov index of a pseudo holomorphic disc when a topological sphere was attached to it. This property is stated in Theorem 3.28. Proving this theorem, and construction of Heegard Floer homology required defining Maslov index of a pseudo holomorphic disc. This is covered in Chapter 3. Proving Theorem 3.28 also required computation of the first Chern class of $\mathrm{Sym}^g(\Sigma)$, which is Corollary 2.10. Computation of Chern class relied heavily on cohomology ring of $\mathrm{Sym}^g(\Sigma)$ as mentioned.

Last, we began investigating Spin^c-structures in Chapter 4. The map s_z is needed in defining generators of the abelian group $\widehat{CF}$ which provides the chain complex for

Heegard Floer homology. The construction of the map s_z is carried in detail in this chapter. We also prove in Proposition 4.2, a property of s_z. This property allows us to conclude that existence of a pseudo holomorphic disk between two intersection points of the totally real tori implies that the map s_z carries these two points to the same spin^c-structure.

2. TOPOLOGY OF $\mathrm{Sym}^g(\Sigma)$

In this chapter computations regarding topology of the symmetric space $\mathrm{Sym}^g(\Sigma)$ are carried out. First we review definition of symmetric spaces. The first result is needed for the observation mentioned in the introduction regarding pseudo holomorphic disks. In order to do this we first review symmetric spaces.

Let X be any topological space. Then for any positive integer n, the symmetric group on n letters, S_n, acts on $X^n = X\times\ldots\times X(n \text{ times})$ by permuting the coordinates:

$$\begin{array}{ccc} S_n \times X^n & \to & X^n \\ (\sigma,(x_1,\ldots,x_n)) & \mapsto & (x_{\sigma(1)},\ldots,x_{\sigma(1)}) \end{array}$$

For each $\sigma \in S_n$, the action of σ on X^n is a homeomorphism.

$\mathrm{Sym}^n(X)$ is defined as the quotient space of X^n under this group action. In other words, elements of $\mathrm{Sym}^n(X)$ are orbits of elements of X^n under the group action. Intuitively points of $\mathrm{Sym}^n(X)$ are simply an unordered collection of n points on X, allowing a point to be taken into a collection more than once. Let $p : X^n \to \mathrm{Sym}^n(X)$ be the projection map, so that $p(x_1,\ldots,x_n) = \{(x_{\sigma(1)},\ldots,x_{\sigma(n)}) : \sigma \in S_n\}$.

When X is a complex 1-manifold, $\mathrm{Sym}^n(X)$ is also a complex manifold of dimension n. This can be seen as follows: Let $(z_1,\ldots,z_n)$ denote coordinates on X^n. Let $(w_1,\ldots,w_n)$ be the coefficients of the monic polynomial whose roots are $z_1,\ldots,z_n$. This means that the vector $(1,w_1,\ldots,w_n)^T$ lies in the kernel of the matrix $\begin{pmatrix} z_1^n & z_1^{n-1} & \ldots & 1 \\ & \vdots & & \\ z_n^n & z_n^{n-1} & \ldots & 1 \end{pmatrix}$

Since this matrix has rank n when z_i's are distinct, its kernel is 1-dimensional. Hence w_i's are determined uniquely from this linear equation. On the other hand, one can choose z_i's to be distinct by choosing a different but equivalent chart on X.

The n-tuple $(w_1, \ldots, w_n)$ is invariant under changing the order of the z_i's. Therefore $(w_1, \ldots, w_n)$ provides coordinates for $\mathrm{Sym}^n(X)$.

A submanifold of $\mathrm{Sym}^n(X)$ that will be involved in the proofs of following propositions is the diagonal $D = \{p(x_1, \ldots, x_n) : x_i = x_j \text{ for some } i \neq j\}$. It consists of points $p(x_1, \ldots, x_n)$ in $Sym^n(X)$, for which at least two of the points of X represented by this element of the symmetric space coincide.

Another tool that is used in the following proofs is the notion of a branched cover. This is defined as follows:

Definition 2.1. *A map $f : X \to Y$ is called a* branched cover *of Y branched along $K \subset X$ if this is a covering map outside K. In other words the map $\tilde{f} : X - K \to Y - f(K)$, which is defined by $\tilde{f}(x) = f(x)$, is a covering map.*

The following lemmas are useful in proof of propositions in this chapter, in particular proposition 2.4.

Lemma 2.2. *Any element of $H^1(Sym^g(\Sigma))$ has a representative that does not intersect the diagonal D.*

Proof. Representatives of elements of the first homology can be chosen as unions of embedded circles. On the other hand, D has codimension 2 in $\mathrm{Sym}^g(\Sigma)$. From differential topology it is known that if a submanifold X has dimension less than the codimension of another submanifold Y, then X can be pulled away from Y by an arbitrarily small deformation. But small deformations do not change the homology class. □

Lemma 2.3. *A surface with boundary in $Sym^g(\Sigma)$ can be chosen transverse to D without changing the boundary.*

Proof. This follows from the same argument as above, by dimension arguments in differential topology. It is a consequence of the principle that transversality is generic. □

2.1. First Homology and Homotopy Groups of $\mathrm{Sym}^g(\Sigma)$

In this section we prove the equivalence of first homology and homotopy groups of $\mathrm{Sym}^g(\Sigma)$ to the first homology group of Σ.

Proposition 2.4. *Let Σ be a surface of genus g. Then*

$$\pi_1(Sym^g(\Sigma)) \cong H_1(Sym^g(\Sigma)) \cong H_1(\Sigma).$$

Proof. Fix $g-1$ distinct points $x_2, \ldots, x_g \in \Sigma$. Let $\iota : \Sigma \hookrightarrow \mathrm{Sym}^g(\Sigma)$ be the inclusion map defined by $\iota(y) = p(y, x_2, \ldots, x_g)$. Then $\iota_* : H_1(\Sigma) \to H_1(\mathrm{Sym}^g(\Sigma))$ is a group homomorphism.

Let the homology class $[\alpha]$ be a generator of the first homology group $H_1(\mathrm{Sym}^g\Sigma)$. Let the map $\alpha : S^1 \to \mathrm{Sym}^g(\Sigma)$ be a representative of the homology class $[\alpha]$. By Lemma 2.2, we can choose α so that it is away from the diagonal. The projection map $p : \Sigma^g \to \mathrm{Sym}^g(\Sigma)$ is a branched cover branched along the diagonal D. Hence α lifts to Σ^g.

Let the map $p_1 : \Sigma^g \to \Sigma$ be the projection on the first coordinate. The set $p_1(p^{-1}(\alpha(S^1)))$ is a 1-cycle in Σ. Call this 1-cycle $\widehat{\alpha}$.

Now suppose that α and α' are homologous in $\mathrm{Sym}^g(\Sigma)$. Then the 1-cycle $\alpha - \alpha'$ bounds a surface Z in $\mathrm{Sym}^g(\Sigma)$. Such a surface could be chosen to intersect the diagonal D transversely in finitely many points, by Lemma 2.3. The subset $\tilde{Z} = p_1(p^{-1}(Z))$ of Σ is a two cycle and a branched g-fold cover of Z. Furthermore boundary of $\tilde{Z}$ is going to be the 1-cycle $\widehat{\alpha - \alpha'}$.

Therefore we have defined a map $j : H_1(\mathrm{Sym}^g\Sigma) \to H_1(\Sigma)$. From the way we defined these maps it follows that $j \circ \iota_* = id$. Hence j is onto.

On the other hand, suppose $j[\alpha] = 0$. Then there is a map, $i : F \to \Sigma$, from a

2-manifold with boundary F into Σ, such that $i|_{\partial F} = \widehat{\alpha}$. By increasing the genus of F, one can find a branched g-fold branched covering of the disk $\pi : F \to D$. Then $\pi^{-1}(z)$ for each $z \in D$ describes g points on F which then describe g points on Σ, hence a point of $\mathrm{Sym}^g(\Sigma)$. Hence we obtain a map from the disk into $\mathrm{Sym}^g(\Sigma)$. This both shows that α is null-homologous, concluding isomorphism of $H_1(\mathrm{Sym}^g(\Sigma))$ and $H_1(\Sigma)$. It also shows that null-homologous curves are contractible, so that $\pi_1(\mathrm{Sym}^g(\Sigma))$ is abelian, hence isomorphic to $H_1(\mathrm{Sym}^g(\Sigma))$. □

As a result of Proposition 2.4 we can now relate first homology group of $\mathrm{Sym}^g(\Sigma)$ to the first homology group of Y. Recall from the introduction that the relation described in the corollary below allows us to see that if there is a pseudo holomorphic disc joining the points $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$, then the the difference of the trajectories $\gamma_{\mathbf{x}} - \gamma_{\mathbf{y}}$ is null homologous in Y.

Corollary 2.5.

$$\frac{H_1(Sym^g(\Sigma))}{H_1(\mathbb{T}_\alpha) \oplus H_1(\mathbb{T}_\beta)} \cong \frac{H_1(\Sigma)}{[\alpha_1], \ldots, [\alpha_g], [\beta_1], \ldots, [\beta_g]} \cong H_1(Y, \mathbb{Z})$$

Proof. The first isomorphism follows from Proposition 2.4, and definition of the tori $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$. The second isomorphism comes from the definition of Heegard decomposition of Y. Since α and β curves are attaching sites for 2-handles, they are null-homologous in Y. □

2.2. Second Homotopy Group of $\mathrm{Sym}^g(\Sigma)$

In this section we compute the second homotopy group of $\mathrm{Sym}^g(\Sigma)$.

For any connected topological space X and a fixed base-point $x_0 \in X$, the first fundamental group $\pi_1(X, x_0)$ acts on the other fundamental groups $\pi_n(X, x_0)$ by shrinking the domain of a representative $f : (I^n, \partial I^n) \to (X, x_0)$ of an element of $\pi_n(X, x_0)$ to a smaller n-cube inside I^n and adjoining any fixed loop representing the element of $\pi_1(X, x_0)$ acting, to the radial rays from the boundary of the inside cube to the boundary of I^n. Let $\pi'_n(X)$ denote the quotient of $\pi_n(X, x_0)$ by the action of $\pi_1(X, x_0)$. Note that this quotient is independent of the choice of the basepoint x_0. When X is connected, different basepoint choices give isomorphic groups.

Now we give the description of second homotopy group of $\mathrm{Sym}^g(\Sigma)$. Let η be the poincare dual to the submanifold $I_x = x \times \mathrm{Sym}^{g-1}(\Sigma)$. Let $\xi_1, \ldots, \xi_g, \xi'_1, \ldots, \xi'_g$ be the generators of the cohomology group $H^1(\mathrm{Sym}^g; \mathbb{Z})$ described in Section 2.3.

Proposition 2.6. *Let Σ be a genus g surface. Then*

$$\pi'_2(Sym^g(\Sigma)) \cong \mathbb{Z}.$$

Furthermore there is a generator S of $\pi'_2(Sym^g(\Sigma))$ whose image under the Hurewicz homomorphism is Poincaré dual to

$$(1-g)\eta^{g-1} + \sum_{i=1}^{g} \xi_i \xi'_i \eta^{g-2}.$$

When $g > 2$, the homotopy group $\pi_1(Sym^g(\Sigma))$ acts trivially on $\pi_2(Sym^g(\Sigma))$ and thus

$$\pi_2(Sym^g(\Sigma)) \cong \mathbb{Z}.$$

Proof. Let $\kappa_x : \pi'_2(\mathrm{Sym}^g(\Sigma)) \to \mathbb{Z}$ be the homomorphism given by the intersection number with the submanifold I_x for generic point $x \in \Sigma$. Hyperelliptic involution $\iota : \Sigma \to \Sigma$ gives rise to a sphere $S_0 \subset \mathrm{Sym}^2\Sigma$. Let $S = S_0 \times x_3 \times \ldots \times x_g \subset \mathrm{Sym}^g\Sigma$, where $x_3, \ldots, x_g$ are arbitrary fixed points of Σ. Since $S \cap I_x = \{x, \iota(x), x_3, \ldots x_g\}$, it follows that $\kappa_x(S) = 1$.

If $Z \in \ker(\kappa_x)$, then one can find a sphere representing the class Z that meets I_x transversally in finitely number of points. We can insert spheres homotopic to S or $-S$ at these intersection points without changing the homotopy class of Z, to obtain a new sphere Z', that does not intersect I_x at all. In other words, $Z' \subset \mathrm{Sym}^g(\Sigma - x)$.

An argument using a theorem in [4] shows that $\pi_2(\mathrm{Sym}^g(\Sigma - x)) = 0$ for $g > 2$. The space $\Sigma - x$ is homotopy equivalent to the wedge of $2g$ circles, which is equivalent to the space $\mathbb{C} - \{z_1, \ldots, z_{2g}\}$ for $2g$ points. On the other hand, the space $\mathrm{Sym}^g(\mathbb{C} - \{z_1, \ldots, z_{2g}\})$ is in one-to-one correspondence with monic polynomials p of degree g satisfying $p(z_i) \neq 0$. Considering the coefficients of polynomials as elements of $\mathbb{C}^g$, the condition $p(z_i) \neq 0$, can be seen as excluding $2g$ hyperplanes from C^g. A theorem in [4] states that homology groups of the universal covering space of $\mathbb{C}^g$ minus $2g$ generic hyperplanes is trivial except in dimension zero or g. This theorem together with Hurewicz theorem shows that $\pi_2'(\mathrm{Sym}^g(\Sigma - x) = 0$. Consequently the map κ_x is injective. Thus we have the first equality of the proposition.

Let σ be the top homology class of $\mathrm{Sym}^g(\Sigma)$. Then

$$\begin{aligned}(PD(S) \cup \eta)(\sigma) &= \eta(\sigma \cap PD(S)) \\ &= \eta(S) \\ &= PD(I_x)(S).\end{aligned}$$

Since S intersects I_x only at one point, the above calculation implies that $PD[S] \cup \eta = PD[1]$. On the other hand from study of cohomology ring $H^*(\mathrm{Sym}^g(\Sigma))$ in Section 2.3, we know that ξ_i and ξ_i' come from elements that reside on all components. However S resides on only two components of Σ. Therefore $PD[S] \cup \xi_j \cup \xi_k' = 0$.

These two equations characterize $PD[S]$, because $\xi_i, \xi_i' \in H^1(\mathrm{Sym}^g(\Sigma))$ for $1 \leq i \leq g$ and $\eta \in H^2(\mathrm{Sym}^g(\Sigma))$ are generators of the cohomology ring $H^*(\mathrm{Sym}^g(\Sigma))$, as explained in Section 2.3.

To see that $(1-g)\eta^{g-1} + \sum_{i=1}^{g} \xi_i\xi_i'\eta^{g-2}$ is indeed $PD[S]$, we utilize conclusions that are discussed in the Section 2.3, specifically Proposition 2.7. We first verify the first identity that $PD(S)$ should verify.

$$[(1-g)\eta^{g-1} + \sum_{i=1}^{g} \xi_i\xi_i'\eta^{g-2}] \cup \eta = \eta^g + \eta^{g-1}\sum_{i=1}^{g}(\xi_i\xi_i' - \eta)$$

But

$$\eta^{g-1}\sum_{i=1}^{g}(\xi_i\xi_i' - \eta) = 0$$

by the mentioned proposition. The cohomology class η^g equals the top cohomology class $PD(1)$.

The second identity follows from Proposition 2.7 as well.

$$[(1-g)\eta^{g-1} + \sum_{i=1}^{g}\xi_i\xi_i'\eta^{g-2}] \cup \xi_j \cup \xi_k' = 0$$

By Proposition 2.4, any element of the homotopy group $\pi_1(\mathrm{Sym}^g(\Sigma))$ has a representative $\gamma : S^1 \to \mathrm{Sym}^g(\Sigma)$ of the form

$$\gamma = \gamma_1 \times x_2, \ldots, x_g$$

where γ_1 is a map from S^1 into Σ.

According to the calculation above of $\pi_2'(\mathrm{Sym}^g(\Sigma))$, it can be arranged that an element of $\pi_2{}'(\mathrm{Sym}^g(\Sigma))$ is represented by a map $\sigma : S^2 \to \mathrm{Sym}^g(\Sigma)$ of the form

$$\sigma = x_1 \times \sigma_1$$

where $\sigma_1 : S^2 \to \mathrm{Sym}^{g-1}(\Sigma)$.

Using these maps, we define

$$\gamma_1 \times \sigma_1 : S^1 \times S^2 \to \mathrm{Sym}^g(\Sigma).$$

Since the action of $\pi_1(S^1 \times S^2)$ on $\pi_2(S^1 \times S^2)$ is trivial, we reach the conclusion.

The case $g = 2$ is settled in [1].

$\square$

2.3. Cohomology Ring $H^*(\mathbf{Sym}^g(\Sigma))$

The computation of cohomology ring of $\mathrm{Sym}^g(\Sigma)$ is carried out in this section. These computations were needed both in the proofs in previous section. They are needed for the computation of the first Chern class of $\mathrm{Sym}^g(\Sigma)$ as well. For a compact connected Riemann surface Σ of genus g, it is easy to compute that

$$H^0(\Sigma, \mathbb{Z}) \cong \mathbb{Z}, \qquad H^1(\Sigma, \mathbb{Z}) \cong \mathbb{Z}^{2g}, \qquad H^2(\Sigma, \mathbb{Z}) \cong \mathbb{Z}.$$

Let β be the generator of $H^2(\Sigma, Z)$ induced by the orientation on Σ. Choose $2g$ generators of $H^1(\Sigma, Z)$, $\alpha_1, \ldots, \alpha_{2g}$ satisfying the relations

$$\alpha_i\alpha_j = 0 \text{ for } |i - j| \neq g; \qquad \alpha_i\alpha_{i+g} = -\alpha_{i+g}\alpha_i = \beta \qquad \text{for } 1 \leq i \leq g$$

where multiplication is the cup product.

From Künneth formula it follows that cohomology ring of Σ^n is given by nth tensor power of cohomology ring of Σ. First this is considered as cohomology ring with coefficients in a field. However the result about generators of the cohomology ring of $\mathrm{Sym}^n(\Sigma)$ is valid for cohomology with coefficients in $\mathbb{Z}$, since it turns out that $H^*(\mathrm{Sym}^n(\Sigma), Z)$ is torsion-free as shown in [5].

Let K be a field. By Künneth formula, $H^*(\Sigma^n, K) \cong H^*(\Sigma, K)^{\otimes n}$. Let

$$\alpha_{ik} = 1 \otimes \ldots \otimes 1 \otimes \alpha_i \otimes 1 \otimes \ldots \otimes 1 \in H^1(\Sigma^n, K)$$

$$\beta_k = 1 \otimes \ldots \otimes 1 \otimes \beta \otimes 1 \otimes \ldots \otimes 1 \in H^2(\Sigma^n, K)$$

where α_i and β are in the k-th position. The ring $H^*(\Sigma^n, K)$ is generated over K by α_{ik} and β_k for $1 \leq i \leq 2g, 1 \leq k \leq n$ with the relations

$$\begin{aligned} \alpha_{ik}\alpha_{jk} &= 0 \text{ for } |i-j| \neq g \\ \alpha_{ik}\alpha_{i+g,k} &= -\alpha_{i+g,k}\alpha_{ik} = \beta_k \qquad \text{for } 1 \leq i \leq g \end{aligned}$$

inherited from $H^*(\Sigma, K)$, and the additional anti-commutativity relation:

$$\alpha_{ik}\alpha_{jl} = -\alpha_{jl}\alpha_{ik}$$

which comes from the cup product.

Monomials of the form $\mu = \pm\alpha_{i_1k_1}\ldots\alpha_{i_pk_p}\beta_{l_1}\ldots\beta_{l_q}$ where $1 \leq k_1 < \ldots < k_p \leq n$ and $1 \leq l_1 < \ldots < l_1 \leq n$ with each k_i different than each l_j and $p + 2q = r$ form a basis of the K-vector space $H^r(\Sigma^n, K)$.

The symmetric group S_n acts on Σ^n by permuting coordinates, hence also acts on $H^*(\Sigma^n, K)$ as follows. Let $\sigma \in S_n$. Then $\sigma(\alpha_{ij})$ is defined to be equal to $\alpha_{i\sigma(j)}$, and $\sigma(\beta_k)$ is defined to be equal to $\beta_{\sigma(k)}$.

The set of elements of $H^*(\Sigma^n, K)$ that are invariant under the action of S_n constitute a subring denoted by $H^*(\Sigma^n, K)^{S_n}$, which is isomorphic to $H^*(\mathrm{Sym}^n(\Sigma), K)$ as shown in [5]. The ring $H^*(\Sigma^n, K)^{S_n}$ is generated by $\xi_1, \ldots, \xi_{2g}$ and η where

$$\xi_i = \alpha_{i1} + \ldots + \alpha_{in},$$

$$\eta = \beta_1 + \ldots + \beta_n.$$

From the definition of these elements and action of S_n, it follows that elements of the form above are left invariant under the action of S_n. To see that they generate $H^*(\Sigma^n, K)^{S_n}$, note that any element $\omega \in H^*(\Sigma^n, K)^{S_n}$ satisfies the equality

$$n!\omega = \sum_{\sigma \in S_n} \sigma\omega.$$

On the other hand ω is a sum of monomials of the form $\mu = \pm\alpha_{i_1k_1} \dots \alpha_{i_pk_p}\beta_{l_1} \dots \beta_{l_q}$ as mentioned before. Note that

$$\xi_{i_1} \dots \xi_{i_p}\eta^q = \sum_{j_1,\dots,j_p,k_1,\dots,k_q=1,\dots,n} \alpha_{i_1j_1} \dots \alpha_{i_pj_p}\beta_{k_1} \dots \beta_{k_q}.$$

Note that in this sum terms for which j's and k's are not distinct are zero. Since specifying $p+q$ indices determines a permutation of S_n up to a permutation of the remaining $n-p-q$ indices, the above sum is equal to $\frac{1}{(n-p-q)!}\sum_{\sigma \in S_n} \sigma\mu$ for any fixed μ with the same number of α and β terms. Consequently ω, can be written as a polynomial in ξ_i's and η with coefficients in K.

It follows from anti-commutativity of α's that $\xi_i\xi_j = -\xi_j\xi_i$ and that $\xi_i\eta = \eta\xi_i$.

For convenience introduce the notation $\xi'_i = \xi_{i+g}$ for $1 \le i \le g$.

The trivial elements of $H^*(\mathrm{Sym}^n(\Sigma))$ are characterized in the following proposition.

Proposition 2.7. *If $i_1, \dots, i_a, j_1, \dots, j_b, k_1, \dots, k_c$ are distinct integers between 1 and g inclusive,*

$$\xi_{i_1} \dots \xi_{i_a}\xi'_{j_1} \dots \xi'_{j_b}(\xi_{k_1}\xi'_{k_1} - \eta) \dots (\xi_{k_c}\xi'_{k_c} - \eta)\eta^q = 0$$

provided that

$$a + b + 2c + q = n + 1.$$

Proof. The terms in the proposition can be written in terms of generators of the ring $H^*(\Sigma^n, K)$, as a sum of terms of the form

$$(\alpha_{i_1p_1}\dots\alpha_{i_ap_a})(\alpha_{j_1q_1}\dots\alpha_{j_bq_b})(\alpha_{k_1r_1}\alpha'_{k_1s_1}\dots\alpha_{k_cr_c}\alpha'_{k_cs_c})\beta_{t_1}\dots\beta_{t_q}$$

because of the way ξ_i and η are defined. Call these terms χ-terms.

First note the following computation:

$$\xi_\nu\xi'_\nu - \eta = \sum_{k,l=1}^{n}\alpha_{\nu k}\alpha'_{\nu l} - \sum_{m=1}^{n}\beta_m = \sum_{m=1}^{n}(\alpha_{\nu m}\alpha'_{\nu m} - \beta_m) + \sum_{k,l=1,\dots,n;k\neq l}\alpha_{\nu k}\alpha'_{\nu l}.$$

Also recall that $\beta_m = \alpha_{\nu m}\alpha'_{\nu m}$ for any ν. For the two reasons just mentioned, in a nonzero χ-term, it must be true that $r_i \neq s_i$ for any i.

On the other hand, indices i_*, j_*, k_* are distinct. We also have the following relations:

$$\alpha_{dk}\alpha_{ek} = \alpha'_{dk}\alpha'_{ek} = 0, \text{ for all } d, e, \tag{2.1}$$

and

$$\alpha_{dk}\alpha'_{ek} = 0, \text{ for } d \neq e. \tag{2.2}$$

Equations 2.1 and 2.2, imply that indices p, q, r, s must all be distinct, and that

$$\alpha_{dk}\beta_k = 0$$

for all d. Then t indices must all be different from p, q, r, s indices. But there are in total $a + b + 2c + q = n + 1$ indices selected from $\{1, \dots, n\}$. Hence there must be at least one pair of indices that are equal to each other. Hence all χ-terms are zero. □

2.3.1. First Chern Class of Sym$^g(\Sigma)$

Theorem 2.8. *Total Chern Class of* $Sym^n(\Sigma)$ *is*

$$(1+\eta)^{n-2g+1}\prod_{i=1}^{g}(1+\eta-\xi_i\xi_i')$$

.

We postpone the proof until after the following corollaries.

Corollary 2.9. *First Chern class of* $Sym^n(\Sigma)$ *is given by:*

$$c_1(Sym^n(\Sigma)) = (n-g+1)\eta - \sum_{i=1}^{g}\xi_i\xi_i'.$$

Proof. Reading the term of second grade of the product gives the formula. □

Corollary 2.10. *The first Chern class of* $Sym^g(\Sigma)$ *is given by*

$$c_1(Sym^g(\Sigma)) = \eta - \sum_{i=1}^{g}\mu(\alpha_i)\mu(\alpha_i').$$

Hence we have,

$$\langle c_1, S\rangle = 1$$

where S *is the generator of* $\pi_2(Sym^g(\Sigma))$.

Proof. The first Chern class comes simply from setting $n=g$ in Corollary 2.9. The second part follows from proposition 2.6. □

We will need the following lemmas proved in [5] in the computation of total Chern class.

Lemma 2.11. *Let X be a complex manifold, V a holomorphic vector bundle on X with fibre C^q, $\bar{X}$ the projective bundle on X associated with V, W line bundle on $\bar{X}$ of vectors lying in the line of the projective space element in $\bar{X}$. Let η be the element in $H^2(\bar{X}, Z)$ such that $1+\eta$ is the Chern class of W^*, dual of W. Then*

$$H^*(\bar{X}, Z) = H^*(X, Z)[\eta].$$

Let J be the jacobian corresponding to a riemann surface Σ. It is shown in [5] that the cohomology ring $H^*(J, \mathbb{Z})$ is modelled after the same exterior algebra E defining cohomology of $\mathrm{Sym}^n(\Sigma)$. The element s_i of the exterior algebra E is mapped to $\xi_i x i_i'$.

Lemma 2.12. *Tangent bundle of jacobian J is trivial, so that $c(J) = 1$*

Lemma 2.13. *Let X be a complex manifold, V a holomorphic vector bundle on X with fibre C^q, $\bar{X}$ the projective bundle on X associated with V. If $c(V) = \prod_{i=1}^{q}(1+\gamma_i)$ and $c(X) = \prod_{j=1}^{n}(1+\tau_j)$, then*

$$c(\bar{X}) = \prod_{i=1}^{q}(1+\eta+\gamma_i)\prod_{j=1}^{n}(1+\tau_j)$$

Proof. Let X be a complex manifold, V a holomorphic vector bundle on X with fibre C^q, $\bar{X}$ the projective bundle on X associated with V. If $c(V) = \prod_{i=1}^{q}(1+\gamma_i)$ and $c(X) = \prod_{j=1}^{n}(1+\tau_j)$, then

$$c(\bar{X}) = \prod_{i=1}^{q}(1+\eta+\gamma_i)\prod_{j=1}^{n}(1+\tau_j)$$

Let $\psi : \bar{X} \to X$ be the projection, and $\psi^{-1}(V)$ the pullback bundle on $\bar{X}$.

Let W be the line bundle on $\bar{X}$ formed by $((x, l), v)$ where $l \in P(V_x)$, i.e. in projective space of the vector space fiber over $x \in X$, and $v \in l$. Let W' denote the quotient bundle $\psi^{-1}(V)/W$. We have an exact sequence of vector bundles over $\bar{X}$:

$$0 \to W \to \psi^{-1}(V) \to W' \to 0$$

We get another exact sequence

$$0 \to Hom(W, W') \to T\bar{X} \to \psi^{-1}(TX) \to 0.$$

Hence we compute the total Chern class as

$$c(T\bar{X}) = c(Hom(W, W'))c(\psi^{-1}(TX)) = c(W^* \otimes W')\psi^* c(TX).$$

Taking tensor product with W^* in the first exact sequence gives the exact sequence:

$$0 \to 1 \to W^* \bigotimes \psi^{-1}(V) \to W^* \bigotimes W' \to 0$$

by utilizing that $1 = \bar{X} \times \mathbb{C} = Hom(W, W) = W^* \otimes W$. Hence from this exact sequence we get that $c(W^* \otimes \psi^{-1}(V)) = c(1)c(W^* \otimes W') = c(W^* \otimes W')$. Then it follows that

$$c(T\bar{X}) = c(W^* \otimes \psi^{-1}(V))\psi^* c(TX).$$

Hence by multiplicative property of Chern classes the result follows. □

Lemma 2.14. *If $n \geq 2g$, then there is a canonically defined vector bundle V of rank $n - g + 1$ on the Jacobian J such that $Sym^n(\Sigma)$ is the projective bundle associated with V and the Chern class of V is*

$$\prod_{i=1}^{g} 1 - \theta(s_i)$$

Proof of Theorem 2.8

Proof. Case 1: $n \geq 2g$. Since $rankV = n - g + 1$, rewrite Chern class of V as a product of $n - g + 1$ terms so that lemma 2.11 could be applied:

$$c(V) = \prod_{i=1}^{n-g+1} 1 + \gamma_i$$

where $\gamma_i = -\theta(s_i)$ for $1 \leq i \leq g$ and $\gamma_i = 0$ for

$$g + 1 \leq i \leq n - g + 1.$$

Since tangent bundle of J is trivial, applying Lemma 2.14 gives

$$c(\mathrm{Sym}^g(\Sigma)) = \prod_{i=1}^{n-g+1} (1 + \eta + \gamma_i) = (1 + \eta)^{n-2g+1} \prod_{i=1}^{g} (1 + \eta - \sigma_i),$$

where σ_i is image of s_i to be in the right space.

Case 2: $n < 2g$. Let $j : \mathrm{Sym}^{n-1}(\Sigma) \to \mathrm{Sym}^n(\Sigma)$ be the inclusion map. Let T_n be the tangent bundle of $\mathrm{Sym}^n(\Sigma)$. Let $j^{-1}(T_n)$ be the pullback bundle on $\mathrm{Sym}^{n-1}(\Sigma)$. T_{n-1} is a subbundle of T_n, via j. Let L be the quotient line bundle of T_n by T_{n-1} over $\mathrm{Sym}^n(\Sigma)$. Then $j^{-1}(L)$ is a line bundle over $\mathrm{Sym}^{n-1}(\Sigma)$. Furthermore we have an exact sequence of vector bundles over $\mathrm{Sym}^{n-1}(\Sigma)$:

$$0 \to T_{n-1} \to j^{-1}(T_n) \to j^{-1}(L) \to 0$$

so that

$$j^* c(T_n) = c(j^{-1}(T_n) = c(T_{n-1}) c(j^{-1}(L))$$

Since Chern class of L is η, and $j^*\eta = \eta$, we obtain that

$$j^*c(T_n) = (1+\eta)c(T_{n-1}).$$

Reading the above equation with $n = 2g$, and applying recursively gives the desired result. $\square$

3. MASLOV INDEX

In this chapter, we define the Maslov index. We needed the definition of Maslov index of a pseudoholomorphic disc for defining Heegard Floer homology. After the preliminaries, first we define Maslov index for a path of Lagrangian subspaces following [6]. Then the necessary modifications to define the Maslov index of a pseudoholomorphic disc are explained. This work is to reach Theorem 3.28 at the end of the chapter. Recall that this result is needed for showing well-definedness of the relative grading gr on the abelian group $\widehat{CF}$.

Definition 3.1. *Let ω be the* standard symplectic structure *on $\mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n$ defined by*

$$\omega(z_1, z_2) = \langle x_1, y_2 \rangle - \langle x_2, y_1 \rangle$$

for $z_k = (x_k, y_k) \in \mathbb{R}^n \times \mathbb{R}^n$, and $\langle \quad , \quad \rangle$ the standard inner product on $\mathbb{R}^n$.

Definition 3.2. *A subspace Λ of $\mathbb{R}^{2n}$, equipped with the standard symplectic structure ω, is called* Lagrangian *iff*

(i) dimension of Λ is n,

(ii) $\omega|_{\Lambda \times \Lambda} = 0$, i.e. $\omega(z_1, z_2) = 0$ for all $z_1, z_2 \in \Lambda$.

Definition 3.3. *A* Lagrangian frame *for a Lagrangian subspace Λ is a linear map $Z : \mathbb{R}^n \to \mathbb{R}^{2n}$ such whose image is Λ.*

Lemma 3.4. *A linear map $Z : \mathbb{R}^n \to \mathbb{R}^{2n}$ is a Lagrangian frame if and only if*

(i) Z is injective

(ii) Z is of the form

$$Z = \begin{pmatrix} X \\ Y \end{pmatrix}$$

where X, Y are $n \times n$ matrices with real entries satisfying

$$Y^T X = X^T Y$$

Proof. Clearly a linear map $Z : \mathbb{R}^n \to \mathbb{R}^{2n}$ is injective if and only if dimension of its image is n.

Any linear map Z from $\mathbb{R}^n$ into $\mathbb{R}^{2n}$ is of the form

$$Z = \begin{pmatrix} X \\ Y \end{pmatrix}$$

with $X, Y \in M_{n \times n}(\mathbb{R})$. To see that these matrices satisfy the stated equality if and only if $\omega|_{\Lambda \times \Lambda} = 0$ where Λ is the image of Z, simply note that

$$\begin{aligned} \omega(Ze_i, Ze_j) &= = \langle Xe_i, Ye_j \rangle - \langle Xe_j, Ye_i \rangle \\ &= (X^T Y - Y^T X)_{ij} \end{aligned}$$

since $Ze_i = (Xe_i, Ye_i)$ where e_i denotes the standard i-th basis vector of $\mathbb{R}^n$, and $(M)_{ij}$ is the entry in the i-th row and j-th column of a matrix M. □

Lemma 3.5. *The graph*

$$Gr(A) = \{(x, Ax) : x \in \mathbb{R}^n\}$$

of $A \in M_{n \times n}(\mathbb{R})$ is Lagrangian if and only if A is symmetric.

Proof. Note that $Gr(A)$ is the image of $Z = \begin{pmatrix} Id \\ A \end{pmatrix}$ where Id is the identity matrix in $M_{n \times n}(\mathbb{R})$. Hence the map Z is injective. So by Lemma 3.4, the map Z is a lagrangian frame if and only if A is symmetric. □

Definition 3.6. *If $\mathbb{R}^{2n} = V \oplus W$, this is called a* Lagrangian Splitting *iff both V and*

W are Lagrangian subspaces of R^{2n} with the standard symplectic structure. W is called a Lagrangian complement *of V.*

Remark 3.7. By proposition 8.2 in [7], if $\mathbb{R}^{2n} = V \oplus W$ where V is Lagrangian, then one can always find a Lagrangian splitting of the form $V \oplus W'$.

Being a Lagrangian complement is equivalent to being a Lagrangian subspace that is transverse to the original Lagrangian subspace.

Lemma 3.8. *Any Lagrangian complement U of $0 \times \mathbb{R}^n$ is the graph*

$$\{(x, Ax) : x \in \mathbb{R}^n\}$$

of a symmetric matrix $A \in M_{n\times n}(\mathbb{R})$. Furthermore if $Z = (X, Y)$ is a Lagrangian frame of U, then X is invertible and $A = YX^{-1}$.

Proof. For $i = 1, 2$, let $p_i : U \to \mathbb{R}^n$ be the projection maps using the standard Lagrangian splitting $\mathbb{R}^{2n} = \mathbb{R}^n \times 0 \oplus 0 \times \mathbb{R}^n$, so that for all $u \in U$,

$$u = (p_1(u), p_2(u))$$

If $p_1(u) = 0$, then $u \in 0 \times \mathbb{R}^n$ as well. Since U is a complement of $0 \times \mathbb{R}^n$, u must be zero as well. Then p_1 has rank n, hence is onto, and is a linear isomorphism. Then let $A = p_2 \circ p_1^{-1}$. It is clear that U is the graph of A. On the other hand, by Lemma 3.5, A is symmetric. By what we have just shown, range of X must cover $\mathbb{R}^n$. Thus X is of full rank, hence invertible. On the other hand $AX = Y$. □

Let $\mathcal{L}(n)$ denote the set of Lagrangian subspaces of $\mathbb{R}^{2n}$ with respect to the standard symplectic structure. $\mathcal{L}(n)$ is a differentiable manifold. Its smooth structure can be understood by Lagrangian frames.

Theorem 3.9. *Let $\Lambda(t)$ be a curve in $\mathcal{L}(n)$ with $\Lambda(0) = \Lambda$ and $\dot{\Lambda}(0) = \hat{\Lambda}$.*

(i) Let W be a fixed Lagrangian complement of Λ and for $v \in \Lambda$ and small t define $w(t) \in W$ so that $v + w(t) \in \Lambda(t)$. Then the form

$$Q(\Lambda, \hat{\Lambda})(v) = Q(v) = \left.\frac{d}{dt}\right|_{t=0} \omega(v, w(t))$$

is independent of choice of W.

(ii) If $Z(t) = (X(t), Y(t))$ is a frame for $\Lambda(t)$, then

$$Q(v) = \langle X(0)u, \dot{Y}(0)u\rangle - \langle Y(0)u, \dot{X}(0)u\rangle$$

where $v = Z(0)u$.

Proof. (i) Since $\mathbb{R}^{2n} = \Lambda \oplus W$, for each t, every element of $\Lambda(t)$ can be written as a sum of elements from Λ and W.

The shifted subspace $v + W$ intersects $\Lambda(0)$ at least at v. The subspace W intersects Λ at 0 transversally hence must intersect $v + W$ transversally. Since transversal intersection is a stable property, for small t, $v + W$ intersects $\Lambda(t)$ transversally and we can find a continuous or even smooth section $w(t)$ of that intersection in that interval.

Choose basis so that $\Lambda(0) = \mathbb{R}^n \times 0$. Then any Lagrangian complement of $\Lambda(0)$ is the graph of a symmetric matrix $B \in M_{n\times n}(\mathbb{R})$ given by

$$W = \{(By, y) : y \in \mathbb{R}^n\}$$

by Lemma 3.8.

On the other hand, Λ is a Lagrangian complement of $0 \times \mathbb{R}^n$, so that for small t, so is $\Lambda(t)$. Hence by Lemma 3.8, $\Lambda(t)$ is the graph of a symmetric matrix $A(t) \in M_{n\times n}(\mathbb{R})$:

$$\Lambda(t) = \{(x, A(t)x) : x \in (R)^n\}$$

Now $v=(x,0)$, $w(t)=(By(t),y(t))$. The condition $v+w(t)\in\Lambda(t)$ implies that

$$A(t)(x+By(t))=y(t)$$

Since $\omega(v,w(t))=\langle x,y(t)\rangle$,

$$\left.\frac{d}{dt}\right|_{t=0}\omega(v,w(t))=\langle x,\dot{y}(0)\rangle$$

But

$$\dot{y}(0)=\dot{A}(0)(x+By(0))+A(0)(x+B\dot{y}(0)=\dot{A}(0)x$$

since $y(0)=0$ and $A(0)=0$. Then $\left.\frac{d}{dt}\right|_{t=0}\omega(v,w(t))$ is independent of B.

(ii) This time assume that $W=0\times\mathbb{R}^n$. Choose frame $Z(t)=(X(t),Y(t))$ for $\Lambda(t)$. Since $0\times\mathbb{R}^n$ is Lagrangian complement of $\Lambda(0)$ this is also true for small t. Hence by Lemma 3.8, $X(t)$ is invertible. Then

$$\begin{aligned} v &= (X(0)u,Y(0)u) \\ w(t) &= (0,y(t)) \end{aligned}$$

The condition that $v+w(t)\in\Lambda(t)$ implies that

$$Y(0)u+y(t)=Y(t)X(t)^{-1}X(0)u$$

Since

$$\omega(v,w(t))=\langle X(0)u,y(t)\rangle$$

and

$$\begin{aligned} \dot{y}(0) &= \dot{Y}(0)X(0)^{-1}X(0)u+Y(0)(-X(0)^{-1}\dot{X}(0)X(0)^{-1})X(0)u \\ &= \dot{Y}(0)u-Y(0)X(0)^{-1}\dot{X}(0)u \end{aligned}$$

the following holds:

$$\begin{aligned}
Q(v) &= \langle X(0)u, \dot{y}(0)\rangle \\
&= \langle X(0)u, \dot{Y}(0)u\rangle - \langle X(0)u, Y(0)X(0)^{-1}\dot{X}(0)u\rangle \\
&= \langle X(0)u, \dot{Y}(0)u\rangle - \langle u, X(0)^{T}Y(0)X(0)^{-1}\dot{X}(0)u\rangle \\
&= \langle X(0)u, \dot{Y}(0)u\rangle - \langle u, Y(0)^{T}X(0)X(0)^{-1}\dot{X}(0)u\rangle \\
&= \langle X(0)u, \dot{Y}(0)u\rangle - \langle u, Y(0)^{T}\dot{X}(0)u\rangle \\
&= \langle X(0)u, \dot{Y}(0)u\rangle - \langle Y(0)u, \dot{X}(0)u\rangle
\end{aligned}$$

(iii) Since $\omega(\Psi v, \Psi w) = \omega(v, w)$ for any symplectic matrix Ψ, this property follows from the definition.

□

Remark 3.10. The form Q described in the theorem above is dependent on Λ and $\hat{\Lambda}$ only.

Let $L_k(V)$ denote the submanifold of $\mathcal{L}(n)$ consisting of Lagrangian subspaces which intersect V in a subspace of dimension k. In this fashion, one obtains a decomposition of Lagrangian subspaces into a disjoint union

$$\mathcal{L}(n) = \cup_{k=0}^{n} L_k(V)$$

Definition 3.11. *The* Maslov cycle *determined by a Lagrangian subspace V is the set*

$$L(V) = \cup_{k=1}^{n} L_k(V).$$

For the following let $\Lambda : [a, b] \to \mathcal{L}(n)$ be a smooth curve of Lagrangian subspaces.

Definition 3.12. *A* crossing *for Λ is a number $t \in [a, b]$ for which $\Lambda(t) \in L(V)$, i.e. $\Lambda(t)$ intersects V nontrivially.*

Let $\chi(\Lambda, V)$ denote the set of crossings. This set is compact, since the condition for being a crossing is a closed condition.

Definition 3.13. *At each crossing* $t \in [a, b]$ *the* crossing form *is defined by*

$$\Gamma(\Lambda, V, t) = Q(\Lambda(t), \dot{\Lambda}(t))|_{\Lambda(t) \cap V}$$

Definition 3.14. *A crossing* t *is* regular *if the crossing form at* t *is nonsingular.*

Definition 3.15. *A crossing* t *is* simple *if it is regular and* $\Lambda(t) \in L_1(V)$*.*

Definition 3.16. *For a curve* $\Lambda : [a, b] \to \mathcal{L}(n)$ *with only regular crossings, its* Maslov index *is defined to be:*

$$\mu(\Lambda, V) = \frac{1}{2} sign \Gamma(\Lambda, V, a) + \sum_{t \in (a,b) \text{ and } t \in \chi(\Lambda, V)} sign \Gamma(\Lambda, V, t) + \frac{1}{2} sign \Gamma(\Lambda, V, b)$$

Remark 3.17. Regular crossings are isolated. So the sum in the definition above is finite.

The following theorem is proven in [6].

Theorem 3.18. *(i) Every Lagrangian path is homotopic with fixed endpoints to one having only regular crossings.*

(ii) For each symplectic matrix Ψ*,*

$$\mu(\Psi\Lambda, \Psi V) = \mu(\Lambda, V)$$

(iii) ***(Catenation)*** *For* $c \in (a, b)$*,*

$$\mu(\Lambda, V) = \mu(\Lambda|_{[a,c]}, V) + \mu(\Lambda|_{[c,b]}, V)$$

(iv) ***(Product)*** *If* $n' + n'' = n$*, and* $\mathcal{L}(n') \times \mathcal{L}(n'')$ *is identified as a submanifold of*

$\mathcal{L}(n)$, then

$$\mu(\Lambda' \oplus \Lambda'', V' \oplus V'') = \mu(\Lambda', V') + \mu(\Lambda'', V'')$$

(v) ***(Localization)*** *If $V = \mathbb{R}^n \times 0$ and $\Lambda(t) = Gr(A(t))$ then*

$$\mu(\Lambda, V) = \frac{1}{2}(signA(b) - signA(a))$$

(vi) Two paths Λ_0, Λ_1 with the same endpoints are homotopic with fixed endpoints if and only if they have the same Maslov index.

(vii) Every path $\Lambda : [a, b] \to L_k(V)$ has Maslov index zero.

Definition 3.19. *Let $\Lambda, \Lambda' : [a, b] \to \mathcal{L}(n)$ be a pair of curves. The* relative crossing form $\Gamma(\Lambda, \Lambda', t)$ *on $\Lambda(t) \cap \Lambda'(t)$ is defined by:*

$$\Gamma(\Lambda, \Lambda', t) = \Gamma(\Lambda, \Lambda'(t), t) - \Gamma(\Lambda', \Lambda(t), t)$$

A crossing for a relative crossing form form is defined in the same way as in the non-relative case. Being a regular or simple crossing is also defined in the same way.

Definition 3.20. *For a pair with only regular crossings the* relative Maslov index *is defined by*

$$\mu(\Lambda, \Lambda') = \frac{1}{2}sign\Gamma(\Lambda, \Lambda', a) + \sum_{t \in (a,b) \text{ and } t \in \chi(\Lambda, \Lambda')} sign\Gamma(\Lambda, \Lambda', t) + \frac{1}{2}sign\Gamma(\Lambda, \Lambda', b)$$

Definition 3.21. *A* symplectic structure *on a vector space V is a non-degenerate skew-symmetric bilinear form on V.*

Definition 3.22. *A basis $u_1, \ldots, u_n, v_1, \ldots, v_n$ of a vector space V equipped with a symplectic structure ω is called a symplectic basis if $\omega(u_i, u_j) = \omega(v_i, v_j) = 0$ and*

$$\omega(u_i, v_j) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

Definition 3.23. *A manifold M equipped with a symplectic structure ω_p on each of the tangent spaces T_pM such that in each coordinate neighborhood, ω varies smoothly on $p \in M$ is called a* symplectic manifold.

Definition 3.24. *A submanifold L of a symplectic manifold (M, ω) is called* Lagrangian *if for every $p \in L$, T_pL is a Lagrangian subspace of T_pM.*

Definition 3.25. *Two Lagrangian submanifolds L, L' of a symplectic manifold (M, ω) are said to intersect transversally at $p \in M$ if T_pL and T_pL' as vector subspaces of T_pM intersect only at* 0.

Although this is not the definition of transversality in differential topology, this formulation coincides with that formulation as well: $T_pL \oplus T_pL' = T_pM$ if and only if $T_pL \cap T_pL' = \{0\}$.

Let (M, ω) be a symplectic manifold of dimension n. Let L_1, L_2 be two Lagrangian submanifolds of M that intersect transversally at two distinct points a, b. Suppose that $f : D^2 \to M$ is a holomorphic map such that boundary of D^2 is mapped onto two arcs in L_1 and L_2. Suppose that these arcs can be parametrized by paths γ_1, γ_2 from $[0, 1]$ into L_1 and L_2 respectively.

Let f^*TM denote the pullback bundle, forming a symplectic vector bundle over D^2, by associating the vector space $T_{f(x)}M$ to each point $x \in D^2$. Since D^2 is contractible, f^*TM is a trivial vector bundle. In other words f^*TM is isomorphic to $D^2 \times \mathbb{R}^{2n}$ with the standard symplectic structure on $\mathbb{R}^{2n}$.

Note that we can insure that the complex structure over TM can be carried to the standard complex structure. One can construct a trivialization of f^*TM, by choosing a symplectic basis of f^*TM at 0, and carrying it by parallel transport. To insure that this process gives a trivialization that takes the symplectic structure on f^*TM induced by the symplectic structure on M, one needs to adjust the utilized connection.

Let the map $\tau_i : [0, 1] \to \Lambda(f^*TM)$ be given by: $t \longmapsto T_{\gamma_i(t)}L_i$. Let $t : f^*TM \to$

$D^2 \times \mathbb{R}^{2n}$ be a trivialization of the vector bundle f^*TM over the disc. Then $t \circ \tau_i$ is a path of Lagrangian subspaces of $\mathbb{R}^{2n}$.

Lemma 3.26. *The relative maslov index*

$$\mu(t \circ \tau_1, \circ\tau_2)$$

is independent of trivialization t.

Proof. Let t, s be two trivializations. Since complex vector bundles over the disk are unique up to isotopy, $t \circ \tau_i$ and $s \circ \tau_i$ are isotopic. Due to preservation of Maslov index under homotopies:

$$\mu(t \circ \tau_1, \circ\tau_2) = \mu(s \circ \tau_1, \circ\tau_2).$$

□

Because of the above lemma, we can now define the Maslov index of a pseudo holomorphic disc $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$. as the quantity appearing in this lemma.

Now we go on to define another version of this Maslov index and show that the two are equivalent. This version is constructed in [8].

Define $\tau_2' : [0,1] \to \Lambda(f^*TM)$ so that:

1. for $0 < t < 1$, $\tau_2(t)$ is a Lagrangian subspace of $T_{\gamma_2(t)}M$ that is transverse to $T_{\gamma_2(t)}L_2$;
2. $\tau_2(0) = T_{\gamma_2(0)}L_1$ and $\tau_2(1) = T_{\gamma_2(1)}L_1$.

Such an arc exists and is unique up to homotopy. This is argued in [8]. Then $\tau_1 \circ (\tau_2')^{-1}$ defines a loop of Lagrangian subspaces of f^*TM. Choose any trivialization t of the pull back bundle f^*TM. Let $\tau = \tau_1(\tau_2')^{-1}$ be the concatenation of the paths τ_1

and $(\tau_2')^{-1}$ of Lagrangian subspaces of f^*TM. Choosing any trivialization t of the pull back bundle f^*TM, can be identified with a loop of Lagrangian subspaces, the loop $t \circ \tau$ is a loop of Lagrangian subspaces of $\mathbb{R}^{2n}$ with the standard symplectic structure. We can compute the Maslov index of this loop (relative to any Lagrangian subspace of $\mathbb{R}^{2n}$). This Maslov index is equivalent to the previous one as proved in the following proposition.

Proposition 3.27.

$$\mu(\tau_1 \circ (\tau_2')^{-1}, V) = \mu(\tau_1, \tau_2)$$

Proof. By small deformations we can insure that, $\tau_1(0) = j \circ \tau_2(0)$ and $\tau_1(1) = j \circ \tau_2(1)$. Because this is done by small deformations, this does not change the Maslov index. One can choose a trivialization of f^*TM so that τ_2 is trivialized as the constant path V, and $\tau_2' = j \circ \tau_1$. Under these choices $\tau_1 \circ (\tau_2')^1$ is homotopic to τ_1. Hence the result follows. □

After having defined the Maslov index of a pseudo holomorphic disc, we can now prove the following theorem which is used in establishing well-definedness of the relative grading on $\widehat{CF}$.

Theorem 3.28. *Let $S \in \pi_2'(Sym^g(\Sigma))$ be the positive generator. Then for any $\phi \in \pi_2(x, y)$ the following holds:*

$$\mu(\phi + k[S]) = \mu(\phi) + 2k$$

Proof. As demonstrated in [8], attaching a topological sphere Z to a disk changes the Maslov index by $2\langle c_1, [Z]\rangle$. But $\langle c_1, S\rangle = 1$ as we have shown in Corollary 2.10. □

4. Spin^c STRUCTURE

In this chapter we define Spin^c-structures on a Riemannian manifold. Recall from Chapter 1 that Heegard Floer homology is defined for a 3-manifold equipped with a Spin^c-structure. The construction of Heegard Floer homology in Chapter 1 involved a map s_z. In this chapter this map is explained in more detail. We also sketch a proof of a property of this map which allows us to see the relationship between spin^c-structures and pseudo holomorphic disks.

Although there is a formulation of Spin^c structures in other terms the following definition is the one used in this context. The formulation below follows [9].

Definition 4.1. *Two unit vector fields on a Riemannian manifold are* homologous *if they are homotopic in the complement of a ball, or equivalently in the complement of finitely many balls.*

Spin^c(Y) *denotes the set of homology classes on the manifold* Y*, and is referred as the set of spin*^{c}*-structures on* Y*.*

Since a 3-manifold is trivializable, there exist unit vector fields, hence spin^c-structures, on 3-manifolds.

Let f be a Morse function on Y compatible with a Heegard diagram (Σ, α, β) for Y. (Fix a Riemannian metric g on Y, to be able to speak of compatibility and ascending, descending manifolds of critical points.) Let the point $\mathbf{x}$ be in the intersection $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$. Then g points of Σ represented in $\mathbf{x}$ lie both on the ascending and descending manifolds of critical points of index 1 and 2 of Σ respectively. Since the trajectory of a gradient flow through a point is unique, these points in fact lie on g trajectories connecting critical points of index 1 and 2 in Y. Now consider a pointed Heegard diagram with a basepoint z lying on Σ but away from α and β curves. Then z does not lie on any of the ascending manifolds of index 1 critical points. It does not lie on descending manifolds of index 2 critical points either. Therefore it lies on a trajectory connecting the two critical points of index 0 and index 3. Once tubular neighborhoods of these $g+1$ trajectories are removed from Y, one obtains a subset of Y, which is a manifold with boundary. On this submanifold gradient of f is nonvanishing. Because this vector field has index zero on the boundary spheres, it can be extended to a nonvanishing vector field on Y.

$$s_z : \mathbb{T}_\alpha \cap \mathbb{T}_\beta \to Spin^c(Y)$$

maps $\mathbf{x}$ to the homology class of this vector field on Y.

By choosing a trivialization, a unit vector field on Y can be viewed as a map, $v : Y \to S^2$, into the sphere. Let ϱ be a generator of the cohomology group $H^2(S^2;\mathbb{Z})$. It can be shown [1] that, the pullback cohomology class $v^*(\varrho) \in H^2(Y;\mathbb{Z})$, is the same for two homologous vector fields. Therefore spin^c-structures on a 3-manifold Y can be identified with cohomology classes in $H^2(Y;\mathbb{Z})$. The statement of the following proposition uses this identification.

For a pair of points $\mathbf{x}, \mathbf{y}$ in the set $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$, let $a : [0,1] \to \mathbb{T}_\alpha$ and $b : [0,1] \to \mathbb{T}_\beta$ be a pair of paths from $\mathbf{x}$ to $\mathbf{y}$. Let $\epsilon(\mathbf{x}, \mathbf{y})$ be the homology class in Y represented by the loop $a - b$ under the identification in Corollary 2.5. It is argued in [1] that this identification is independent from the choice of paths a and b.

Proposition 4.2. *Let* $\mathbf{x}, \mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$*. Let* $\epsilon(\mathbf{x}, \mathbf{y})$ *denote the first homology class in* Y*, formed by the gradient trajectories through the points of* Y *represented in* $\mathbf{x}$ *and* $\mathbf{y}$*. Then*

$$s_z(\mathbf{y}) - s_z(\mathbf{x}) = PD[\epsilon(\mathbf{x}, \mathbf{y})].$$

We outline a sketch of proof.

Proof. For a point $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, let $\gamma_\mathbf{x}$ denote the g trajectories through the points of Y represented in $\mathbf{x}$. For a pair of points $\mathbf{x}, \mathbf{y}$ in the set $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$, the link $\gamma_\mathbf{x} - \gamma_\mathbf{y}$ is a 1-cycle in Y. The vector fields $s_z(\mathbf{y})$ and $s_z(\mathbf{x})$ agree outside the trajectories $\gamma_\mathbf{x} \cup \gamma_\mathbf{y}$, because the map s_z is constructed from the gradient of the same Morse function f on Y. Therefore the difference $s_z(\mathbf{x}) - s_z(\mathbf{y})$ represents by a cohomology class supported in a neighborhood of $\gamma_\mathbf{x} - \gamma_\mathbf{y}$. Consequently it comes from algebraic topology that the cohomology class $s_z(\mathbf{x}) - s_z(\mathbf{y})$ is a multiple of the Poincare dual of $\gamma_\mathbf{x} - \gamma_\mathbf{y}$.

Let $x_i \in \mathbf{x} - \mathbf{y}$. Take a small disk D_0 in Σ around x_i. The vector field v_x representing $s_z(\mathbf{x})$ can be chosen to agree with gradient of f near the boundary of D_0. The vector field v_y representing $s_z(\mathbf{y})$ can be chosen to agree with gradient of f in D_0. It is then stated in [1] that

$$s_z(\mathbf{x}) - s_z(\mathbf{y}) = (\deg_{D_0}(v_\mathbf{x}) - \deg_{D_0}(v_\mathbf{y}))\mathrm{PD}(\gamma_\mathbf{x} - \gamma_\mathbf{y}).$$

Take another disk D_1 with the same boundary as D_0, so that the sphere $D_0 \cup D_1$ bounds a 3-ball in Y containing the index 1 critical point corresponding to x_i, and no other critical point. Then v_x can be chosen to agree with the gradient of f on D_1. The vector field v_x does not vanish inside the 3-ball bounded by the sphere $D_0 \cup D_1$. Therefore

$$0 = \deg_{D_0}(v_x) + \deg_{D_1}(v_x) = \deg_{D_0}(v_x) + \deg_{D_1}(\nabla f).$$

The gradient of f vanishes with a contribution of -1 around index 1 critical points. Therefore we obtain that

$$\deg_{D_0}(v_\mathbf{x}) - \deg_{D_0}(v_\mathbf{y}) = -\deg_{D_1}(\nabla f) - \deg_{D_0}(\nabla f) = 1.$$

Let a_i be one of the arcs on Σ joining x_i to some y_j. This arc a_i is homotopic with endpoints fixed to the gradient trajectories connecting x_i and y_j to the index 1 critical point associated, by the gradient trajectories descending to this critical. Since this argument could be repeated for arcs in the path b, the 1-cycle $a - b$ is homologous to the 1-cycle $\gamma_{\mathbf{x}} - \gamma_{\mathbf{y}}$. □

Now note that if a pseudo holomorphic disk exists between the points $\mathbf{x}$ and $\mathbf{y}$, then the 1-cycle $a - b$ is null-homologous in $\mathrm{Sym}^g \Sigma$. In this case, $\epsilon(\mathbf{x}, \mathbf{y}) = 0$. As a result of the property we have proved above, the points $\mathbf{x}$ and $\mathbf{y}$ are mapped to the same spin^c-structure under the map s_z.

REFERENCES

1. Ozsváth, Peter; Szabó, Zoltán, *Holomorphic disks and topological invariants for closed three-manifolds*, Ann. of Math. (2) 159 (2004), no. 3, 1027–1158.

2. Ozsváth, Peter; Szabó, Zoltán, *On Heegaard diagrams and holomorphic disks*, European Congress of Mathematics, 769–781, Eur. Math. Soc., Zürich, 2005.

3. Ozsváth, Peter; Szabó, Zoltán; *Holomorphic disks and three-manifold invariants: properties and applications.* Ann. of Math. (2) 159 (2004), no. 3, 1159–1245.

4. Hattori, Akio; *Topology of* $\mathbb{C}^n$ *minus a finite number of affine hyperplanes in general position.* J. Fac. Sci. Univ. Tokyo Sect. IA Math. 22 (1975), no. 2, 205–219.

5. Macdonald, I. G. *Symmetric products of an algebraic curve.* Topology 1 1962 319–343.

6. Robbin, Joel; Salamon, Dietmar *The Maslov index for paths.* Topology 32 (1993), no. 4, 827–844.

7. Cannas da Silva, Ana; Lectures on symplectic geometry. Lecture Notes in Mathematics, 1764. Springer-Verlag, Berlin, 2001.

8. Viterbo, Claude *Intersection de sous-variétés lagrangiennes, fonctionnelles d'action et indice des systèmes hamiltoniens.* Bull. Soc. Math. France 115 (1987), no. 3, 361–390.

9. Turaev, Vladimir; *Torsion invariants of* $Spin^c$*-structures on 3-manifolds.* Math. Res. Lett. 4 (1997), no. 5, 679–695.