Knot homology of $(3, m)$ torus knots

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ABSTRACT
We give a direct computation of the Khovanov knot homology of the $(3, m)$ torus knots/links. Our computation yields complete results with $\mathbb{Z}[\frac{1}{2}]$ coefficients, though we leave a slight ambiguity concerning 2-torsion when integer coefficients are used. Our computation uses only the basic long exact sequence in knot homology and Rasmussen’s result on the triviality of the embedded surface invariant.

Keywords: Khovanov knot homology, torus knot

Mathematics Subject Classification 2000: 57M25, 57M27

1. Introduction and Setup

In [MK1], Khovanov introduced a knot homology theory which associates a finitely generated bigraded abelian group $KH(L) = \bigoplus_{i,j} KH^i_j(L)$ to an oriented link $L$. The Jones polynomial of $L$ is recovered from $KH(L)$ by the formula

$$ (q + q^{-1}) \text{Jones}(L)(\tau)|_{\sqrt{\tau} = -q} = \sum_{i,j} (-1)^i q^j \text{rk} KH^i_j(L). \quad (1.1) $$

In the same paper, Khovanov computes the knot homology groups $KH(T_{2,m})$ of the $(2, m)$ torus knots/links. In the present paper, we perform the next obvious computation: the knot homology of $(3, m)$ torus knots/links. This does not seem to appear anywhere in the literature (probably because it is long and tedious) but I have heard that it is probably known to J. Rasmussen and others.\(^*\) The correct answer can be easily predicted using computer computations for small $m$.

A key feature of Khovanov’s knot homology theory is that link cobordisms induce maps on knot homology (see §2). We use these maps in an essential way in our computation. Our (fairly simplistic) computation method involves a careful examination of various fundamental sequences (see §1.5) relating (via a long exact sequence) the knot homology of three links with diagrams $D_0, D, D_1$, where $D_0$ is obtained from $D$ by the “0-resolution” of a crossing and $D_1$ is obtained from $D$ by

\(^*\)Since this paper was first written early in my graduate student days, the papers [MS] and [PT] have appeared. Among other things, these papers compute the knot homology of the $(3, m)$ torus knots with $\mathbb{Q}$ coefficients (by completely different methods).
the “1-resolution” of the same crossing. By making some rather delicate diagram chases, we reduce the computation of the maps in these long exact sequences to computing maps on knot homology induced by link cobordisms between trivial links. To compute these maps we apply a result of Rasmussen [JR2] on the triviality of the surface invariant coming from Khovanov’s theory to show that among all connected link cobordisms from the unknot to the two component trivial link, there are exactly two distinct non-zero maps induced on knot homology. This helps to account for the frequent appearance of 2-torsion in knot homology.

1.2. Notation

All groups are finitely generated and abelian. For a group $G$ and a prime power $s$, let $\text{rk}(G, s)$ denote the rank of the $(\mathbb{Z}/s\mathbb{Z})$-summand in the representation of $G$ as a finite sum of cyclic groups, with the convention that $\text{rk}(G, 0) := \text{rk}(G)$ is the usual rank of $G$. If $G = \bigoplus_n G_n$ is a graded group, we use $G_n$ to denote the $n$th graded part of $G$, and we use a similar subscript notation for maps of graded abelian groups.

For a group $G$ and integer $n$ we write $G^n$ for the graded group supported in grading $n$ whose $n$th graded part is $G$. For example, $\mathbb{Z}_2$ is not the two element group (the latter is denoted $\mathbb{F}$ throughout), but is rather the graded group whose degree 2 part is $\mathbb{Z}$. If $G = \bigoplus_{i,j} G_{i,j}$ is a bigraded group, we refer to the summand $G_{i,j}$ as the degree $i$ grading $j$ part of $G$ and we define the Poincaré polynomial of $G$ by

$$P(G)(t, q_0, q_2, q_4, \ldots) := \sum_{i,j \in \mathbb{Z}_s} t^i q_j^s \text{rk}(G_{i,j}, s).$$

(The $s$ in the sum runs over prime powers and $s = 0$.) Clearly the Poincaré polynomial determines $G$ up to isomorphism of bigraded groups. It is customary to define $q := q_0$ and $Q := q_2$. Recall [MK1 §4.2] that the knot homology $\text{KH}(L)$ of an oriented link $L$ is related to the unshifted homology $H(D)$ of an oriented diagram $D$ of $L$ by

$$\text{KH}^i(L) = H^{i+x(D)}(D)\{2x(D) - y(D)\},$$

(1.3)

where $x(D)$ and $y(D)$ are the number of $x$-type (“negative”) and $y$-type (“positive”) crossings in $D$ (Figure 1).

Let $O$, $OO$, etc. denote the standard diagrams of the unknot, two component trivial link, etc. Let $A := \mathbb{Z}_1 \oplus \mathbb{Z}_{-1}$ denote the knot homology of the unknot; its generators are denoted $1 \in \mathbb{Z}_1$ and $X \in \mathbb{Z}_{-1}$ (aka $v_+$ and $v_-$ in [JR1],[JR2]).

\[\text{This agrees with the output format of Shumakovitch's computer program. Note that the torsion in knot homology tends to be only of order 2 for small knots, so the variables } q_3, q_4, \ldots \text{ rarely appear.} \]
The knot homology of the trivial \( n \) component link \( O_n \) is supported in degree 0 and given by
\[
KH^0(O_n) = A^\otimes n = \mathbb{Z}_{-n} \oplus \mathbb{Z}_{n+2} \oplus \cdots \oplus \mathbb{Z}_{n-2} \oplus \mathbb{Z}_n
\]
(regardless of orientation). Given a “non-standard” diagram \( D \) of \( O_n \), one can compute the unshifted homology \( H(D) \) by choosing an orientation of \( D \) (the choice does not actually matter), counting the number of \( x \)- and \( y \)-type crossings in \( D \), then applying (1.3).

1.5. **Fundamental sequence**

Suppose \( D_0, D, D_1 \) are three link diagrams which agree away from a crossing \( c \) of \( D \), where \( D_0 \) is obtained from \( D \) by taking the 0-resolution of \( c \) and \( D_1 \) is obtained from \( D \) by taking the 1-resolution of \( c \) (Figure 2). In this situation, we say \( (D_0, D, D_1) \) is a **triple**. It is immediate from the construction of \( H \) that the unshifted homologies of \( D_0, D, D_1 \) are related by a long exact sequence
\[
0 \to H^0_0(D) \to H^0_0(D_0) \to H^0_0(D_1) \{ -1 \} \to H^1_0(D) \to \cdots
\]
(1.6) called the **fundamental sequence** associated to the triple \( (D_0, D, D_1) \). See the proof of Proposition 9 in [MK1] or §3.4 in [G] for details. In the case that \( D_0, D, D_1 \) are all trivial links (but possibly nontrivial diagrams), as will be the case in many of our calculations, the fundamental sequence (1.6) is simply a short exact sequence.
1.7. Main result

Our main result is the computation of the knot homology of the \((3, m)\) torus knots/links with integer coefficients, modulo a certain ambiguity about the presence of \(P\) summands:

**Theorem 1.1.** For \(k \geq 1\), the Khovanov knot homology \(\text{KH}(T_{3,m})\) of the \((3, m)\) torus knot/link takes the form:

\[
P(\text{KH}^*(T_{3,3k-1})) = q^{6k-5} + q^{6k-3} + q^{6k-1}t^2 + (Q^{6k+1} + q^{6k+3})t^3 + (q^{6k+1} + q^{6k+3})t^4 \\
+ (q^{6k+5} + Q^{6k+5} + q^{6k+7})t^5 + q^{6k+5}t^6 + (Q^{6k+7} + q^{6k+9})t^7 \\
+ (q^{6k+7} + q^{6k+9})t^8 + (q^{6k+11} + Q^{6k+11} + q^{6k+13})t^9 \\
+ \cdots + q^{12k-7}t^{4k-2} + (Q^{12k-5} + q^{12k-3})t^{4k-1} \\
P(\text{KH}(T_{3,3k})) = q^{6k-3} + q^{6k-1} + (Q^{6k+3} + q^{6k+5})t^3 + (q^{6k+3} + q^{6k+5})t^4 \\
+ (q^{6k+7} + Q^{6k+7} + q^{6k+9})t^5 + q^{6k+7}t^6 + (Q^{6k+9} + q^{6k+11})t^7 \\
+ \cdots + q^{12k-5}t^{4k-2} + (Q^{12k-3} + q^{12k-1})t^{4k-1} \\
+ (q^{12k-3} + 3q^{12k-1} + 2q^{12k+1})t^{4k} \\
P(\text{KH}(T_{3,3k+1})) = q^{6k-1} + q^{6k+3}t^2 + (Q^{6k+5} + q^{6k+7})t^3 + (q^{6k+5} + q^{6k+7})t^4 \\
+ (q^{6k+9} + Q^{6k+9} + q^{6k+11})t^5 + q^{6k+9}t^6 + (Q^{6k+11} + q^{6k+13})t^7 \\
+ \cdots + q^{12k-3}t^{4k-2} + (Q^{12k-1} + q^{12k+1})t^{4k-1} \\
+ (q^{12k-1} + q^{12k+1})t^{4k} + (q^{12k+3} + Q^{12k+3} + q^{12k+5})t^{4k+1},
\]

where the underlined terms \(Q\) may or may not be present.

Our convention throughout the paper is that anything underlined may or may not be present. Computer calculations suggest that none of the underlined terms is present, though we do not prove this in the present paper. Our proof will show, however, that the presence of the underlined terms is not “dependent on \(m\)”, so we know, for example (by computer calculations), that the underlined term in the coefficient of \(t^5\) is never present, though we have chosen not to put this in the statement of our theorem. In any case, our theorem does prove that all torsion is 2 torsion, and it yields complete results with \(\mathbb{Z}[\frac{1}{2}]\) coefficients, as well as substantial information about the 2 torsion with integer coefficients.

Recall that the Jones polynomial of the \((3, m)\) torus knot is given by

\[
\text{Jones}(T_{3,m})(\tau) = \begin{cases} 
\tau^{m-1} + \tau^{m+1} + 2\tau^{2m}, & 3|m \\
\tau^{m-1} + \tau^{m+1} - \tau^{2m}, & \text{otherwise}
\end{cases}
\]
so the result claimed above at least has the correct Euler characteristic since

\[(q + q^{-1})\text{Jones}(T_{3,m})(\tau)|_{\sqrt{\tau} = -q} = \begin{cases} 
q^{6k-5} + q^{6k-3} + q^{6k-1} + q^{6k+1} - q^{12k-5} - q^{12k-3}, & m = 3k - 1 \\
q^{6k-3} + q^{6k-1} + q^{6k+1} + q^{6k+3} + 2q^{12k-1} + 2q^{12k+1}, & m = 3k \\
q^{6k-1} + q^{6k+1} + q^{6k+3} + q^{6k+5} - q^{12k+3} - q^{12k+5}, & m = 3k + 1.
\end{cases}\]

**Proof overview.** To prove Theorem 1.1, we consider the two triples depicted in Figure 3, and the associated fundamental sequences:

\[0 \to H^0(T_m) \to H^0(K_m) \to H^0(N_m) \{ -1 \} \to H^1(T_m) \to \cdots \] (1.8)

\[0 \to H^0(K_m) \to H^0(T_{m-1}) \to H^0(M_m) \{ -1 \} \to H^1(K_m) \to \cdots \] (1.9)

It is understood that the braids in Figure 3 (and subsequent such figures) are closed up in the usual way, although this is not shown. The diagram $T_m$ is a diagram representing $T_{3,m}$ with $2m$ crossings. When each strand is oriented bottom-to-top (the “usual” orientation of $T_{3,m}$), each crossing is of $y$-type (c.f. Figure 1), so we must show that the unshifted Poincaré polynomials $P(T_m) := P(H^*(T_m))$ are given

![Diagram](image-url)
Using this method we find:

\[ P(T_{3k-1}) = q^{-3} + q^{-1} + qt^2 + (Q^3 + q^5)t^3 + (q^3 + q^5)t^4 + (q^7 + Q^7 + q^9)t^5 + \cdots + (Q^{6k-3} + q^{6k-1})t^{4k-1} \]

\[ P(T_{3k}) = q^{-3} + q^{-1} + qt^2 + (Q^3 + q^5)t^3 + (q^3 + q^5)t^4 + (q^7 + Q^7 + q^9)t^5 + \cdots + q^{6k-5}q^{4k-2} + (Q^{6k-3} + q^{6k-1})t^{4k-1} \]

\[ P(T_{3k+1}) = q^{-3} + q^{-1} + qt^2 + (Q^3 + q^5)t^3 + (q^3 + q^5)t^4 + (q^7 + Q^7 + q^9)t^5 + \cdots + q^{6k-5}q^{4k-2} + (Q^{6k-3} + q^{6k-1})t^{4k-1} \]

\[ + q^{6k-3} + q^{6k-1}t^{4k} + (q^{6k+1} + Q^{6k+1} + q^{6k+3})t^{4k+1} \]

We work with unshifted homology of diagrams throughout the rest of the paper (this is defined without an orientation, so there is no longer any concern about how our diagrams are oriented). We systematically use labelled arrows to indicate that one diagram is obtained from another by resolving a crossing in a particular way, so that, for example, \( N_m \) is obtained from \( T_m \) by taking the 1-resolution of crossing \( 2m \).

Notice that the diagrams \( N_m \) and \( M_m \) are trivial links, so their homology can be computed as in §1.4. For example, the diagram \( M_{3k+2} \) is a diagram of the unknot with \( 4k + 1 \) crossings of \( x \)-type and \( 2k + 1 \) crossings of \( y \)-type (regardless of how it is oriented), so we have

\[
H^i(M_{3k+2}) = KH^{i-(4k+1)}(O)\{(2k + 1) - 2(4k + 1)\}
\]

\[
= \begin{cases} 
\mathbb{Z}_{6k} \oplus \mathbb{Z}_{6k+2}, & i = 4k + 1 \\
0, & i \neq 4k + 1.
\end{cases}
\]

Using this method we find:

\[ P(M_{3k+2}) = (q^{6k} + q^{6k+2})t^{4k+1} \quad \text{(1.10)} \]
\[ P(N_{3k+2}) = (q^{6k+2} + q^{6k+4})t^{4k+2} \]
\[ P(M_{3k}) = (q^{6k-2} + q^{6k})t^{4k-1} \]
\[ P(N_{3k}) = (q^{6k-4} + 2q^{6k-2} + q^{6k})t^{4k-1} \]
\[ P(M_{3k+1}) = (q^{6k-2} + 2q^{6k} + q^{6k+2})t^{4k} \]
\[ P(N_{3k+1}) = (q^{6k-2} + q^{6k})t^{4k} \]

We prove Theorem 1.1 by induction on \( m \) (the case \( m = 2 \) is well-known [MK1]). The induction step from \( m \) to \( m + 1 \) is different depending on the 3-modulus of \( m \), but we will always make heavy use of (1.9) and (1.8) above, and the formulas of (1.10).
1.11. From 3k-1 to 3k

From (1.10) we see that $H(M_{3k})$ is supported in dimension $4k - 1$, so from (1.9), we find

$$H^i(K_{3k}) = H^i(T_{3k-1}) \quad (i \neq 4k - 1, 4k).$$

We know the latter groups by the induction hypothesis. The only interesting part of (1.9) looks like

$$0 \to \mathbb{H}^{4k-1}(K_{3k}) \to \mathbb{H}^{4k-1}(T_{3k-1}) \to \mathbb{H}^{4k-1}(M_{3k}) \{ -1 \} \to \mathbb{H}^{4k}(K_{3k}) \to 0$$
$$\mathbb{F}_{6k-3} \oplus \mathbb{Z}_{6k-1} \to \mathbb{Z}_{6k-1} \oplus \mathbb{Z}_{6k+1}. $$

We will show in §3.1 that $f_{6k-1}$ is zero, hence we have

$$H^{4k-1}(K_{3k}) = \mathbb{F}_{6k-3} \oplus \mathbb{Z}_{6k-1}$$
$$H^{4k}(K_{3k}) = \mathbb{Z}_{6k-1} \oplus \mathbb{Z}_{6k+1}. $$

From (1.10) we see that $H(N_{3k})$ is supported in degree $4k - 1$, so the sequence (1.8) yields isomorphisms

$$H^i(T_{3k}) = H^i(K_{3k})$$
$$= H^i(T_{3k-1}) \quad (i \neq 4k - 1, 4k)$$

and its interesting part looks like

$$H^{4k-1}(T_{3k}) \to H^{4k-1}(K_{3k}) \to \mathbb{H}^{4k-1}(N_{3k}) \{ -1 \} \to \mathbb{H}^{4k}(T_{3k}) \to \mathbb{H}^{4k}(K_{3k})$$
$$\mathbb{F}_{6k-3} \oplus \mathbb{Z}_{6k-1} \to \mathbb{Z}_{6k-1} \oplus \mathbb{Z}_{6k-1} \oplus \mathbb{Z}_{6k+1} \oplus \mathbb{Z}_{6k+1} \oplus \mathbb{Z}_{6k+1} \oplus \mathbb{Z}_{6k+1}$$

(with zeros to the left and right), so we can complete the induction step by showing $g_{6k-1}$ is zero (hence $g$ is zero), which is done in §3.2.

1.12. From 3k to 3k+1

From (1.10) we see that $H(M_{3k+1})$ is supported in degree $4k$, so (1.9) yields isomorphisms

$$H^i(K_{3k+1}) = H^i(T_{3k}) \quad (i \neq 4k, 4k + 1)$$

and its interesting part looks like

$$H^{4k}(K_{3k+1}) \to H^{4k}(T_{3k}) \to H^{4k}(M_{3k+1}) \{ -1 \} \to H^{4k+1}(K_{3k+1})$$
$$\mathbb{Z}_{6k-3} \oplus \mathbb{Z}_{6k-1} \oplus \mathbb{Z}_{6k-1} \oplus \mathbb{Z}_{6k+1} \oplus \mathbb{Z}_{6k+1} \oplus \mathbb{Z}_{6k+1} \oplus \mathbb{Z}_{6k+3}.$$
(with zeros to the left and right). In §3.3 we will show that $f_{6k+1}$ has cokernel $F$ and $f_{6k-1}$ is surjective. Since $H(N_{3k+1})$ is supported in degree $4k$ (1.10), the sequence (1.8) then yields isomorphisms

$$H^i(T_{3k+1}) = H^i(K_{3k+1})$$

and its relevant part looks like

$$H^{4k}(T_{3k+1}) \rightarrow H^{4k}(K_{3k+1}) \rightarrow H^{4k}(N_{3k+1}) \{ -1 \} \rightarrow H^{4k+1}(T_{3k+1}) \rightarrow H^{4k+1}(K_{3k+1})$$

(with zeros to the left and right). In §3.4 we will show that $g_{6k-1} : Z^2 \rightarrow Z$ is surjective, hence

$$H^{4k}(T_{3k+1}) = Z_{6k-3} \oplus Z_{6k-1}$$

and its relevant part looks like

$$H^{4k+1}(T_{3k+1}) = Z_{6k-3} \oplus Z_{6k-1}$$

In degree $6k+1$ we have an exact sequence

$$0 \rightarrow Z \rightarrow H^{4k+1}(T_{3k+1})_{6k+1} \rightarrow F \rightarrow 0,$$

so either

$$H^{4k+1}(T_{3k+1})_{6k+1} = Z,$$

or

$$H^{4k+1}(T_{3k+1})_{6k+1} = Z \oplus F,$$

which completes the induction step.

We write $H^{4k+1}(T_{3k+1}) = Z_{6k+1} \oplus Z_{6k+2} \oplus Z_{6k+3}$ according to our convention that underlined terms may or may not be present. We strongly suspect that $H^{4k+1}(T_{3k+1})_{6k+1} = Z$, but it does not seem possible to establish this by the methods of this paper. In principle, one could check this by showing that the image of a generator of $H^k(N_{3k+1}) \{ -1 \}_{6k+1} = Z$ under $h$ is divisible by 2 in $H^{4k+1}(T_{3k+1})_{6k+1}$. This sort of calculation seems to require us to actually “go in by hand” to the complexes computing $H$. It turns out that it is not particularly difficult to write down an explicit cycle $\alpha$ whose image in homology generates $H^k(N_{3k+1}) \{ -1 \}_{6k+1} = Z$ (hence it is not hard to write down an explicit cycle representative for $h(\alpha)$—the map $h$ on the chain level is induced by the inclusion of a subcomplex). The difficulty is in writing down “out of the blue” a cycle $\beta$ with $2\beta \simeq h(\alpha)$ (mod boundary).

### 1.13. From $3k+1$ to $3k+2$

From (1.10), we see that $H(M_{3k+2})$ is supported in degree $4k+1$, so (1.9) yields isomorphisms

$$H^i(K_{3k+2}) = H^i(T_{3k+1})$$

and its relevant part looks like

$$H_{4k+1}(T_{3k+1}) = Z_{6k-3} \oplus Z_{6k-1} \oplus Z_{6k+1}$$

In degree $6k+1$ we have an exact sequence

$$0 \rightarrow Z \rightarrow H^{4k+1}(T_{3k+1})_{6k+1} \rightarrow F \rightarrow 0,$$

so either

$$H^{4k+1}(T_{3k+1})_{6k+1} = Z,$$
and its interesting part looks like

\[ H^{4k+1}(K_{3k+2}) \xrightarrow{f} H^{4k+1}(T_{3k+1}) \xrightarrow{g} H^{4k+3}(T_{3k+2}) \]

(with zeros to the left and right). In §3.5 we will show that \( f \) is zero. Since \( H(N_{3k+2}) \) is supported in degree \( 4k + 2 \), the sequence (1.8) yields isomorphisms

\[ H^i(T_{3k+2}) = H^i(K_{3k+2}) \]
\[ = H^i(T_{3k+1}) \quad (i \neq 4k + 2, 4k + 3) \]

and its interesting part looks like

\[ H^{4k+2}(T_{3k+2}) \xrightarrow{g} H^{4k+3}(T_{3k+2}) \]

(with zeros to the left and right). In §3.6, we show that \( g_{6k+3} : \mathbb{Z} \to \mathbb{Z} \) is multiplication by 2, hence we find

\[ H^{4k+2}(T_{3k+2}) = \mathbb{Z}_{6k+1} \]
\[ H^{4k+3}(T_{3k+2}) = \mathbb{F}_{6k+3} \oplus \mathbb{Z}_{6k+5}, \]

which completes the induction step.

This completes the proof of Theorem 1.1, modulo the boundary map computations of §3, which will be done after a brief interlude.

2. Link cobordisms

Recall that a link cobordism \( \Sigma = [\Sigma, f] \) is an equivalence class of embeddings \( f : \Sigma \to I \times \mathbb{R}^3 \) from an oriented surface \( \Sigma \) (possibly with boundary) satisfying \( \delta \Sigma = f^{-1}[\{0,1\}\mathbb{R}^3] \) up to isotopy (homotopy through such maps). These can be composed in an obvious way to form the morphisms of a category whose objects are links. Taking a sufficiently fine partition

\[ 0 = t_0 < t_1 < \cdots < t_n = 1 \]

of the interval, and a generic projection \( p : \mathbb{R}^3 \to \mathbb{R}^2 \), such an embedding determines a sequence of knot diagrams \( D_0, \ldots, D_n \) called frames representing the abstract links \( f[\Sigma] \cap \{t_i\} \times \mathbb{R}^3 \), each of which differs from the previous by planar isotopy, a Reidemeister move, or a Morse move. Thus, applying the maps associated to these “elementary” cobordisms in [MK1] we get a graded map of degree \( \chi(\Sigma) \) from the knot homology of the link depicted in \( D_0 \) to the knot homology of the link depicted in \( D_n \). It is shown in [MJ], [MK2], and [DB] that (up to multiplication by \( \pm 1 \)) this map does not actually depend on the choices made in representing the link cobordism by a sequence of frames. In light of this annoying \( \pm 1 \) factor, for
the remainder of this paper we identify any two maps between abelian groups that differ by $\pm 1$, so that “commutes” means “commutes up to $\pm 1$.” The proof of the following lemma was suggested by M. Khovanov.

**Lemma 2.1.** Let $\Sigma$ be a connected link cobordism with no closed components, with $\chi(\Sigma) = -1$, with a frame sequence $D_0 = O, \ldots, D_n = OO$. Then

$$\text{KH}^0(\Sigma) : \text{KH}^0(D_0) = A \to \text{KH}^0(D_n) = A \otimes A$$

is given by

1. $\mapsto 1 \otimes X + X \otimes 1$
2. $X \mapsto X \otimes X$

if $\Sigma$ can be oriented so that the induced orientations on the two circles in $D_n$ agree and otherwise by

1. $\mapsto 1 \otimes X - X \otimes 1$
2. $X \mapsto X \otimes X$.

**Proof.** The notation of [JR1],[JR2] is used heavily here (Lee’s generators are denoted $s_o$, Rasmussen’s canonical generators are denoted $\pi_o$, $a := X + 1, b := X - 1$). Let $o$ (resp. $\overline{o}$) be the standard counter-clockwise (resp. non-standard) orientation on the circle. Since the links in question are trivial, Lee’s spectral sequence [EL] collapses immediately and is compatible with link cobordisms (Lemma 2.1 in [JR2]), so we may compute in Lee’s theory. There is an error in [JR2] so that his Equation 1 should read

$$\phi_S'(\pi_0) = 2^{-\chi(S)/2} \sum_i \pm \pi_{0i},$$

since this is easily seen to be the convention one needs to prove his Lemma 3.3 and, indeed, is the convention he is using in the calculation at the bottom of Page 4. Now use the corrected [JR2, Proposition 3.2] to compute in the first case:

$$\phi_{\Sigma}(1) = \phi_{\Sigma}(2^{-1}(1 + X) + 2^{-1}(1 - X)) = 2^{-1/2} \phi_{\Sigma}(\overline{\pi}_o + \overline{\pi}_{\pi}) = 2^{-1/2} 2^{1/2} (\pm \overline{\pi}_{oo} \pm \overline{\pi}_{\pi}) = \pm 2^{-1/2} 2^{1/2} 2^{-1}(a \otimes a + b \otimes b) = 2^{-1/2} 2^{1/2} 2^{-1}(21 \otimes X + 2X \otimes 1) = \pm (1 \otimes X + X \otimes 1).$$

The other computations are similar. We switch to Rasmussen's notation. His $\phi_{\Sigma}$ is our $\text{KH}(\Sigma)$. \hfill $\Box$
Notice that \(1 \otimes X + X \otimes 1\) is two times a generator of
\[
((A \otimes A)/(1 \otimes X - X \otimes 1))_0 \cong \mathbb{Z}.
\]
Furthermore, if \(G \cong \mathbb{Z}\), then the question of whether two group homomorphisms \(f, g : G \to H\) have the same image, or whether the image of \(f\) generates twice the cokernel of \(g\), etc. is independent of pre-composing \(f, g\) with (possibly different) automorphisms of \(G\) and post-composing \(f, g\) with the same automorphism of \(H\), so we can remove the necessity of having the diagrams \(D_0, D_n\) in the standard form \(O, OO\) in Lemma 2.1 and restate it as follows:

**Lemma 2.2.** If \(\Sigma, \Sigma'\) are two connected link cobordisms with no closed components, with \(\chi = -1\), and with frame sequences \(D_0, \ldots, D_n\) and \(E_0, E_1, \ldots, E_n = D_n\), where \(D_0\) and \(E_0\) are any diagrams of the unknot and \(D_n = E_n\) is any diagram of a two-component trivial link, then
\[
\text{Im} \ KH^0(\Sigma) = \text{Im} \ KH^0(\Sigma')
\]
if \(\Sigma\) and \(\Sigma'\) can be oriented to induce the same orientations on the two components of \(D_n = E_n\) and otherwise \(\text{Im} \ KH^0(\Sigma)\) generates twice the cokernel of \(KH^0(\Sigma')\).

In practice, it is always easy to decide, by direct inspection, which case of Lemma 2.2 applies to a given cobordism.

### 3. Boundary maps

The basic idea here is to reduce to computing maps induced on knot homology by link cobordisms between trivial links, then use Lemma 2.2. To do this, we use the induction hypothesis to trace all knot homology back to trivial links by complicated diagram chases (together with a certain amount of luck and persistence).

#### 3.1. \(f_{6k-1} : H^{4k-1}(T_{3k-1})_{6k-1} \to H^{4k-1}(M_{3k})\{1\}_{6k-1}\) is zero.

**Proof.** Consider the diagrams and cobordisms shown in Figure 4. We abusively denote a cobordism and the induced map on (unshifted) homology by the same letter. The diagrams on the right of Figure 4 represent trivial links, so their unshifted homology groups can be computed as in §1.4 (the only non-zero homology groups are as indicated in the figure). The map
\[
j : H^{4k-2}(N_{3k-1}) \to H^{4k-1}(T_{3k-1})\{1\}
\]
is an isomorphism in grading \(6k - 1\) because it is part of the fundamental sequence associated to the triple \((K_{3k-1}, N_{3k-1}, T_{3k-1})\) of Figure 3 whose terms in grading \(6k - 1\) before and after \(j\) are zero.\(^{\dagger}\)

\(^{\dagger}\)This sequence is the second sequence discussed in §1.13 after the reindexing \(k \mapsto k - 1\).
The map \( f_{6k - 1} \) of interest to us fits into a commutative diagram

\[
\begin{array}{c}
\mathbb{Z} \\
\downarrow j_{6k - 1} \\
\mathbb{Z} \\
\end{array}
\xrightarrow{f_{6k - 1}}
\begin{array}{c}
\mathbb{Z} \\
\downarrow j_{6k - 1} \\
\mathbb{Z} \\
\end{array}
\]

where the bottom row is exact (it is the fundamental sequence of the triple \((M_{0k}^0, M_{3k}, M_{1k}^1)\) in grading \(6k - 1\)), so to prove the claim, it suffices to show that \( \text{Im} \ h_{6k - 1} = \text{Im} \ f_{6k - 1} \).

The maps \( h_{6k - 1} \) and \( f_{6k - 1} \) are both induced by cobordisms from the unknot to the two component trivial link, so we are in a position to apply Lemma 2.2. The first case of that lemma applies because the orientation of \( f \) inducing the orientation of \( N_{3k - 1} \) in Figure 4 induces the orientation of \( M_{1k}^1 \) in Figure 4 and the orientation of \( h \) inducing the orientation of \( M_{0k}^0 \) in Figure 4 also induces the orientation of \( M_{1k}^1 \) in Figure 4.

In the rest of §3 we will be more brief in our application of Lemma 2.2, leaving it to the reader to orient the trivial links and decide which case of the lemma to apply.

3.2. \( g_{6k - 1} : H^{4k - 1}(K_{3k})_{6k - 1} \to H^{4k - 1}(N_{3k})\{1\}_{6k - 1} \) is zero.

Proof. First consider the diagrams in Figure 5. The arrows in this figure show the direction of the induced maps on homology (as opposed to indicating the direction...
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The triples $(M_{3k}, N_{3k}, N_{13k})$, $(K_{3k}, T_{3k}, N_{3k})$, and $(T_{3k-1}, T_{3k}, N_{03k})$ (horizontal) and the triples $(T_{3k-1}, K_{3k}, M_{3k})$, $(N_{03k}, N_{3k}, N_{13k})$, and $(T_{3k}, T_{3k}, N_{3k})$ (vertical) of the crossing resolutions. The knot homology of every diagram in Figure 5, except $T_{3k}$ and $T_{3k}^0$, can be computed by the result of §3.1, the induction hypothesis, or because it is a trivial link. The fundamental sequences associated to the six triples in Figure 5 fit into a large commutative diagram with exact columns and rows. The
relevant part of this diagram looks like:

\[
\begin{array}{c}
0 \\
\downarrow b \\
H^{4k-1}(T_{3k-1}) \\
\downarrow 0 \\
H^{4k-1}(N_{3k}) \\
\downarrow 0 \\
H^{4k-2}(N_{3k}) \{ -2 \} \\
\downarrow 0 \\
H^{4k-1}(T_{3k-1}) \\
\downarrow 0 \\
H^{4k-1}(N_{3k-1}) \{ -1 \} \\
\downarrow 0 \\
H^{4k-1}(N_{3k}) \{ 0 \} \\
\downarrow 0 \\
H^{4k}(M_{3k+1}) \{ 1 \}
\end{array}
\]

We will show momentarily that \( g_{6k-1} \) is zero, so \( a \) yields an isomorphism

\[
H^{4k-1}(T_{3k-1}) = H^{4k-1}(N_{3k}) \{ -1 \} = \mathbb{Z}.
\]

It follows that \( b_{6k-1} \) is also an isomorphism because its domain and codomain are both \( \mathbb{Z} \) and its cokernel is torsion-free because this cokernel injects into \( H^{4k-1}(N_{3k}) \{ -1 \} \), which is torsion-free (\( N_{3k} \) is a trivial link). Since \( b_{6k-1} \) is an isomorphism, \( c_{6k-1} \) is injective by exactness. It now follows from a diagram chase that \( g_{6k-1} \) is zero.

It remains to prove that \( g_{6k-1} = 0 \). To do this, consider the diagrams in Figure 6. The map

\[
j : H^{4k-2}(N_{3k}) \{ -1 \} \to H^{4k-1}(T_{3k-1}) \{ -1 \}
\]

is an isomorphism in grading \( 6k - 1 \) by the induction hypothesis. The rest of the argument is identical to the one in §3.1 (using Figure 6 in place of Figure 4). \( \square \)

**3.3.** \( f : H^{4k}(T_{3k}) \to H^{4k}(M_{3k+1}) \{ -1 \} \) is surjective in grading \( 6k - 1 \) and has cokernel \( F \) in grading \( 6k + 1 \).

**Proof.** Consider the diagrams in Figure 7. First notice that the map \( a \) is surjective in gradings \( 6k - 1 \) and \( 6k + 1 \) because it is part of a fundamental sequence associated to the triple \( (N_{3k}, E_{3k}, M_{3k+1}) \), where \( E_{3k} \) (not shown!) is a diagram of the unknot obtained from either \( N_{3k} \) or \( M_{3k+1} \) by creating a positive crossing in the lower
right. In gradings $6k - 1$ and $6k + 1$, the maps $b, c$ form the kernel and cokernel map in a short exact sequence by the induction hypothesis. The maps $\delta, \varphi$ are also kernel and cokernel maps in short exact sequences in these gradings because they appear in the fundamental sequence associated to the triple $(M^0_{3k+1}, M_{3k+1}, M^1_{3k+1})$ of trivial links. The first statement is now clear, and the second statement is now reduced to showing that $d_{6k+1}$ is multiplication by two.

The map $t$ is an isomorphism in grading $6k + 1$ by the induction hypothesis, and the maps $\bar{t}, h$ form a short exact sequence in grading $6k + 1$ since they are part of the fundamental sequence associated to the triple $(M^0_{3k+1}, M_{3k+1}, M^1_{3k+1})$ of trivial links. Thus we reduce to showing that a generator of $H^{4k - 2}(D_{3k})\{-1\}$ maps via $e_{6k+1}$ to twice a generator of the cokernel of $h_{6k+1}$. This follows from Lemma 2.2.

3.4. $g_{6k-1} : H^{4k}(K_{3k+1})_{6k-1} \rightarrow H^{4k}(N_{3k+1})\{-1\}_{6k-1}$ is surjective.

**Proof.** First consider the diagrams in Figure 8. The diagrams are shifted “around the torus” to show symmetry, so that what would usually be rendered as the bottom/left two crossings now appears as the top/right two crossings. The diagram of $N_{3k+1}$ is not shown. The map $a$ induces an isomorphism in grading $6k - 1$ because it is part of a fundamental sequence surrounded by 0 in grading $6k - 1$. In grading $6k - 1$, the maps $h, f$ are the kernel and cokernel maps in a short exact

---

C.f. the second sequence in §1.11.
sequence by what we just proved in §3.3 (c.f. the first exact sequence in §1.12), and thus $h, f$ also form a short exact sequence in grading $6k - 1$ by symmetry. Consequently, it suffices to prove that there is $\alpha \in \text{Ker } f_{6k-1}$ such that $f_{6k-1}(\alpha)$ generates $H^{4k}(M_{3k})\{-1\}$ by symmetry.

To find such an $\alpha$ we should break up the large abelian group $H^{4k}(T_{3k})\{\sim\} = \mathbb{Z}^3$. There is a perfect way to do this, since we know by the induction hypothesis\(^\dagger\) that there is an exact sequence

\[
0 \longrightarrow H^{4k-1}(N_{3k})\{-1\} \overset{i}{\longrightarrow} H^{4k}(T_{3k}) \overset{\pi}{\longrightarrow} H^{4k}(K_{3k}) \longrightarrow 0
\]

\(^\dagger\text{C.f. the second exact sequence in §1.11.}\)
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\[ H_4^k(K_{3k+1}) = \mathbb{Z}_{6k} \oplus \mathbb{Z}_{6k-1} \]

\[ H_4^k(T_{3k}) = \mathbb{Z}_{6k} \oplus \mathbb{Z}_{6k-1}^2 \oplus \mathbb{Z}_{6k+1}^2 \]

\[ H_4^k(M_{3k+1}) = \mathbb{Z}_{6k} \oplus \mathbb{Z}_{6k}^2 \oplus \mathbb{Z}_{6k+1} \]

Fig. 8. The triple \((T_{3k}, K_{3k+1}, M_{3k+1})\) and its relationship with the triple \((T_{3k}, K_{3k+1}, M_{3k+1})\) of Figure 3.

(we are interested in this sequence only in grading \(6k - 1\)) of free abelian groups, and hence a (non-canonical) splitting

\[ H_4^k(T_{3k})_{6k-1} = i[H_4^{k-1}(N_{3k})\{1\}_{6k-1}] \oplus \langle \alpha \rangle, \tag{3.1} \]

where \(\alpha \in H_4^k(T_{3k})_{6k-1}\) is chosen so that \(\pi(\alpha)\) generates \(H_4^k(K_{3k})_{6k-1} = \mathbb{Z}\).

We claim that any such \(\alpha\) will be as desired. To see this, we first consider the map of triples in Figure 9. The induced map between the corresponding fundamental sequences yields an exact diagram:

\[
\begin{array}{cccc}
0 & 0 & 0 & 0 \\
H_4^k(K_{3k}) & \rightarrow & H_4^k(M_{3k+1}^0) & \{1\} \\
\pi & & \downarrow & \\
H_4^k(T_{3k}) & \rightarrow & H_4^k(M_{3k+1}) & \{1\} \\
\downarrow & f & \downarrow & j \\
H_4^{k-1}(N_{3k})\{1\} & \rightarrow & H_4^{k-1}(M_{3k+1}^1) & \{2\} \\
\downarrow & \alpha & \downarrow & \downarrow \\
0 & 0 & 0 & 0
\end{array}
\]
The map $a$ is easily seen to be surjective because it is part of the fundamental sequence associated to a triple $(N_{3k}, E_{3k}, M_{13k+1})$, where $E_{3k}$ (not shown!) is the diagram of the unknot obtained from either $N_{3k}$ or $M_{13k+1}$ by creating a positive crossing at the dotted circle in Figure 9. Furthermore, the map $l$ is an isomorphism in grading $6k - 1$ because

$$H^{4k}(M_{03k+1})\{-1\} = Z_{6k+1} \oplus Z_{6k+3}$$

vanishes in degree $6k - 1$. Since $f$ and $a$ are surjective in grading $6k - 1$, our $\alpha$ must be in the kernel of $f_{6k-1}$ because it is not in the image of $i$.

It remains only to prove that $f_{6k-1}(\alpha)$ generates $H^{4k}(M_{3k+1})\{-1\}_{6k-1} = Z$. To do this, we consider the “barred” version of Figure 9 shown in Figure 10. This time, the induced map between the corresponding fundamental sequences yields an
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Fig. 10. The triple $(\mathcal{M}_{3k+1}^0, \mathcal{M}_{3k+1}^1, \mathcal{M}_{3k+1}^1)$ of trivial links (right) and its relationship with the triple $(T_{3k}, K_{3k+1}, M_{3k+1})$ of Figure 3 (left)

exact diagram:

\[
\begin{array}{cccc}
0 & \rightarrow & H^4_k(K_{3k}) & \rightarrow \rightarrow 0 \\
& \downarrow & \downarrow & \downarrow \\
H^4_k(T_{3k}) & \rightarrow & H^4_k(M_{3k+1}) \{ -1 \} & \rightarrow \rightarrow \\
& \downarrow i & \downarrow \pi & \downarrow d \\
H^{4k-1}(N_{3k}) \{ -1 \} & \rightarrow & H^{4k-1}(M_{3k+1}) \{ -2 \} & \rightarrow \rightarrow 0 \\
& \downarrow \pi & \downarrow d & \\
& 0 & H^{4k-1}(M_{3k+1}) \{ -1 \} & \\
& \downarrow & \downarrow & 0
\end{array}
\]

The kernel of $e$ is equal to the image of $d$. By applying Lemma 2.2 in Figure 10 we find that $d$ and $\pi$ have the same image, so commutativity implies that

\[i[H^{4k-1}(N_{3k}) \{ -1 \}] \subseteq \text{Ker } f_{6k-1},\]

hence $f(\alpha)$ must generate $H^{4k}(M_{3k+1}) \{ -1 \}$ because $f_{6k-1}$ is surjective and we have the splitting (3.1). \qed
3.5. \( f : H^{4k+1}(T_{3k+1}) \rightarrow H^{4k+1}(M_{3k+2})\{-1\} \) is zero.

Proof. Consider the diagrams in Figure 11. The commutativity of the bottom square implies that \( f_{6k+1} = 0 \) because \( h_{6k+1} \) is either multiplication by 2 or an isomorphism on the free part (c.f. the second sequence in §1.12) and the codomain of \( f \) is torsion-free (so certainly \( f_{6k+1} \) kills the torsion summand if it is present). It remains only to prove that \( f_{6k+3} = 0 \). The map \( k_{6k+3} \) is an isomorphism by the induction hypothesis (again see the second sequence in §1.12), so it suffices to show that \( f_{6k+3} = 0 \). The map \( k_{6k+3} \) is an isomorphism (see the first sequence in §1.12),
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and the map \(a_{6k+3}\) is injective (examine the fundamental sequence associated to the triple \((M^0_{3k+2}, M_{3k+2}, M^1_{3k+2})\) of trivial links), so it suffices to show that \((\overline{e}a)_{6k+3}\) is zero. The maps \(d_{6k+3}\) and \(\overline{e}a_{6k+3}\) are the kernel and cokernel maps in a short exact sequence (namely, the fundamental sequence for the triple \((M^0_{3k+1}, M^0_{3k+2}, M^1_{3k+1})\) of trivial links), so it suffices to show that \(e_{6k+3}\) and \(d_{6k+3}\) have the same image, which can be done with Lemma 2.2.

3.6. \(g_{6k+3}: H^{4k+2}(K_{3k+2})_{6k+3} \rightarrow H^{4k+2}(N_{3k+2})\{-1\}_{6k+3}\) is multiplication by 2.

Proof. Consider the diagrams in Figure 12. The map \(t\) is an isomorphism by what

we just proved in §3.5 (see the first sequence in §1.13). The maps \(a, b\) are the kernel and cokernel maps in a short exact sequence (the fundamental sequence associated to the triple of trivial links \((N^0_{3k+2}, N_{3k+2}, N^1_{3k+2})\)), so it suffices to prove that \(g_{6k+3}\) takes a generator of \(H^{4k+1}(M_{3k+2})\{-1\}_{6k+3}\) to two times a generator of the cokernel of \(a_{6k+3}\), which follows from Lemma 2.2.

\[\text{Fig. 12. The triple } (N^0_{3k+2}, N_{3k+2}, N^1_{3k+2}) \text{ (right) of trivial links and its relationship with the triple } (T_{3k+1}, K_{3k+2}, M_{3k+2}) \text{ of Figure 3}\]

\[\text{we just proved in §3.5 (see the first sequence in §1.13). The maps } a, b \text{ are the kernel and cokernel maps in a short exact sequence (the fundamental sequence associated to the triple of trivial links } (N^0_{3k+2}, N_{3k+2}, N^1_{3k+2})\text{), so it suffices to prove that } g_{6k+3} \text{ takes a generator of } H^{4k+1}(M_{3k+2})\{-1\}_{6k+3} \cong \mathbb{Z} \text{ to two times a generator of the cokernel of } a_{6k+3}, \text{ which follows from Lemma 2.2.}\]

\[\text{\footnote{Technically, } d \text{ and } e \text{ are cobordisms from the two component trivial link to the three component trivial link, but there is one component on which they clearly agree, so we can disregard that component everywhere.}}\]
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