Sheaf Theory

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Introduction

The aim of this document is to give a concise treatment of the “classical” theory of sheaves on topological spaces, with the following philosophical guidelines:

(1) Appeal to standard facts about limits in the category of sets whenever possible, e.g. exactness of filtered direct limits.
(2) When possible, formulate constructions and results about sheaves on a topological space so that make sense on a site or topos with little modification.
(3) Appeal to general homological algebra whenever possible.
(4) Follow Godement [G] closely as long as this does not conflict with the other guidelines.

Regarding (1) and (2), my feeling is that the general philosophy of topos theory is that a topos is a category that behaves like the category of sets, so, for example, abelian group objects in a topos should behave like abelian groups, etc. Almost any theorem about sets that can be stated and proved using category theoretic language should also hold true in a topos. In particular, the category of sheaves on a topological space should be a lot like the category of sets. This philosophy is usually manifested by bootstrapping up from the category of presheaves, which is “very much” like the category of sets from the perspective of category theory. A related philosophical point is that, since our object of study is the category of sheaves on a topological space, we should try to phrase our constructions intrinsically using this category, rather than referring back to the topological space, its category of open sets, etc. The manifestation of this is the usage of the representable sheaf associated to an open set, rather than the open set itself.

Regarding (3), my feeling is that basically every spectral sequence arises as a special case of the Grothendieck spectral sequence. One should never have to write down a double complex out of the blue and analyse what its horizontal and vertical cohomology mean, etc. Grothendieck already did that once and for all.
CHAPTER I

Sheaves

1. Open sets

For a topological space $X$, let $\textbf{Ouv}(X)$ denote the category of open subsets of $X$, where the only morphisms are inclusions. The category $\textbf{Ouv}(X)$ has arbitrary direct limits given by the formula

$$\lim_{\longrightarrow} (f : C \to \textbf{Ouv}(X)) = \bigcup_{C \in C} f(C)$$

and inverse limits over any category with finitely many objects, given by the formula

$$\lim_{\longleftarrow} (f : C \to \textbf{Ouv}(X)) = \bigcap_{C \in C} f(C).$$

In particular, $\emptyset \in \textbf{Ouv}(X)$ is initial (the direct limit of the empty functor, or the empty union) and $X \in \textbf{Ouv}(X)$ is final (the inverse limit of the empty functor).

2. Presheaves

A presheaf on a topological space $X$ is a functor

$$\mathcal{F} : \textbf{Ouv}(X)^{\text{op}} \to \textbf{Ens}$$

$$U \mapsto \mathcal{F}(U).$$

Presheaves on $X$ form a category $\textbf{PSh}(X)$ where a morphism of presheaves is a natural transformation of functors. For $U \in \textbf{Ouv}(X)$, elements of $\mathcal{F}(U)$ are called sections of $\mathcal{F}$ over $U$. For an inclusion of open sets $V \subseteq U$ and $s \in \mathcal{F}(U)$ we often write $s|_V$ for the image of $s$ under the morphism $\mathcal{F}_{V \subseteq U} : \mathcal{F}(U) \to \mathcal{F}(V)$. We refer to these morphisms as restriction morphisms. For $U \in \textbf{Ouv}(X)$, the functor

$$\textbf{PSh}(X) \to \textbf{Ens}$$

$$\mathcal{F} \mapsto \mathcal{F}(U)$$

is often denoted $\Gamma(U, \_ )$. We sometimes write $\Gamma(U, \mathcal{F})$ in lieu of $\mathcal{F}(U)$, especially when $U = X$, in which case $\Gamma(X, \_ )$ has a special name: the global sections functor.

The category $\textbf{PSh}(X)$, being defined as a category of functors to $\textbf{Ens}$, inherits various properties from $\textbf{Ens}$. For example, $\textbf{PSh}$ has all small direct and inverse limits. These limits are formed “objectwise.” For example, for a direct limit system
$i \mapsto \mathcal{F}_i$ in $\mathbf{PSh}(X)$, the direct limit presheaf $\mathcal{F} = \lim_{\longrightarrow} \mathcal{F}_i$ is given by

$$\mathcal{F}(U) = \lim_{\longrightarrow} \mathcal{F}_i(U).$$

(2.1)

In particular, notice that the section functors

$$\Gamma(U, \_): \mathbf{PSh}(X) \rightarrow \mathbf{Ens}$$

commute with all limits.

3. Representable presheaves

The prototypical presheaf is the representable presheaf $h_U$ associated to $U \in \mathbf{Ouv}(X)$, defined by $h_U(V) = \text{Hom}_{\mathbf{Ouv}(X)}(V, U)$. By abuse of notation, we often write $U$ instead of $h_U$. Formation of $h_U$ is functorial in $U$ and defines a functor $h: \mathbf{Ouv}(X) \rightarrow \mathbf{PSh}(X)$ called the Yoneda functor.

**Lemma 3.1.** The Yoneda functor $h$ is fully faithful and commutes with arbitrary inverse limits. There is a bijection

$$\text{Hom}_{\mathbf{PSh}(X)}(h_U, \mathcal{F}) = \mathcal{F}(U)$$

natural in $U \in \mathbf{Ouv}(X)$ and $\mathcal{F} \in \mathbf{PSh}(X)$.

**Proof.** This is a straightforward variant of Yoneda’s Lemma. $\square$

**Remark 3.2.** The Yoneda functor does not generally commute with direct limits, or even with finite direct sums. In particular, the presheaves $h_U \oplus h_V$ and $h_{U \cup V}$ do not generally coincide. Indeed, for $W \in \mathbf{Ouv}(X)$, by definition of direct sums, we have

$$\text{Hom}_{\mathbf{PSh}(X)}(h_U \oplus h_V, h_W) = \text{Hom}_{\mathbf{PSh}(X)}(h_U, h_W) \coprod \text{Hom}_{\mathbf{PSh}(X)}(h_V, h_W) = h_W(U) \coprod h_W(V) = \text{Hom}_{\mathbf{Ouv}(X)}(U, W) \coprod \text{Hom}_{\mathbf{Ouv}(X)}(V, W),$$

while

$$\text{Hom}_{\mathbf{PSh}(X)}(h_{U \cup V}, h_W) = \text{Hom}_{\mathbf{Ouv}(X)}(U \cup V, W).$$

These two sets do not generally have the same cardinality. Now, we could remedy this particular failure, as some authors do, by using a variant of $\mathbf{Ouv}(X)$ where we consider not only open sets of $X$, but all maps $f: Y \rightarrow X$ which are local homeomorphisms. From our point of view, there is no reason to do this. We are only using $\mathbf{Ouv}(X)$ as a stepping stone to get to another category; we do not particularly care about the category $\mathbf{Ouv}(X)$ or the behaviour of its Yoneda functor.
4. Generators

Recall that a set \( G \) of objects of a category \( C \) is called a \textit{generating set} if, for any parallel \( C \) morphisms \( f, g : C \to D \), the following are equivalent:

1. \( f = g \)
2. \( fh = gh \) for every \( G \in G \) and every \( h \in \text{Hom}_C(G, C) \). That is, \( f_*, g_* : \text{Hom}_C(G, C) \to \text{Hom}_C(G, D) \) are equal for every \( G \in G \).

If \( C \) has direct sums, then \( G \) is a generating set iff the “evaluation map”

\[
\bigoplus_{G \in G} \bigoplus_{\text{Hom}_C(G, C)} G \to C
\]

is an epimorphism for every \( C \in C \). An object \( G \) of \( C \) is called a \textit{generator} if \( \{G\} \) is a generating set. Dually, \( G \) is called a \textit{cogenerating set} if \( G \) is a generating set for \( C^{\text{op}} \).

The paradigm example here is: any nonempty set is a generator of \text{Ens}.

**Lemma 4.1.** The representable presheaves \( \{h_U : U \in \text{Ouv}(X)\} \) form a generating set for \( \text{PSh}(X) \).

**Proof.** Two morphisms \( f, g : \mathcal{F} \to \mathcal{G} \) in \( \text{PSh}(X) \) are equal iff \( f(U), g(U) : \mathcal{F}(U) \to \mathcal{G}(U) \) are equal for every \( U \in \text{Ouv}(X) \). On the other hand, \( f(U), g(U) \) are identified with

\[
f_*, g_* : \text{Hom}_{\text{PSh}(X)}(h_U, \mathcal{F}) \to \text{Hom}_{\text{PSh}(X)}(h_U, \mathcal{G})
\]

by (3.1). \( \square \)

5. Sheaves

Let \( \mathcal{U} = \{U_i : i \in I\} \) be a family of open subsets of \( X \) with union \( U \) (a cover of \( U \)). A presheaf \( \mathcal{F} \) is \textit{separated} for \( \mathcal{U} \) if the product of the restriction maps

\[
\mathcal{F}(U) \to \prod_{i \in I} \mathcal{F}(U_i)
\]

is injective (that is, if a section of \( \mathcal{F} \) is completely determined by its restrictions to the open sets in a cover). \( \mathcal{F} \) has the \textit{gluing property} (or \textit{descent property}) with respect to \( \mathcal{U} \) if, for any set of sections \( \{s_i \in \mathcal{F}(U_i) : i \in I\} \) with \( s_i|_{U_{ij}} = s_j|_{U_{ij}} \) for all \( i, j \in I \), there is a section \( s \in \mathcal{F}(U) \) with \( s|_{U_i} = s_i \) for all \( i \in I \). \( \mathcal{F} \) is \textit{separated} (resp. has the gluing property) if it is separated (resp. has the gluing property) for every such family \( \mathcal{U} \). A presheaf is a \textit{sheaf} if it is separated and it has the gluing property.
We can sum this up by saying that $\mathcal{F}$ is a sheaf if and only if
\[
\mathcal{F}(U) \to \prod_{i \in I} \mathcal{F}(U_i) \Rightarrow \prod_{(i,j) \in I \times I} \mathcal{F}(U_{ij})
\]
is an equalizer diagram of sets for any $U \in \text{Ouv}(X)$ and any cover $U = \{U_i : i \in I\}$ of $U$.

**Remark 5.1.** If $\mathcal{F}$ is a separated presheaf, then the set $\mathcal{F}(\emptyset)$ is punctual (has one element) because the empty set is covered by the empty cover and the empty product in the category of sets is punctual.

The following criterion is often useful:

**Lemma 5.2.** For $\mathcal{F} \in \text{PSh}(X)$, the following are equivalent:

1. $\mathcal{F}$ is a sheaf.
2. $\mathcal{F}$ commutes with filtered inverse limits and has the sheaf property for covers of the form $U = \{U, V\}$. That is,
   a. For any filtered union $U = \bigcup_i U_i$ of open sets of $X$, the natural map
      
      $\mathcal{F}(U) \to \varprojlim \mathcal{F}(U_i)$

      is an isomorphism, and
   b. For any $U, V \in \text{Ouv}(X)$, the diagram
      
      $\mathcal{F}(U \cup V) \to \mathcal{F}(U) \times \mathcal{F}(V) \Rightarrow \mathcal{F}(U \cap V)$

      is an equalizer diagram.

**Proof.** (1) $\Rightarrow$ (2). The natural map $\mathcal{F}(U) \to \varprojlim \mathcal{F}(U_i)$ is monic because $\mathcal{F}$ is separated for the cover $\{U_i\}$ of $U$ and epic because $\mathcal{F}$ has the gluing property for this cover.

(2) $\Rightarrow$ (1). Let $U = \{U_i : i \in I\}$ be a family of open subsets of $X$ with union $U$. To prove $\mathcal{F}$ is separated for $U$, suppose $s, t \in \mathcal{F}(U)$ have the same restriction to every $U_i$. By Zorn’s Lemma and the fact that $\mathcal{F}$ has the sheaf property for filtered unions, there is a maximal $J \subseteq I$ such that $s$ and $t$ have the same restriction to $V := \bigcup_{i \in J} U_i$. If $J \neq I$, then there is an $i \in I \setminus J$ and, using the separation of $\mathcal{F}$ for the cover $\{U_i, V\}$ we find that $s$ and $t$ agree on $V \cup U_i$, contradicting maximality of $J$. The gluing property is proved similarly. \[\square\]

**Corollary 5.3.** If $X$ is a noetherian topological space, then $\mathcal{F} \in \text{PSh}(X)$ is a sheaf iff

$\mathcal{F}(U \cup V) \to \mathcal{F}(U) \times \mathcal{F}(V) \Rightarrow \mathcal{F}(U \cap V)$

is an equalizer diagram for every $U, V \in \text{Ouv}(X)$.

**Proof.** By definition, $X$ is noetherian iff any filtered direct limit system in $\text{Ouv}(X)$ is essentially constant, so any presheaf $\mathcal{F}$ trivially commutes with filtered inverse limits. \[\square\]
A morphism of sheaves is defined so that the category $\text{Sh}(X)$ of sheaves on $X$ forms a full subcategory of the category $\text{PSh}(X)$.

**Lemma 5.4.** Every representable presheaf $h_U$ is a sheaf and
\[
\{h_U \in \text{Sh}(X) : U \in \text{Ouv}(X)\}
\]
is a generating set for $\text{Sh}(X)$.

**Proof.** The first statement is straightforward, and the second is obvious from (4.1). □

### 6. Stalks

The **stalk** of a presheaf $\mathcal{F}$ at a point $x \in X$ is
\[
\mathcal{F}_x := \lim_{\rightarrow U \ni x} \mathcal{F}(U).
\]
The direct limit is taken over open neighborhoods of $x$ and restriction maps between them. Given a section $s \in \mathcal{F}(U)$ and a point $x \in U$ we let $s_x \in \mathcal{F}_x$ denote the image of $s$ under the natural map
\[
\mathcal{F}(U) \to \lim_{\rightarrow V \ni x} \mathcal{F}(V) = \mathcal{F}_x.
\]
Observe that a presheaf $\mathcal{F}$ is separated if and only if, for any object $U$ of $\text{Top}(X)$ and any $s, t \in \mathcal{F}(U)$, we have $s = t$ if and only if $s_x = t_x$ for all $x \in U$. A morphism of presheaves $f : \mathcal{F} \to \mathcal{G}$ induces a morphism $f_x : \mathcal{F}_x \to \mathcal{G}_x$ on stalks for each $x \in X$ by taking the direct limit of the morphisms $f(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ over all open neighborhoods $U$ of $x$.

For example, the stalk $h_{U,x}$ of the representable sheaf $h_U$ at $x$ is a punctual set if $x \in U$ and empty otherwise.

In contrast to the situation for presheaves, the stalks of a map $f : \mathcal{F} \to \mathcal{G}$ of sheaves are a good reflection of $f$:

**Lemma 6.1.** For a morphism $f : \mathcal{F} \to \mathcal{G}$ in $\text{Sh}(X)$, the following are equivalent:

1. $f$ is a monomorphism in $\text{Sh}(X)$.
2. $f(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ is a monomorphism in $\text{Ens}$ for every $U \in \text{Ouv}(X)$.
3. $f_x : \mathcal{F}_x \to \mathcal{G}_x$ is a monomorphism in $\text{Ens}$ for every $x \in X$.

The following are equivalent:

4. $f$ is an epimorphism in $\text{Sh}(X)$.
5. $f_x : \mathcal{F}_x \to \mathcal{G}_x$ is an epimorphism in $\text{Ens}$ for every $x \in X$.

The following are equivalent:

6. $f$ is an isomorphism in $\text{Sh}(X)$.
7. $f(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ is an isomorphism in $\text{Ens}$ for every $U \in \text{Ouv}(X)$.
8. $f_x : \mathcal{F}_x \to \mathcal{G}_x$ is an isomorphism in $\text{Ens}$ for every $x \in X$.
(3) \implies (2) If \( f(U)(s) = f(U)(t) \), then \( f_x(s(x)) = f_x(t(x)) \) for every \( x \in U \). Since the maps on stalks are monic, we have \( s_x = t_x \) for all \( x \in U \), hence \( s = t \) because a sheaf is separated, so \( f(U) \) is monic.

(2) \implies (3) A filtered direct limit of monic maps of sets is monic.

(2) \iff (1) By definition, \( f \) is monic in \( \text{Sh}(X) \) iff \( f^*: \text{Hom}_{\text{Sh}(X)}(H, F) \to \text{Hom}_{\text{Sh}(X)}(H, G) \) is a monomorphism of sets for every \( H \in \text{Sh}(X) \). In any category, it suffices to check this for \( H \) running over a generating set, so by (4.1), we conclude that \( f \) is monic iff \( f^*: \text{Hom}_{\text{Sh}(X)}(h_U, F) \to \text{Hom}_{\text{Sh}(X)}(h_U, G) \) is monic for every \( U \in \text{Ouv}(X) \). But this map of sets is identified with the map \( f_x \) by Yoneda (3.1).

(5) \implies (4) Suppose \( g_1, g_2: G \to \mathcal{H} \) satisfy \( g_1f = g_2f \). Then \( g_{1,x}f_x = g_{2,x}f_x \) for every \( x \in X \), hence \( g_{1,x} = g_{2,x} \) for every \( x \in X \) because \( f \) is surjective on stalks. We conclude \( g_1 = g_2 \) using separation for \( H \).

(4) \implies (5) Since \( f \) is epic, by definition, the map

\[ f^*: \text{Hom}_{\text{Sh}(X)}(\mathcal{G}, \mathcal{H}) \to \text{Hom}_{\text{Sh}(X)}(\mathcal{G}, \mathcal{H}) \]

is a monomorphism of sets for every \( \mathcal{H} \in \text{Sh}(X) \). If \( \mathcal{H} = x_*A \) (c.f. (3) below) for \( A \in \text{Ens} \), then this monomorphism is identified with

\[ f^*_x: \text{Hom}_{\text{Ens}}(\mathcal{G}_x, A) \to \text{Hom}_{\text{Ens}}(\mathcal{G}_x, A). \]

Since this is a monomorphism for every \( x, A \), \( f_x \) is an epimorphism in \( \text{Ens} \) for every \( x \).

(8) \implies (7) Consider a \( t \in \mathcal{G}(U) \). Since the maps on stalks are epic, there are neighborhoods \( U_x \) of every \( x \in U \) and sections \( s(x) \in \mathcal{F}(U_x) \) such that \( f(U_x)(s(x)) = t|_{U_x} \). After restricting to \( U \cap U_x \), we may assume every \( U_x \) is contained in \( U \), so that the \( U_x \) form a cover of \( U \). Since \( f \) commutes with restriction maps, we have

\[ f(U_x \cap U_y)(s(x)|_{U_x \cap U_y}) = t|_{U_x \cap U_y} = f(U_x \cap U_y)(s_y|_{U_x \cap U_y}). \]

Since each \( f(U_x \cap U_y) \) is monic by ((3) \implies (2) above), the \( s(x) \) agree on overlaps, so they glue, by the sheaf property for \( \mathcal{F} \), to an \( s \in \mathcal{F}(U) \). It then follows that \( f(U)(s) = t \) because this equality can be checked on each \( U_x \) by separation for \( \mathcal{G} \).

(6) \iff (7) Obvious.

(7) \iff (8) A map of sets is an isomorphism iff it is both epic and monic, so this follows from the previous results.

\[ \square \]
7. Sheafification

The main goal of this section is to prove that $\text{Sh}(X)$ is a reflexive subcategory of $\text{PSh}(X)$, i.e. the inclusion functor $\text{Sh}(X) \rightarrow \text{PSh}(X)$ has a left adjoint, called sheafification.

For a presheaf $\mathcal{F}$, we will construct a sheaf $\mathcal{F}^+$ (the sheafification of $\mathcal{F}$) and a morphism of presheaves $\mathcal{F} \rightarrow \mathcal{F}^+$, such that any morphism of presheaves $\mathcal{F} \rightarrow \mathcal{G}$ with $\mathcal{G}$ a sheaf will factor uniquely through $\mathcal{F} \rightarrow \mathcal{F}^+$. We set $\mathcal{F}^+(U)$ equal to the set of all $t = (t(x))_{x \in U} \in \prod_{x \in U} \mathcal{F}_x$ satisfying the following local coherence condition: There are a cover $U = \{U_i\}$ of $U$ and sections $s(i) \in \mathcal{F}(U_i)$ such that $t(x) = s(i)_x \in \mathcal{F}_x$ for all $x \in U_i$.

The presheaf $\mathcal{F}^+$ has natural restriction maps given by restriction of functions and it is easy to check that $\mathcal{F}^+$ is a sheaf. Indeed, we may actually glue functions together in the usual sense if they agree on overlaps. The construction is functorial because a map of presheaves $f : \mathcal{F} \rightarrow \mathcal{G}$ induces maps of stalks $f_x : \mathcal{F}_x \rightarrow \mathcal{G}_x$ which we may use to define maps (pullback) between the sets of functions $f^+(U) : \mathcal{F}^+(U) \rightarrow \mathcal{G}^+(U)$ as above. The map $f^+(U)$ takes a function $t$ satisfying the local coherence condition to another such function because if $s(i) \in \mathcal{F}(U_i)$ witness it for $t$ then $f(U_i)(s(i)) \in \mathcal{G}(U_i)$ witness it for $f^+(U)(t)$.

Notice that the map $\mathcal{F} \rightarrow \mathcal{F}^+$ induces an isomorphism on every stalk. It then follows from (6.1) that if $\mathcal{F}$ is already a sheaf, then the natural map $\mathcal{F} \rightarrow \mathcal{F}^+$ is an isomorphism. In particular, if $\mathcal{F} \rightarrow \mathcal{G}$ is a morphism of presheaves with $\mathcal{G}$ a sheaf, then we obtain the claimed factorization $\mathcal{F} \rightarrow \mathcal{F}^+ \rightarrow \mathcal{G}^+ \cong \mathcal{G}$. This factorization is unique because one can check easily that a morphism of sheaves is determined by its values on stalks.

The fundamental facts about sheafification are summed up in the following

**Theorem 7.1.** The inclusion functor $\text{Sh}(X) \rightarrow \text{PSh}(X)$ commutes with arbitrary inverse limits and has a left adjoint: the sheafification functor

$$\text{PSh}(X) \rightarrow \text{Sh}(X)$$

$$\mathcal{F} \mapsto \mathcal{F}^+.$$ 

The adjunction morphism $\mathcal{F}^+ \rightarrow \mathcal{F}$ is an isomorphism for any sheaf $\mathcal{F}$.

**Proof.** We proved the statements about adjointness in the discussion above. Saying that the forgetful functor commutes with inverse limits means that an inverse limit of sheaves (taken in the category of presheaves) is already a sheaf, which can be checked directly from the fact that the sheaf property is defined in terms of a certain diagram being an equalizer diagram (a certain kind of inverse limit) and the fact that “inverse limits commute amongst themselves”. □
Corollary 7.2. The category $\text{Sh}(X)$ has all small direct and inverse limits. The sheafification functor commutes with arbitrary direct limits and finite inverse limits.

Proof. We have already mentioned that $\text{Sh}(X)$ has all inverse limits and that these coincide with the inverse limit taken in $\text{PSh}(X)$. Using the adjointness properties of the sheafification functor, it follows from a formal exercise in category theory that the direct limit of a direct system of sheaves $i \mapsto F_i$ can be obtained by first taking the direct limit in the category of presheaves (c.f. (2.1)), then applying the sheafification functor to the result.

Sheafification commutes with arbitrary direct limits simply because it is a right adjoint (formal category theory). To prove that it commutes with finite inverse limits, let $f : C \to \text{PSh}(X)$ be a functor with $C$ a finite category (finitely many morphisms and finitely many objects). We must check that the natural map

$$\left( \lim_{\leftarrow} (f : C \to \text{PSh}(X)) \right)^+ \to \lim_{\leftarrow} (f^+ : C \to \text{Sh}(X))$$

is an isomorphism. By (6.1), it is enough to check this on stalks. On the other hand, stalks commute with sheafification, so this map is identified with the natural map

$$\lim_{U \ni x} \lim_{C \in C} f(C)(U) \to \lim_{C \in C} \lim_{U \ni x} f(C)(U),$$

which is an isomorphism because finite inverse limits commute with filtered direct limits in the category of sets. \(\square\)

For example, the representable sheaf $h_X$ is the terminal object of $\text{Sh}(X)$ (and $\text{PSh}(X)$). Indeed, $h_X(U)$ is punctual (terminal in $\text{Ens}$) for every $U \in \text{Ouv}(X)$ so there is clearly a unique map $F \to h_X$ as there is a unique map $F(U) \to h_X(U)$ for every $U \in \text{Ens}$.

8. Espaces étalé

Let $F$ be a presheaf on a space $X$. We define a topological space $\text{Et} F$, called the espace étalé of $F$ and a continuous map $\pi : \text{Et} F \to X$ as follows: As a set, we let

$$\text{Et} F := \coprod_{x \in X} F_x$$

be the disjoint union of the stalks of $F$. The set $\text{Et} F$ is given the topology where a basic open subset is a set of the form

$$U_s := \{ s_x \in F_x \subseteq \text{Et} X : x \in U \},$$

where $U \in \text{Ouv}(X)$, $s \in F(U)$. To show that these sets form the basic opens for a topology we should show that every $s_x \in \text{Et} F$ is in one of the $U_s$ (this is clear) and
that an intersection $U_s \cap V_t$ of such basic opens is again such a basic open. Indeed, 
the set 
\[ W := \{ x \in U \cap V : s_x = t_x \} \]
is open in $U \cap V$ (hence also in $X$) because equality at a stalk by definition implies 
equality on a neighborhood, and we have $U_s \cap V_t = W_r$, where $r := s|_W$, or $r := t|_W$ 
(note that $s, t$ have the same stalk everywhere in $W$, but they might not be equal 
in $\mathcal{F}(W)$ because $\mathcal{F}$ might not be a sheaf). Define $\pi : \text{Et} \mathcal{F} \to X$ so that $\pi^{-1}(x) = 
\mathcal{F}_x \subseteq \text{Et} \mathcal{F}$ for each $x \in \mathcal{F}$. The map $\pi$ is continuous because 
\[ \pi^{-1}(U) = \bigcup_{V \subseteq U, s \in \mathcal{F}(V)} V_s \]
for any $U \in \text{Ouv}(X)$.

Given any $U \in \text{Ouv}(X)$ and any $s \in \mathcal{F}(U)$, we define a map $\phi(U)(s) : U \to \text{Et} \mathcal{F}$ by $\phi(U)(s)(x) := s_x \in \mathcal{F}_x \subseteq \text{Et} \mathcal{F}$. This map clearly 
sections $\pi : \text{Et} \mathcal{F} \to X$ and is continuous because 
\[ \phi(U)(s)^{-1}(V_t) = \{ x \in U : x \in V \text{ and } s_x = t_x \} \]
\[ = \{ x \in U \cap V : s_x = t_x \} \]
is open in $U$ as we argued above. Our maps 
\[ \phi(U) : \mathcal{F}(U) \to \{ s \in \text{Hom}_{\text{Top}}(U, \text{Et} \mathcal{F}) : \pi s = \text{Id}_U \} \]
are clearly compatible with restricting to smaller $U$, hence define a morphism $\phi$ from 
$\mathcal{F}$ to the sheaf of local sections of $\pi : \text{Et} \mathcal{F} \to X$ (see (11.2)).

**Lemma 8.1.** For any space $X$ and presheaf $\mathcal{F}$ on $X$, the map $\phi$ from $\mathcal{F}$ to 
the sheaf of local sections of the espace étalé $\pi : \text{Et} \mathcal{F} \to X$ identifies the latter sheaf 
with the sheafification of $\mathcal{F}$. In particular, if $\mathcal{F}$ is a sheaf, then $\mathcal{F}$ is isomorphic to 
the sheaf of local sections of $\pi : \text{Et} \mathcal{F} \to X$.

**Proof.** By the stalk criterion (6.1) it suffices to prove that $\phi$ is an isomorphism 
on stalks. To see that $\phi_x$ is surjective, consider a local section $f : U \to \text{Et} \mathcal{F}$ of $\pi$ on 
a neighborhood $U$ of $x \in U$. We must prove that $f|V$ agrees with $\phi(V)(t) : V \to \text{Et} \mathcal{F}$ 
for some $t \in \mathcal{F}(V)$ for some neighborhood $V$ of $x$ in $U$. By definition of the stalk, 
we can find some neighborhood $W$ of $x$ in $U$ and some $r \in \mathcal{F}(W)$ such that 
$r_x = f(x) \in \mathcal{F}_x$. Since $f$ is continuous, the set 
\[ V := f^{-1}W_r := \{ y \in W \cap U : r_y = f(y) \} \]
is open in $U$ and clearly contains $x$, hence $V$ and $t := r|V$ are as desired.

To see that $\phi_x$ is injective, suppose $s, t \in \mathcal{F}(U)$ for some neighborhood $U$ of $x$ 
and the functions $\phi(U)(s), \phi(U)(t) : U \to \text{Et} X$ agree on a neighborhood of $X$. We 
have to show that $s_x = t_x \in \mathcal{F}_x$. But this is obvious from the fact that $\phi(U)(s)$ and 
$\phi(U)(t)$ agree at $x$ because 
\[ s_x = \phi(U)(s)(x) = \phi(U)(t)(x) = t_x. \]
Evidently then, the theory of the espace étalé gives another means of sheafifying a presheaf. It also gives a means of translating various sheaf-theoretic questions into questions of topology.

**Remark 8.2.** Given a map $p : W \to X$, the espace étalé of the sheaf $\mathcal{F}$ of local sections of $p$ is a topological space $\pi : \text{Et } \mathcal{F} \to X$ over $X$ that generally has little to do with the space $W$. One only knows that the sheaf of local sections of $\pi$ is isomorphic to the sheaf of local sections of $p$. The map $\pi$ is built solely from the sheaf of local sections of $p$, but one cannot hope to recover the map $p$ from this sheaf. Another very special property of the espace étalé $\pi$ is that the map $[f] \mapsto f(x)$ identifies germs of local sections of $\pi$ near $x$ with the fiber $\pi^{-1}(x)$. For a general space $p : W \to X$ over $X$, the map $[f] \mapsto f(x)$ from germs of local sections of $p$ near $x$ to the fiber $p^{-1}(x)$ is neither surjective nor injective.

The formation of the espace étalé is functorial in $\mathcal{F}$, so we may think of $\text{Et}$ as a functor

$$\text{Et} : \text{Sh}(X) \to \text{Top}/X.$$ (It is actually defined on $\text{PSh}(X)$, but we saw above that the espace étalé of a presheaf is naturally isomorphic to the espace étale of its sheafification.) Formation of the espace étalé is also functorial in $X$, in a sense that we will make precise in Section 8 once we discuss the functorial dependence of the category of sheaves $\text{Sh}(X)$ on $X$.

9. Sieves

The following categorical constructions are used to define sheaves in more general settings. The constructions are not needed elsewhere in any serious way, though some of the terminology will be natural in our study of Čech cohomology in Chapter VIII.

Given a category $\mathcal{C}$, a **sieve** in $\mathcal{C}$ is a full subcategory $\mathcal{U} \subseteq \mathcal{C}$ such that, for any $U \in \mathcal{U}$ and any $V \in \mathcal{C}$,

$$\text{Hom}_\mathcal{C}(V, U) \neq \emptyset \implies V \in \mathcal{U}.$$ Let $\text{Crib}(\mathcal{C})$ denote the set of sieves in $\mathcal{C}$. $\text{Crib}(\mathcal{C})$ has a natural ordering by inclusion. A set of objects $\mathcal{G} \subseteq \mathcal{C}$ determines a sieve—namely the sieve consisting of all $C \in \mathcal{C}$ which admit a morphism to some $G \in \mathcal{G}$. For a sieve $\mathcal{U}$ and subset $\mathcal{G} \subseteq \mathcal{U}$, we say that $\mathcal{G}$ is a generating set for $\mathcal{U}$ if the sieve generated by $\mathcal{G}$ (which is clearly contained in $\mathcal{U}$) is equal to $\mathcal{U}$. For an object $X \in \mathcal{C}$, we set $\text{Crib}(X) := \text{Crib}(\mathcal{C}/X)$. The **trivial sieve** is the sieve in $\text{Crib}(X)$ generating by $\{\text{Id}_X\}$ (i.e. the entire category $\mathcal{C}/X$).
Given a sieve \( U \in \text{Crib}(Y) \), and \( \mathbf{C} \) morphisms \( Y \to Z, W \to Z \), let \( W \times_Z U \in \text{Crib}(W) \) be the sieve consisting of all \( V \to W \) fitting into a commutative diagram

\[
\begin{array}{ccc}
V & \longrightarrow & U \\
| & & | \\
\downarrow & & \downarrow \\
W & \longrightarrow & Z
\end{array}
\]

for some \((U \to Y) \in U\).

For an object \( Y \in \mathbf{C} \), a sieve \( U \in \text{Crib}(Y) \), and a family of sieves

\[
V = \{ V(U) \in \text{Crib}(U) : U \in U \},
\]

let \( V/U \in \text{Crib}(Y) \) be the sieve consisting of all morphisms \( W \to Y \) which can be factored \( W \to V \to U \to Y \) with \((U \to Y) \in U\) and \((V \to U) \in V(U)\).

For a topological space \( X \), and \( U \in \text{Ouv}(X) \), notice that \( U \in \text{Crib}(U) \) is nothing more than a family of open subsets of \( U \) closed under passage to smaller open subsets. We declare a sieve \( U \in \text{Crib}(U) \) to be a covering sieve if \( U \) is a cover of \( U \), and we let \( \text{Cov}(U) \) denote the set of covering sieves (it inherits an ordering from \( \text{Crib}(U) \)). A covering sieve \( U \in \text{Cov}(U) \) is nothing more than a cover of \( U \) closed under passage to smaller sets, and a generating set for \( U \) is a subset of \( \mathcal{G} \subseteq U \) such that every \( V \in U \) is contained in some \( G \in \mathcal{G} \). This clearly implies that \( \mathcal{G} \) is a subcover, but not every subcover is a generating set—\( \mathcal{G} \) must contain all the largest open sets in \( U \).

The covering sieves in \( \text{Ouv}(X) \) enjoy the following properties:

1. For all \( U \in \text{Ouv}(X) \), the trivial sieve is in \( \text{Cov}(U) \).
2. For all \( \text{Ouv}(X) \) morphisms \( Y \to Z, W \to Z \) and all \( U \in \text{Cov}(Y), W \times_Z U \in \text{Cov}(W) \).
3. For all \( Y \in \text{Ouv}(X) \), all \( U \in \text{Cov}(Y) \), and all families

\[
V = \{ V(U) \in \text{Cov}(U) : U \in U \},
\]

\( V/U \in \text{Cov}(Y) \).

We remark that the sheafification of \( \mathcal{F} \in \text{PSh}(X) \) can be constructed as follows. First define \( \mathcal{F}' \in \text{PSh}(X) \) by setting

\[
\mathcal{F}'(U, U) := \lim_{\longleftarrow} \left( \prod_{U_i \in U} \mathcal{F}(U_i) \rightrightarrows \prod_{(i, j)} \mathcal{F}(U_i \cap U_j) \right)
\]

for a covering sieve

\[
U = \{ U_i : i \in I \} \in \text{Cov}(U).
\]

Next note, for

\[
U' = \{ U_i : i \in I' \} \subseteq U,
\]
with \( U' \in \text{Cov}(U) \) (i.e. for a \( \text{Cov}(U)^{\text{op}} \) morphisms \( U \to U' \)), we have an obvious map \( \mathcal{F}'(U, U) \to \mathcal{F}'(U', U) \) induced by the restriction map
\[
\prod_{i \in I} \mathcal{F}(U_i) \to \prod_{i' \in I'} \mathcal{F}(U_i).
\]
Set
\[
\mathcal{F}'(U) := \lim_{U \in \text{Cov}(U)^{\text{op}}} \mathcal{F}'(U, U).
\]
Note that \( \text{Cov}(U)^{\text{op}} \) is filtered: any two covering sieves have a common refinement. Note that, for an \( \text{Ouv}(X) \) morphism \( V \to U \), we have a natural morphism \( \mathcal{F}'(U) \to \mathcal{F}'(V) \) induced by the functor
\[
\text{Cov}(U) \to \text{Cov}(V)
\]
\[
U \mapsto V \times_U U.
\]
We leave it as an exercise to check that

1. For any \( \mathcal{F} \in \text{PSh}(X) \), \( \mathcal{F}' \) is separated.
2. For any separated \( \mathcal{F} \in \text{PSh}(X) \), \( \mathcal{F}' \) is a sheaf.
3. For any \( \mathcal{F} \in \text{PSh}(X) \), there is a natural isomorphism \( (\mathcal{F}')' = \mathcal{F}^+ \).

Notice that this construction avoids the use of stalks.

10. Properties of \( \text{Sh}(X) \)

In this section we establish some facts about the category \( \text{Sh}(X) \) that will allow us to work more directly with \( \text{Sh}(X) \) as a category per se, without so much explicit reference to the topological space \( X \).

**Lemma 10.1.** Epimorphisms in \( \text{Sh}(X) \) are closed under base change.

**Proof.** Base change commutes with stalks (finite inverse limits of sets commute with filtered direct limits), so by (6.1) we reduce to proving that epimorphisms in \( \text{Ens} \) are closed under base change, which is easy. \( \square \)

**Theorem 10.2.** A morphism \( f : \mathcal{F} \to \mathcal{G} \) in \( \text{Sh}(X) \) is an epimorphism iff, for every \( \mathcal{H} \in \text{Sh}(X) \), the diagram
\[
\text{Hom}_{\text{Sh}(X)}(\mathcal{G}, \mathcal{H}) \to \text{Hom}_{\text{Sh}(X)}(\mathcal{F}, \mathcal{H}) \Rightarrow \text{Hom}_{\text{Sh}(X)}(\mathcal{F} \times_{\mathcal{G}} \mathcal{F}, \mathcal{H})
\]
is an equalizer diagram of sets.

**Proof.** Injectivity of the left map in the diagram for all \( \mathcal{H} \) is the definition of “epimorphism”, so the implication \( (\Leftarrow) \) is obvious, and to establish \( (\Rightarrow) \), it remains only to show the following: suppose \( g : \mathcal{F} \to \mathcal{H} \) is a \( \text{Sh}(X) \) morphism such that \( g\pi_1 = g\pi_2 : \mathcal{F} \times_{\mathcal{G}} \mathcal{F} \to \mathcal{H} \), then \( g = hf \) for some \( h : \mathcal{G} \to \mathcal{H} \). The fastest way to prove this is to use stalks: the theorem we are trying to prove is clear in \( \text{Ens} \) and stalks preserve epimorphism (6.1) and commute with base change,
so we at least have a (unique) map $h_x : \mathcal{G}_x \to \mathcal{H}_x$ (for each $x \in X$) such that $g_x = h_x f_x$. Since $\mathcal{H}$ is a sheaf, we can view a section $t \in \mathcal{H}(U)$ as an element $t = (t(x)) \in \prod_{x \in U} \mathcal{H}_x$ satisfying the local coherence condition (c.f. (7)). For $U \in \text{Ouv}(X)$, and $s \in \mathcal{G}(U)$, it remains only to prove that $h(U)(s)(x) := h_x(s_x)$ actually satisfies the local coherence condition (the desired equality $g = h f$ will then follow immediately by checking on stalks). To check local coherence of $h(U)(s)$ at a point $y \in U$, we use surjectivity of $f_y$ to find an open neighborhood $V$ of $y$ in $U$ and a section $t \in \mathcal{F}(V)$ such that $f(V)(t) = s|_V$ (equivalently, $f_z(t_z) = s_z$ for all $z \in V$). Then $g(V)(t) \in \mathcal{H}(V)$ witnesses local consistency:

$$g(V)(t)_z = g_z(t_z) = h_z f_z(t_z) = h_z(s_z) = h(U)(s)(z)$$

for all $z \in V$. \hfill \Box

In highfalutin language, the next result says that the “usual covers of $X$ are cofinal in the category of epimorphisms to the terminal object of $\text{Sh}(X)$.”

**Lemma 10.3.** Let $\mathcal{G} \in \text{Sh}(X)$. The morphism $\mathcal{G} \to h_X$ to the terminal object is an epimorphism iff there is a cover $U = \{U_i : i \in I\}$ of $X$ and a morphism

$$\prod_{i \in I} h_{U_i} \to \mathcal{G}.$$

**Proof.** ($\Rightarrow$). By (5.4), the $h_U$ form a generating set for $\text{Sh}(X)$, so we can certainly find an epimorphism of the form

$$\prod_{U_i \in \text{Ouv}(X)} \prod_{j \in J_i} h_{U_i} \to \mathcal{G}$$

(take $J_i = \text{Hom}_{\text{Sh}(X)}(h_{U_i}, \mathcal{G}) = \mathcal{G}(U_i)$ and use the evaluation map, for example). If $x \in X$ is any point, then since

1. the composition of epimorphisms is an epimorphism,
2. coproducts (or any direct limits) commute with stalks, and
3. an epimorphism in $\text{Sh}(X)$ induces an epimorphism on stalks (6.1),

we have an epimorphism

$$\prod_{U_i \in \text{Ouv}(X)} \prod_{j \in J_i} h_{U_i,x} \to h_{X,x}.$$

Note $h_{X,x}$ is punctual, so this simply says there is some $U_{i(x)}$ such that $h_{U_{i(x)},x} \neq (\text{i.e.} x \in U_{i(x)}$, see (6)). The open sets $U_{i(x)}$ hence clearly cover $X$ and the coproduct of the $h_{U_{i(x)}}$ clearly maps to $\mathcal{G}$ by first mapping to the “big” coproduct above, which can be done in many ways.

($\Leftarrow$). Check on stalks. \hfill \Box
11. Examples

For example, if $X$ is a one point space, then $\mathbf{PSh}(X)$ is isomorphic to the arrow category of $\mathbf{Ens}$ via the map

$$\mathbf{PSh}(X) \to \mathbf{Ens} \quad \mathcal{F} \mapsto (\mathcal{F}(X) \to \mathcal{F}(\emptyset)),$$

while $\mathbf{Sh}(X)$ is equivalent to $\mathbf{Ens}$ via the map

$$\mathbf{Sh}(X) \to \mathbf{Ens} \quad \mathcal{F} \mapsto \mathcal{F}(X),$$

(c.f. Remark 5.1).

**Example 11.1.** For any topological space $X$, the presheaf $\mathcal{C}(X, \mathbb{R})$ which associates to every $U \in \mathbf{Ouv}(X)$ the set of continuous real-valued functions on $U$ (with the obvious restriction maps) is clearly a sheaf because continuity is local in nature (can be checked on each open set in a cover). This sheaf contains the presheaf $\mathcal{C}^b(X, \mathbb{R})$ of bounded functions. This latter presheaf is separated, but it is not generally a sheaf because being bounded is global in nature and one cannot glue together functions which are locally bounded into a function which is globally bounded. Indeed, any continuous map $f : X \to \mathbb{R}$ is locally bounded.

**Example 11.2.** Generalizing the previous example, for any topological spaces $X, Y$, the presheaf $U \mapsto \hom_{\mathbf{Top}}(U, Y)$ (with the obvious restriction maps) is a sheaf on $X$ because continuity is local. Similarly, if $\pi : W \to X$ is a map of topological spaces the presheaf

$$U \mapsto \{ s \in \hom_{\mathbf{Top}}(U, W) : \pi s = \text{Id}_U \}$$

of local sections of $\pi$ is a sheaf on $X$. We will see in Section ?? that every sheaf is isomorphic to a sheaf of this form.

12. Exercises

**Exercise 12.1.** Let $X$ be a topological space, $\mathcal{F} \in \mathbf{PSh}(X)$, $\mathcal{U}$ a cover of $X$, $\mathcal{V}$ a refinement of $\mathcal{U}$. Show that, if $\mathcal{F}$ is a sheaf for $\mathcal{V}$, then it is a sheaf for $\mathcal{U}$.

**Exercise 12.2.** A *basis* is a subset (full subcategory) $\mathcal{B} \subseteq \mathbf{Ouv}(X)$ such that every $U \in \mathbf{Ouv}(X)$ is a union (direct sum) of open sets in $\mathcal{B}$. Set $\mathbf{PSh}(\mathcal{B}) = \hom_{\mathbf{Cat}}(\mathcal{B}^{\text{op}}, \mathbf{Ens})$. Say $\mathcal{F} \in \mathbf{PSh}(\mathcal{B})$ is a sheaf if, for every $U \in \mathcal{B}$ and every cover $\{ U_i \in \mathcal{B} : i \in I \}$ of $U$, the diagram

$$\mathcal{F}(U) \to \prod_{i \in I} \mathcal{F}(U_i) \Rightarrow \prod_{(i,j) \in I \times I} \mathcal{F}(U_{ij})$$

is an equalizer diagram of sets. Let $\mathbf{Sh}(\mathcal{B})$ be the full subcategory of $\mathbf{PSh}(\mathcal{B})$ consisting of sheaves. Observe that the functor $\mathbf{PSh}(X) \to \mathbf{PSh}(\mathcal{B})$ induced by the inclusion $\mathcal{B}^{\text{op}} \subseteq \mathbf{Ouv}(X)^{\text{op}}$ takes $\mathbf{Sh}(X)$ into $\mathbf{Sh}(\mathcal{B})$. Show that $\mathbf{Sh}(X) \to \mathbf{Sh}(\mathcal{B})$ is an equivalence of categories.
CHAPTER II

Direct and Inverse Images

So far we have only considered the category of (pre) sheaves on a fixed topological space $X$. In this section, we study the functoriality of these categories in $X$.

1. Functoriality of Ouv($X$)

First notice that Ouv($X$) is contravariantly functorial in $X$. The functor

$$\text{Ouv} : \text{Esp}^{\text{op}} \to \text{Cat}$$

$$X \mapsto \text{Ouv}(X)$$

is given on a morphism $f : X \to Y$ in Esp by

$$\text{Ouv}(f) : \text{Ouv}(Y) \to \text{Ouv}(X)$$

$$U \mapsto f^{-1}(U).$$

The functor Ouv($f$) commutes with direct limits (unions) and finite inverse limits (intersections). Furthermore, Ouv($f$) takes covers to covers. Though we do not wish to define this, Ouv($f$) is a continuous morphism of sites.

2. Direct images

Let $f : X \to Y$ be a continuous map of topological spaces, $\mathcal{F}$ a presheaf on $X$. The direct image presheaf $f_*\mathcal{F}$ is the presheaf on $Y$ given by

$$f_*\mathcal{F} : \text{Ouv}(Y)^{\text{op}} \to \text{Ens}$$

$$U \mapsto \mathcal{F}(f^{-1}(U)).$$

The restriction maps for this presheaf $f_*\mathcal{F}(U) \to f_*\mathcal{F}(V)$ are given by the restriction maps $\mathcal{F}(f^{-1}(U)) \to \mathcal{F}(f^{-1}(V))$ for $\mathcal{F}$. In other words, $f_*\mathcal{F} : \text{Ouv}(Y)^{\text{op}} \to \text{Ens}$ is the composition of Ouv($f$)$^{\text{op}} : \text{Ouv}(Y)^{\text{op}} \to \text{Ouv}(X)^{\text{op}}$ and $\mathcal{F} : \text{Ouv}(X)^{\text{op}} \to \text{Ens}$.

3. Skyscraper sheaves

For example, if $x : \{x\} \hookrightarrow X$ is the inclusion of a point, and $A$ is a set (regarded as a sheaf on $\{x\}$ as in (11)), then $x_*A$ is the sheaf on $X$ given by

$$U \mapsto \begin{cases} A, & x \in U \\ \{0\}, & \text{otherwise} \end{cases}$$
A sheaf on $X$ isomorphic to such a sheaf is called a skyscraper sheaf. The stalks of $x_\ast A$ are given by

$$(x_\ast A)_y = \begin{cases} A, & y \in \{x\}^- \\ \{0\}, & \text{otherwise} \end{cases}$$

If $\mathcal{F}$ is a sheaf, then so is $f_\ast \mathcal{F}$ because $f_\ast \mathcal{F}$ has the sheaf property with respect to a cover $\mathcal{U}$ if and only if $\mathcal{F}$ has it with respect to the cover $f^{-1}\mathcal{U}$, as is clear from the definitions. The construction $f_\ast$ is clearly functorial in $\mathcal{F}$.

**Example 3.1.** The following example shows why direct limits of sheaves do not typically commute with sections. Let $X$ be any infinite set with the discrete topology and let $\mathbb{Z}_X$ be the constant sheaf. The presheaf direct sum $\bigoplus_{x \in X} x_\ast \mathbb{Z}$ of all stalks of $\mathbb{Z}$ is given by

$$U \mapsto \bigoplus_{x \in U} \mathbb{Z}.$$ 

This presheaf if clearly separated, but it does not have the gluing property for the cover of $X$ by singleton sets. The sheafification $\bigoplus_{x \in X} x_\ast \mathbb{Z}$ of this presheaf is given by

$$U \mapsto \prod_{x \in U} \mathbb{Z}$$

because these are the functions to the stalks which (trivially) satisfy the local coherence condition. If, on the other hand, $X$ is given the cofinite topology (a noetherian topology), then this presheaf if already a sheaf.

**4. Inverse images**

For a continuous map $f : X \to Y$, and a presheaf $\mathcal{G}$ on $Y$, then the inverse image presheaf $f^{-1}_{\text{pre}} \mathcal{G} \in \text{PSh}(X)$ is defined by

$$f^{-1}_{\text{pre}} \mathcal{G} : \text{Ouv}(X)^{\text{op}} \to \text{Ens}$$

$$U \mapsto \lim_{V \supseteq f(U)} \mathcal{G}(V),$$

where the direct limit is taken over open subsets $V$ of $Y$ containing $f(U)$ with respect to the restriction maps of $\mathcal{G}$ (this is a filtered subcategory of $\text{Ouv}(Y)$). Note $V \supseteq f(U)$ iff $U \subseteq f^{-1}(V)$, so the direct limit is over the category $U \downarrow \text{Ouv}(Y)$, whose objects are pairs $(V, g)$ where $V \in \text{Ouv}(Y)$ and $g : U \to \text{Ouv}(f)(V)$ is morphism in $\text{Ouv}(X)$. A morphism $(V, g) \to (V', g')$ in $U \downarrow \text{Ouv}(Y)$ is a morphism $i : V \to V'$ in $\text{Ouv}(X)$ making the diagram:

$$\begin{array}{ccc}
U & \rightarrow & \text{Ouv}(f)(V) \\
q & \downarrow & \downarrow \text{Ouv}(f)(i) \\
\text{Ouv}(f)(V) & \rightarrow & \text{Ouv}(f)(V')
\end{array}$$
in \( \text{Ouv}(Y) \) commute.\(^1\)

For any \( U \in \text{Ouv}(X) \), we have \( f(f^{-1}(U)) \subseteq U \) so a section \( s \in \mathcal{G}(U) \) yields a section \( f_{\text{pre}}^{-1}(s) \in (f_{\text{pre}}^{-1}\mathcal{G})(f^{-1}U) \) given by the class of \( s \) in the direct limit system defining \( (f_{\text{pre}}^{-1}\mathcal{G})(f^{-1}(U)) \). It is clear from the definition of \( f_{\text{pre}}^{-1}\mathcal{G} \) that its stalks are given by

\[
(f_{\text{pre}}^{-1}\mathcal{G})_x = \mathcal{G}_{f(x)}.
\]

The stalk of a presheaf \( \mathcal{F} \) on a space \( X \) at a point \( x \in X \) is nothing but the inverse limit of \( \mathcal{F} \) under the inclusion map \( x : \{x\} \to X \).

The functor \( f_{\text{pre}}^{-1} \) does not generally take sheaves to sheaves. We are led to define

\[
f^{-1} : \text{Sh}(Y) \to \text{Sh}(X)
\]

\[
\mathcal{G} \mapsto (f_{\text{pre}}^{-1}\mathcal{G})^+
\]

by composing \( f_{\text{pre}}^{-1} \) and the sheafification functor:

\[
f^{-1}\mathcal{G} := (f_{\text{pre}}^{-1}\mathcal{G})^+.
\]

As for the presheaf inverse image, a section \( s \in \mathcal{G}(U) \) determines a section \( f^{-1}s \in (f^{-1}\mathcal{G})(f^{-1}(U)) \), namely, the image of \( f_{\text{pre}}^{-1}s \in (f_{\text{pre}}^{-1}\mathcal{G})(f^{-1}(U)) \) under the sheafification morphism

\[
(f_{\text{pre}}^{-1}\mathcal{G})(f^{-1}(U)) \to (f^{-1}\mathcal{G})(f^{-1}(U)).
\]

Examining the construction of sheafification, we see that a section of \( (f^{-1}\mathcal{G})(V) \) can be viewed as an element

\[
s = (s(x)) \in \prod_{x \in V} \mathcal{G}_{f(x)}
\]

satisfying the local coherence condition: For every \( x \in V \), there is a neighborhood \( W \) of \( x \) in \( V \), a neighborhood \( U \) of \( f(W) \) in \( Y \), and a section \( t \in \mathcal{G}(U) \) such that \( s(x') = t_{f(x')} \) for every \( x' \in W \).

The formation of \( f_{\text{pre}}^{-1} \) and \( f^{-1} \) are both functorial in \( f \) in the sense that for \( f : X \to Y \) and \( g : Y \to Z \) the functors

\[
f_{\text{pre}}^{-1}g_{\text{pre}}^{-1}, (gf)_{\text{pre}}^{-1} : \text{PSh}(Z) \to \text{PSh}(X)
\]

are naturally isomorphic, as are the functors

\[
f^{-1}g^{-1}, (gf)^{-1} : \text{PSh}(Z) \to \text{PSh}(X).
\]

(In particular, the case where \( X = \{y\} \) is a point of \( Y \) gives the stalk formula \( (g_{\text{pre}}^{-1}\mathcal{G})_y = \mathcal{G}_{g(y)} \) mentioned above.)

**Theorem 4.1.** The functor \( f_{\text{pre}}^{-1} : \text{PSh}(Y) \to \text{PSh}(X) \) is left adjoint to \( f_* : \text{PSh}(X) \to \text{PSh}(Y) \). The functor \( f_{\text{pre}}^{-1} \) commutes with arbitrary direct limits and finite inverse limits. The functor \( f_* \) commutes with arbitrary inverse limits.

\(^1\)Of course, the usage of the category \( U \downarrow \text{Ouv}(Y) \) is overly complex, but it generalizes well and allows us to link up with various general category theoretic techniques (comma categories, Kan extension).
Proof. The adjointness is a special case of the Kan Extension Theorem and is straightforward to check. The other statements are formal consequences of adjointness, except the fact that $f^{-1}_{pre}$ commutes with finite inverse limits. This is proved by noting that the category of neighborhoods of any subspace is filtered, and filtered direct limits commute with finite inverse limits in $\text{Ens}$. □

Corollary 4.2. The functor $f^{-1} : \text{Sh}(Y) \to \text{Sh}(X)$ is left adjoint to $f_* : \text{Sh}(X) \to \text{Sh}(Y)$. The functor $f^{-1}$ commutes with arbitrary direct limits and finite inverse limits. The functor $f_*$ commutes with arbitrary inverse limits.

Proof. This follows formally from the theorem using the adjointness and exactness properties of the sheafification functor (7.1), (7.2). □

Corollary 4.3. Inverse images commute with sheafification: For $f : X \to Y$ and any presheaf $\mathcal{F} \in \text{PSh}(Y)$ we have a natural isomorphism $f^{-1}(\mathcal{F}^+) = (f_{pre}^{-1} \mathcal{F})^+$ in $\text{Sh}(X)$.

Proof. This is a consequence of Yoneda’s Lemma and the adjointness results just established: For any $\mathcal{G} \in \text{Sh}(X)$ we have

$$
\text{Hom}_{\text{Sh}(Y)}(f^{-1}(\mathcal{F}^+), \mathcal{G}) = \text{Hom}_{\text{Sh}(Y)}(\mathcal{F}^+, f_* \mathcal{G})
= \text{Hom}_{\text{PSh}(Y)}(\mathcal{F}, f_* \mathcal{G})
= \text{Hom}_{\text{PSh}(X)}(f^{-1}_{pre} \mathcal{F}, \mathcal{G})
= \text{Hom}_{\text{PSh}(X)}((f^{-1}_{pre} \mathcal{F})^+, \mathcal{G}).
$$

Remark 4.4. Suppose $i : U \hookrightarrow X$ is the inclusion of an open set and $\mathcal{F}$ is a sheaf on $X$. Then for any $V \in \text{Ouv}(U)$, $V$ is in $\text{Ouv}(X)$ and is cofinal among open neighborhoods of $i(V)$ in $X$, hence $i^{-1}_{pre} \mathcal{F}(V) = \mathcal{F}(V)$. Furthermore, $i^{-1}_{pre} \mathcal{F}$ is already a sheaf, so we have $i^{-1} \mathcal{F}(V) = \mathcal{F}(V)$. Evidently then, $i^{-1} \mathcal{F}$ is nothing but the restriction of the functor $\mathcal{F} : \text{Ouv}(X)^{op} \to \text{Ens}$ to the full subcategory $\text{Ouv}(V)^{op} \subseteq \text{Ouv}(X)^{op}$. Accordingly, we often write $\mathcal{F}|_U$ instead of $i^{-1} \mathcal{F}$.

5. Sheaf Hom

Consider the functor

$$
\text{Hom}(\_ , \_ ) : \text{PSh}(X)^{op} \times \text{PSh}(X) \to \text{PSh}(X)
(\mathcal{F}, \mathcal{G}) \mapsto \text{Hom}(\mathcal{F}, \mathcal{G}),
$$

where $\text{Hom}(\mathcal{F}, \mathcal{G})$ is the presheaf

$$
\text{Hom}(\mathcal{F}, \mathcal{G})(U) := \text{Hom}_{\text{PSh}(U)}(\mathcal{F}|_U, \mathcal{G}|_U).
$$

We use the notation $\mathcal{F}|_U$ of Remark 4.4.

Lemma 5.1. If $\mathcal{G} \in \text{Sh}(X)$, then the presheaf $\text{Hom}(\mathcal{F}, \mathcal{G})$ is a sheaf.
PROOF. Say \( \{U_i \to U\} \) is a cover. If \( f, g \) \( \text{Hom}(\mathcal{F}|_U, \mathcal{G}|_U) \) agree when restricted to \( U_i \) for every \( i \), then for any \( V \in \text{Ouv}(U) \) and any \( s \in \mathcal{F}(V) \), we have \( f(V)(s)|_{U_i \cap V} = g(V)(s)|_{U_i \cap V} \) for all \( i \) and \( \{U_i \cap V \to V\} \) is a cover, so \( f(V)(s) = g(V)(s) \) by separation for \( \mathcal{G} \). This holds for any \( s, V \), so \( f = g \), hence \( \text{Hom}(\mathcal{F}, \mathcal{G}) \) is separated.

Given maps \( f_i \in \text{Hom}(\mathcal{F}|_U, \mathcal{G}|_U) \) which agree on overlaps, define \( f : \mathcal{F}|_U \to \mathcal{G}|_U \) as follows. For any \( V \in \text{Ouv}(U) \), and any \( s \in \mathcal{F}(V) \) consider the \( f_i(U_i \cap V)(s|_{U_i \cap V}) \in \mathcal{G}(U_i \cap V) \). Since \( f_i(U_i \cap U_j \cap V) = f_j(U_i \cap U_j \cap V) \), these agree on overlaps, so by the gluing property for \( \mathcal{G} \) there is an \( f(V)(s) \in \mathcal{F}(V) \) restricting to \( f_i(U_i \cap V)(s|_{U_i \cap V}) \) in \( \mathcal{G}(U_i \cap V) \) for every \( i \). To show that the maps
\[
\begin{align*}
f(V) : \mathcal{F}(V) & \to \mathcal{G}(V) \\
s & \mapsto f(V)(s)
\end{align*}
\]
actually define a map of presheaves \( \mathcal{F}|_U \to \mathcal{G}|_U \), it remains to show that
\[
\begin{array}{ccc}
\mathcal{F}(V) & \longrightarrow & \mathcal{G}(V) \\
\downarrow & & \downarrow \\
\mathcal{F}(W) & \longrightarrow & \mathcal{G}(V)
\end{array}
\]
commutes for any morphism \( W \to V \) in \( \text{Ouv}(U) \). This follows from separation for \( \mathcal{G} \). \( \square \)

In light of the lemma, we may view (5.1) as a functor
\[
\text{Hom}(\_ \_): \text{Sh}(X)^{\text{op}} \times \text{Sh}(X) \to \text{Sh}(X) \\
(\mathcal{F}, \mathcal{G}) \mapsto \text{Hom}(\mathcal{F}, \mathcal{G}).
\]
The sheaf \( \text{Hom}(\mathcal{F}, \mathcal{G}) \) is called \textit{sheaf hom} from \( \mathcal{F} \) to \( \mathcal{G} \).

For a space \( X \) and an open set \( i : U \hookrightarrow X \), the formulae
\[
\begin{align*}
\text{Hom}(h_X, \mathcal{F}) & = \mathcal{F} \\
\text{Hom}(h_U, \mathcal{F}) & = i_*(\mathcal{F}|_U)
\end{align*}
\]
are clear from the Yoneda Lemma (3.1). For a continuous map \( f : X \to Y \), the adjunction \((f^{-1}, f_*)\) “sheafifies” to yield a natural isomorphism
\[
f_* \text{Hom}_X(f^{-1} \mathcal{F}, \mathcal{G}) = \text{Hom}_Y(\mathcal{F}, f_* \mathcal{G}).
\]

6. Constant sheaves

An extreme case of the formation of inverse images occurs when \( f \) is the map from \( X \) to a one point space \( Y \). If we identify \( \text{Sh}(Y) \) with \( \text{Ens} \) as in (11), then the presheaf inverse image functor \( f^{-1} \text{pre} \) is identified with the functor \( A \to A^\text{pre}_X \), where \( A^\text{pre}_X \) is the constant presheaf \( U \mapsto A \), and the inverse image functor \( f^{-1} \) is the functor \( A \mapsto A^\text{pre}_U \), where
\[
A\_X : \text{Ouv}(X)^{\text{op}} \to \text{Ens} \\
U \mapsto A^\pi(U),
\]
where \( \pi_0(U) \) is the set of connected components of \( U \). A sheaf isomorphic to some \( \underline{A}_X \) is called a constant sheaf.

### 7. Representable case

It is instructive to describe \( f_*F \) and \( f^{-1}F \) in the case where \( F = h_U \) is a representable sheaf. The equalities

\[
f^{-1}h_U = f_{\text{pre}}^{-1}h_U = h_{f^{-1}(U)}
\]

are clear for any morphism \( f : X \to Y \) and any \( U \in \text{Ouv}(Y) \). The set \( (f^{-1}h_U)(V) \) is punctual if \( f(V) \subseteq U \) (equivalently \( V \subseteq f^{-1}(U) \)) and empty otherwise. Similarly, for \( V \in \text{Ouv}(X) \), the sheaf \( f_*h_V \) assigns the punctual set to \( U \in \text{Ouv}(Y) \) if \( f^{-1}(U) \subseteq V \), else \( (f_*h_V)(U) \) is empty.

### 8. Inverse images and espaces étalé

The formation of inverse images is compatible with the construction of the espace étalé (§8).

**Lemma 8.1.** Let \( f : X \to Y \) be a continuous map of topological spaces, \( F \) a presheaf on \( Y \), \( \pi : \text{Et}F \to Y \) its espace étalé. Then \( \pi_1 : X \times_Y \text{Et}F \to X \) is naturally isomorphic to the espace étalé of \( f_{\text{pre}}^{-1}F \). Similarly, if \( F \) is a sheaf, \( \pi_1 \) is naturally isomorphic to the espace étalé of \( f^{-1}F \).

**Proof.** Recall from Section ?? that, as a set,

\[
\text{Et} f_{\text{pre}}^{-1}F = \coprod_{x \in X} (f_{\text{pre}}^{-1}F)_x = \coprod_{x \in X} F_{f(x)},
\]

which we prefer to think of as the set of pairs \((x, s)\) where \( x \in X \) and \( s \in F_{f(x)} \). On the other hand, \( X \times_Y \text{Et}F \) is exactly the same set of pairs (the preimage of \( f(x) \in Y \) under \( \pi : \text{Et}F \to Y \) is \( F_{f(x)} \) by construction of \( \pi \)). This identification clearly respects the two maps to \( X \), so it remains to prove that it is continuous.

A basic open set in \( \text{Et} f_{\text{pre}}^{-1}F \) is a set of the form

\[
U_t = \{ (x, s) \in \text{Et} f_{\text{pre}}^{-1}F : x \in U, s = t_x \in F_{f(x)} \}
\]

where \( U \) is an open subset of \( X \) and \( t \in (f_{\text{pre}}^{-1}F)(U) \). Such a \( t \) is represented by a \( V \in \text{Ouv}(Y) \) with \( f(U) \subseteq V \) and a section \( r \in F(V) \). For \( x \in U \) one then has \( t_x = r_{f(x)} \), so this open corresponds to the open subset \( U \times_Y V_r \) under our identification.

A basic open set of \( X \times_Y \text{Et}F \) is given by \( U \times_Y V_s \) where \( U \in \text{Ouv}(X) \), \( V \in \text{Ouv}(Y) \), \( s \in F(V) \) and \( V_s := \{ y \in V : s_y \in F_y \subseteq \text{Et}F \} \). But we have

\[
U \times_Y V_s = f^{-1}(V) \times_Y V_s = f^{-1}(V) f_{\text{pre}}^{-1}s
\]

under our identification, so this basic open in \( X \times_Y \text{Et}F \) is also open in \( \text{Et} f_{\text{pre}}^{-1}F \).
The statements about sheaves follow from the statements about presheaves because both inverse images and espaces étalé commute with sheafification. □

9. Faithfulness of $X \mapsto \text{Sh}(X)$
CHAPTER III

Descent

1. Canonical topology
   canonical cover = universal effective epimorphism

2. Open maps and variants
   open maps, weakly open maps, universal w.o. maps, basic properties (composition, base change)

3. Descent for spaces

4. Descent for sheaves
CHAPTER IV

Cohomology

The purpose of this chapter is to give a concise, self-contained treatment of sheaf cohomology.

1. Abelian categories

2. Enough injectives

3. Homological algebra

- basic theory of delta functors and derived functors, Grothendieck spectral sequence, spectral sequence from complexes

4. Abelian Sheaves

For a topological space $X$, let $\text{Ab}(X)$ denote the category of abelian group objects in $\text{Sh}(X)$. An object $\mathcal{F} \in \text{Ab}(X)$ is often called a sheaf of abelian groups or just an abelian sheaf. Similarly, we let $\text{PAb}(X)$ denote the category of abelian group objects in $\text{PSh}(X)$. Note that $\text{PAb}(X)$ is isomorphic to the category of functors from $\text{Ouv}(X)^{\text{op}}$ to $\text{Ab}$ and that this identifies $\text{Ab}(X)$ with the full subcategory consisting of those $\mathcal{F} : \text{Ouv}(X)^{\text{op}} \to \text{Ab}$ whose composition with the forgetful functor $\text{Ab} \to \text{Ens}$ is a sheaf. For $\mathcal{F} \in \text{PAb}(X)$ and any inclusion $V \subseteq U$ of open subsets of $X$, the restriction map $\mathcal{F}(U) \to \mathcal{F}(V)$ is a homomorphism of (abelian) groups.

The basic theory of sheaves from Chapter ?? translates to the abelian setting in the same way that one can translate various statements about sets into statements about (abelian) groups.

**Lemma 4.1.** The forgetful functor $\text{Ab}(X) \to \text{PAb}(X)$ has a left adjoint (sheafification) which commutes with arbitrary direct limits and finite inverse limits.

**Proof.** The analogous statements are true of $\text{Sh}(X) \to \text{PSh}(X)$ and preservation of finite inverse limits implies that sheafification takes (abelian) group objects to (abelian) group objects. □

The categories $\text{PAb}(X)$ and $\text{Ab}(X)$ clearly carry the structure of additive categories. Kernels and cokernels (in fact all limits) are formed “objectwise” in $\text{PAb}(X)$ just as they were in $\text{PSh}(X)$; in particular, $\text{PAb}(X)$ has all small direct and inverse limits.
limits. Consequently, \( \text{Ab}(X) \) also has all small direct and inverse limits: inverse limits are formed by taking the inverse limit in \( \mathbf{PAb}(X) \) and noting that it is already in \( \text{Ab}(X) \). Indeed, \( \text{Ab} \to \text{Ens} \) commutes with inverse limits, so this is obvious from the corresponding statement about inverse limits in \( \mathbf{Sh}(X) \). However, to form direct limits in \( \text{Ab}(X) \), one must sheafify (i.e. apply the sheafification functor of the lemma to) the direct limit taken in \( \mathbf{PAb}(X) \).

Explicitly, the zero object \( 0 \in \mathbf{PAb}(X) \) is the constant presheaf \( U \mapsto \{0\} \). Note that this is also a sheaf and that it is also the zero object in \( \text{Ab}(X) \). For any morphism \( f : \mathcal{F} \to \mathcal{G} \) in \( \text{Ab}(X) \), the natural map

\[
\text{Cok} \text{Ker} f \to \text{Ker} \text{Cok} f
\]

is an isomorphism because

\[
\text{Cok} \text{Ker} f(U) \to \text{Ker} \text{Cok} f(U)
\]

is an isomorphism for each \( U \in \mathcal{Ouv}(X) \) (\( \text{Ab} \) is an abelian category!). A similar statement holds in \( \text{Ab}(X) \) because the natural map there is obtained by sheafifying this natural map. We have proved:

**Proposition 4.2.** \( \text{Ab}(X) \) and \( \mathbf{PAb}(X) \) are abelian categories with all small direct and inverse limits.

Observe that \( \mathbb{Z} \) is a generator for \( \text{Ab} \) (c.f. §??), and that this is obvious on category theoretic grounds from the fact that a one element set is a generator for \( \text{Ens} \) and the free abelian group functor is left adjoint to the forgetful functor \( \text{Ab} \to \text{Ens} \). These observations are easily promoted to \( \text{Ab}(X) \), by the usual method of bootstrapping through \( \mathbf{PAb}(X) \).

**Lemma 4.3.** The forgetful functor \( \mathbf{PAb}(X) \to \mathbf{PSh}(X) \) has a left adjoint, the “free abelian group” functor

\[
\mathbf{PSh}(X) \to \mathbf{PAb}(X)
\]

\[
\mathcal{F} \mapsto \bigoplus_{\mathcal{F}} \mathbb{Z},
\]

where \( \bigoplus_{\mathcal{F}} \mathbb{Z} \in \mathbf{PAb}(X) \) is given by

\[
\bigoplus_{\mathcal{F}} \mathbb{Z} : \mathcal{Ouv}(X)^{\text{op}} \to \text{Ab}
\]

\[
U \mapsto \bigoplus_{\mathcal{F}(U)} \mathbb{Z}.
\]

**Proof.** Obvious. \( \square \)

**Corollary 4.4.** The forgetful functor \( \text{Ab}(X) \to \text{Sh}(X) \) has a left adjoint, the “free abelian group” functor

\[
\text{Sh}(X) \to \text{Ab}(X)
\]

\[
\mathcal{F} \mapsto (\bigoplus_{\mathcal{F}} \mathbb{Z})^+.
\]

**Proof.** Follows from the lemma and the adjointness property of sheafification (7.1). \( \square \)
Theorem 4.5. \( \{ \bigoplus_{U \in \text{Ouv}(X)} Z : U \in \text{Ouv}(X) \} \) (resp. \( \{ \bigoplus_{U \in \text{Ouv}(X)} Z : U \in \text{Ouv}(X) \} \)) is a generating set for \( \text{PAb}(X) \) (resp. \( \text{Ab}(X) \)). \( \text{PAb}(X) \) and \( \text{Ab}(X) \) have all small direct and inverse limits. Finite inverse limits and filtered direct limits commute in both \( \text{PAb}(X) \) and \( \text{Ab}(X) \).

Proof. The first statement is purely formal; let us prove it for \( \text{Ab}(X) \) for concreteness. For a sheaf \( \mathcal{F} \), we have
\[
\text{Hom}_{\text{Ab}(X)}(\bigoplus_{U \in \text{Ouv}(X)} Z, \mathcal{F}) = \text{Hom}_{\text{Sh}(X)}(h_U, \mathcal{F}) = \mathcal{F}(U).
\]
This is natural in \( \mathcal{F} \), so for parallel arrows \( f, g : \mathcal{F} \to \mathcal{G} \), the maps
\[
f(U), g(U) : \mathcal{F}(U) \to \mathcal{G}(U)
\]
(which are equal for all \( U \) iff \( f = g \)) are identified with
\[
f_\ast, g_\ast : \text{Hom}_{\text{Ab}(X)}(\bigoplus_{U \in \text{Ouv}(X)} Z, \mathcal{F}) \to \text{Hom}_{\text{Ab}(X)}(\bigoplus_{U \in \text{Ouv}(X)} Z, \mathcal{G}).
\]
The existence of limits was discussed above. The last statement is certainly true of \( \text{PAb}(X) \) since these limits are formed objectwise and this statement is true of \( \text{Ab} \) (because it is true of \( \text{Ens} \) and \( \text{Ab} \to \text{Ens} \) is conservative and commutes with filtered direct limits and inverse limits). Hence this is also true of \( \text{Ab}(X) \) by adjointness and exactness properties of sheafification.

Corollary 4.6. Every \( \mathcal{F} \in \text{PAb}(X) \) (resp. \( \text{Ab}(X) \)) admits a monomorphism into an injective object of \( \text{PAb}(X) \) (resp. \( \text{PAb}(X) \)).

Proof. This follows formally from the theorem as in Grothendieck’s Tohoku paper.

The section functors
\[
\text{PAb}(X) \to \text{Ab} \quad \mathcal{F} \mapsto \Gamma(U, \mathcal{F})
\]
commute with all direct and inverse limits, as these are formed “objectwise” in \( \text{PAb}(X) \) just as they were in \( \text{PSh}(X) \). The section functors
\[
\text{Ab}(X) \to \text{Ab} \quad \mathcal{F} \mapsto \Gamma(U, \mathcal{F})
\]
commute with inverse limits (note \( \text{Ab} \to \text{Ens} \) commutes with inverse limits, and the section functors for sheaves commute with inverse limits), but they do not generally commute with direct limits. That is, \( \Gamma(U, \mathcal{F}) \) preserves kernels, but not cokernels. Note that \( \Gamma(U, \_ \) does, however, preserve finite direct sums since these coincide with finite direct products. In other words, \( \Gamma(U, \_ \) is a left exact morphism of abelian categories—its right derived functors will be considered in more generality in \( \S \).
The forgetful functor $\text{Ab}(X) \to \text{PAb}(X)$ is similarly a left exact morphism of abelian categories. Its $n$th right derived functor is denoted $\mathcal{F} \mapsto \mathcal{H}^n(\mathcal{F})$ and is called the $n$th cohomology presheaf of $\mathcal{F}$. It will be studied further in Chapter 7.

A morphism of topological spaces $f : X \to Y$ induces an exact morphism of abelian categories $f^{-1} : \text{Ab}(Y) \to \text{Ab}(X)$ and a left exact morphism of abelian categories $f_* : \text{Ab}(X) \to \text{Ab}(Y)$. The right derived functors $R^n f_*$ are called the higher direct images of $f$. These will be studied further in Section 15.

5. Ringed spaces

Let $\text{An}(X)$ be the category of ring objects of $\text{Sh}(X)$. For $\mathcal{O}_X \in \text{An}(X)$ we can consider the category $\text{Mod}(\mathcal{O}_X)$ of modules (“module objects”) over $\mathcal{O}_X$. Concretely, $\mathcal{O}_X \in \text{An}(X)$ is a functor $\mathcal{O}_X : \text{Ouv}(X)^\text{op} \to \text{An}$ whose composition with the forgetful functor $\text{An} \to \text{Ens}$ is a sheaf. From this point of view, $\mathcal{F} \in \text{Mod}(\mathcal{O}_X)$ is a functor $\mathcal{F} : \text{Ouv}(X)^\text{op} \to \text{Ab}$ (whose composition with $\text{Ab} \to \text{Ens}$ is a sheaf) such that, for each $U \in \text{Ouv}(X)$, the abelian group $\mathcal{F}(U)$ is equipped with an $\mathcal{O}_X(U)$ module structure making the restriction maps $\mathcal{F}(U) \to \mathcal{F}(V)$ linear over $\mathcal{O}_X(U) \to \mathcal{O}_X(V)$. That is, for $s \in \mathcal{F}(U)$ and $t \in \mathcal{O}_X(U)$, we require

\[(t \cdot s)|_V = t|_V \cdot s|_V.\]

Note that $\text{Mod}(\mathbb{Z}_X)$ is isomorphic to $\text{Ab}(X)$. The results of §4 on $\text{Ab}(X)$ carry over, mutatis mutandis, to $\text{Mod}(\mathcal{O}_X)$. In particular, $\text{Mod}(\mathcal{O}_X)$ is an abelian category with all small direct and inverse limits, a set of generators, etc. A pair $(X, \mathcal{O}_X)$ consisting of a topological space $X$ and a sheaf of rings $\mathcal{O}_X \in \text{An}(X)$ is called a ringed space. If $\mathcal{O}_X$ is clear from context, we will simply write $X$ for $(X, \mathcal{O}_X)$ and $\text{Mod}(X)$ for $\text{Mod}(\mathcal{O}_X)$. We will write “$X$ module” instead of “$\mathcal{O}_X$ module,” etc. Note that the stalk $\mathcal{F}_x$ of an $\mathcal{O}_X$ module has a natural module structure over the stalk $\mathcal{O}_{X,x}$ of $\mathcal{O}_X$.

The following direct proof that $\text{Mod}(X)$ has enough injectives is sometimes useful, especially if we need to get our hands on a supply of easily understood injectives.

**Lemma 5.1.** Let $x : \{x\} \hookrightarrow X$ be the inclusion of a point of a topological space $X$ with a sheaf of rings $\mathcal{O}_X$. Let $I$ be an injective $\mathcal{O}_{X,x}$ module. Then $x_*I$ is an injective $\mathcal{O}_X$ module.

**Proof.** We must show $\mathcal{F} \mapsto \text{Hom}_X(\mathcal{F}, x_*I)$. Note $\mathcal{O}_{X,x} = x^{-1}\mathcal{O}_X$, so by the adjointness ($x^{-1}, x_*$),

$$\text{Hom}_X(\mathcal{F}, x_*I) = \text{Hom}_{\mathcal{O}_{X,x}}(\mathcal{F}_x, I),$$

which is exact because the stalk functor is exact and because $\text{Hom}_{\mathcal{O}_{X,x}}(\cdot, I)$ is exact by definition of injective. \[\square\]

**Theorem 5.2.** For any ringed space $X$, the category $\text{Mod}(X)$ has enough injectives.
PROOF. Let \( F \in \text{Mod}(X) \). Since \( \text{Mod}(A) \) has enough injectives for any ring \( A \), we can choose, for each \( x \in X \), a monomorphism \( \mathcal{F}_x \hookrightarrow I(x) \) from the stalk of \( \mathcal{F} \) into an injective \( O_{X,x} \) module. The adjunction morphism for the adjoint functors \( (x^{-1}, x_*) \) furnishes a natural morphism \( \mathcal{F} \rightarrow x_*I(x) \) of \( X \) modules. The product of all these maps is a map \( \mathcal{F} \rightarrow \prod_{x \in X} x_*I(x) \) which is easily seen to be monic. Note \( \prod_{x \in X} x_*I(x) \) is a product of injectives and is hence injective. \( \square \)

6. Sheaf Ext

For \( G \in \text{Ab}(X) \) and \( \mathcal{F} \in \text{Sh}(X) \), observe that \( \text{Hom}(\mathcal{F}, G) \) (c.f. (5)) inherits an abelian group object structure from that of \( G \). We may view sheaf \( \text{Hom} \) as a functor
\[
\text{Hom}(\_ , \_ ) : \text{Sh}(X)^{\text{op}} \times \text{Ab}(X) \rightarrow \text{Ab}(X) \\
(\mathcal{F}, G) \mapsto \text{Hom}(\mathcal{F}, G).
\]

Similarly, if \( X \) is equipped with a sheaf of rings \( O_X \), then we have an “internal \( \text{Hom} \)” or “sheaf \( \text{Hom} \)” functor
\[
\text{Hom}(\_ , \_ ) : \text{Mod}(X)^{\text{op}} \times \text{Mod}(X) \rightarrow \text{Mod}(X) \\
(\mathcal{F}, G) \mapsto \text{Hom}(\mathcal{F}, G).
\]

For any fixed \( \mathcal{F} \in \text{Mod}_X \), the functor \( \text{Hom}(\mathcal{F}, \_ ) \) is left exact. Its right derived functors are denoted
\[
\mathcal{G} \mapsto \mathcal{E}xt^p(\mathcal{F}, \mathcal{G}).
\]

**Lemma 6.1.** Let \( X \) be a ringed space, \( \mathcal{F} \in \text{Mod}(X) \). There is an isomorphism of \( \delta \) functors from \( \text{Mod}(X) \) to \( \text{Mod}(\Gamma(X, O_X)) \) :
\[
H^\bullet(X, \mathcal{F}) = \text{Ext}^\bullet(O_X, \mathcal{F})
\]

**Proof.** It is straightforward to check that mapping \( s \in \Gamma(X, \mathcal{F}) \) to the homomorphism \( \cdot s \in \text{Hom}(O_X, \mathcal{F}) \) given by
\[
(\cdot s)(U) : O_X(U) \rightarrow \mathcal{F}(U) \\
f \mapsto f \cdot s|_U
\]
yields an isomorphism \( \Gamma(X, \mathcal{F}) = \text{Hom}(X, \mathcal{F}) \). The result now follows since both \( \delta \) functors are effaceable, as \( \text{Mod}(X) \) has enough injectives. \( \square \)

The proof of the following lemma will require several constructions that we have not yet discussed, but in any case it is natural to state it now.

**Lemma 6.2.** **Local-to-Global Ext Sequence.** For \( \mathcal{F}, \mathcal{G} \in \text{Mod}(X) \), there is a natural first quadrant spectral sequence
\[
E_2^{p,q} = H^p(X, \mathcal{E}xt^q(\mathcal{F}, \mathcal{G})) \implies \text{Ext}^{p+q}(\mathcal{F}, \mathcal{G}).
\]
Proof. Since $\Gamma(X, \mathcal{H}om(F, G)) = \text{Hom}(F, G)$, this will be obtained as a special case of the Grothendieck spectral sequence as long as we can prove that $\mathcal{H}om(F, G)$ is $\Gamma$-acyclic for any injective sheaf $G \in \text{Mod}(X)$. Since flasque sheaves are $\Gamma$-acyclic by (10.4), below, it suffices to prove that $\mathcal{H}om(F, G)$ is flasque, so what we must show is that, for any open set $\iota : U \to X$ and any homomorphism $f : \iota^{-1}F \to \iota^{-1}G$, there is a homomorphism $F : F \to G$ in $\text{Mod}(X)$ with $\iota^{-1}F = f$. To do this, we apply the extension by zero functor $\iota_!$ (to be discussed in (2), below) to $f$ to get a diagram of solid arrows

$\begin{array}{ccc}
\iota_!\iota^{-1}F & \to & \iota_!\iota^{-1}G \\
\downarrow & & \downarrow \\
\mathcal{F} & \to & G
\end{array}$

in $\text{Mod}(X)$, which completes as indicated by injectivity of $G$ since the vertical arrow is monic. Since $\iota^{-1}\iota = \text{Id}$, the vertical and second horizontal arrows become isomorphisms upon applying $\iota^{-1}$ so this $F$ is as desired. \hfill \Box

7. Supports

Let $\mathcal{F}$ be an abelian presheaf on a topological space $X$, $U$ an open subset of $X$, $s \in \mathcal{F}(U)$ a section. Let $\text{Supp} s := \{x \in U : s(x) \neq 0 \in \mathcal{F}_x\}$ be the support of $s$. The set $\text{Supp} s$ is closed in $U$, hence locally closed in $X$. For any $s, t \in \mathcal{F}(U)$, the set $\{x \in U : s(x) = t(x)\}$ is an open subset of $U$.

The closed set

$\text{Supp} \mathcal{F} := \{x \in X : \mathcal{F}_x \neq 0\}^-$

of $X$ is called the support of $\mathcal{F}$. The support of any direct sum or product of sheaves is the closure of the union of their supports.

A family $\Phi$ of closed subsets of a topological space $X$ is called a family of supports if it satisfies the conditions:

1. If $A \in \Phi$ and $B$ is a closed subset of $A$, then $B \in \Phi$.
2. If $A, B \in \Phi$ then $A \cup B \in \Phi$.

The first axiom in particular implies that $\emptyset \in \Phi$. For example, the family $\Phi$ of all closed subsets of $X$ is a family of supports. Another example is the family $\Phi$ of all closed quasi-compact subsets of $X$ (if $X$ is Hausdorff, then the adjective “closed” is redundant, but this is not true for a general $X$). For a subspace $Z$ of $X$, we could take $\Phi$ to be the family of all closed subsets of $X$ contained in $Z$. We could also take $\Phi$ to be the family of all closed, paracompact subsets of $X$. We say that a family of supports $\Phi$ has neighborhoods if, it satisfies the additional condition

3. Every $A \in \Phi$ has a neighborhood $B$ with $B^c \in \Phi$.

If, furthermore, $\Phi$ satisfies
(4) Every \( A \in \Phi \) has a neighborhood \( B \) such that \( B^{-} \in \Phi \) and \( B^{-} \) is paracompact, then we say \( \Phi \) is paracompactifying.

Recall that a space \( X \) is called paracompact iff \( X \) is Hausdorff and every open cover of \( X \) has a locally finite (open) refinement.

For a subspace \( Y \subseteq X \), and a family of supports \( \Phi \) on \( X \), the families

\[
\Phi \cap Y := \{ A \cap Y : A \in \Phi \}
\]

\[
\Phi_{Y} := \{ A \in \Phi : A \subseteq Y \}
\]

are both families of supports on \( Y \). We emphasize that, in general, these are very different, so we caution the reader to take care in keeping track of which family of supports we are using.

If \( \mathcal{F} \) is an abelian sheaf on \( X \) and \( i : U \hookrightarrow X \) is an open subset of \( X \), then the global sections with support in \( \Phi \) of the abelian sheaf \( \mathcal{F}|_{U} = ii^{-1}\mathcal{F} \) on \( X \) are given by \( \Gamma_{\Phi}(X, \mathcal{F}) = \Gamma_{\Phi_{U}}(U, \mathcal{F}) \). This is immediate from the definitions.

8. Sheaf cohomology

For any abelian presheaf \( \mathcal{F} \) on \( X \), and any family of supports \( \Phi \) on \( X \),

\[
\Gamma_{\Phi}(X, \mathcal{F}) := \{ s \in \mathcal{F}(X) : \text{Supp} \ s \in \Phi \}
\]

is a subgroup of \( \Gamma(X, \mathcal{F}) \) (it contains \( 0 \) because \( \emptyset \in \Phi \), and it is closed under addition because \( \Phi \) is closed under finite unions and passage to smaller closed sets) called the group of global sections of \( \mathcal{F} \) with support in \( \Phi \). It is easy to see that \( \mathcal{F} \mapsto \Gamma_{\Phi}(X, \mathcal{F}) \) is a left exact functor \( \text{Ab}(X) \rightarrow \text{Ab} \). The corresponding right derived functors are denoted \( H_{n}^{\Phi}(X, \mathcal{F}) \) and called the cohomology of \( \mathcal{F} \) with support in \( \Phi \). In case \( \Phi \) is the family of supports consisting of all closed subsets contained in a closed subspace \( Y \subseteq X \), \( H_{n}^{\Phi}(X, \mathcal{F}) \) is usually denoted \( H_{n}^{Y}(X, \mathcal{F}) \) and is called the cohomology of \( \mathcal{F} \) with support in \( Y \).

Unlike the usual global section functor, \( \Gamma_{\Phi} \) does not generally commute with infinite inverse limits, or even with infinite products.

9. Acyclic sheaves

Let \( X \) be a topological space, \( \Phi \) a family of supports on \( X \). A sheaf \( \mathcal{F} \in \text{Ab}(X) \) is called \( \Gamma_{\Phi} \)-acyclic iff \( H_{n}^{\Phi}(X, \mathcal{F}) = 0 \) for \( n > 0 \).

(PUT SOMETHING HERE TO THE EFFECT THAT COHOMOLOGY CAN BE COMPUTED USING GAMMA ACYCLIC RESOLUTIONS)

10. Flasque sheaves

A sheaf \( \mathcal{F} \) on a topological space \( X \) is called flasque iff the restriction map \( \mathcal{F}(U) \rightarrow \mathcal{F}(V) \) is surjective for every inclusion \( V \subseteq U \) of open subsets of \( X \).
Lemma 10.1. The direct image of a flasque sheaf under any morphism of topological spaces is again flasque.

Proof. This is clear from the formula $(f_*\mathcal{F})(U) = \mathcal{F}(f^{-1}U)$ for sections of the direct image sheaf. □

Lemma 10.2. (Gluing sections) Suppose $\mathcal{A} = \{A_i : i \in I\}$ is a locally finite family of subsets of $X$ with union $A$ and we have sections $s_i$ of an abelian sheaf $\mathcal{F}$ over $A_i$ for each $i \in I$ which agree on overlaps ($s_i|_{A_{ij}} = s_j|_{A_{ij}}$ for all $i, j \in I$). Then there is a unique section $s$ of $\mathcal{F}$ over $A$ such that $s|_{A_i} = s_i$ for each $i \in I$.

Proof. Certainly there is a unique map $s : A \to \prod_{x \in A} \mathcal{F}_x$ with

$$s(x) = s_i(x) \in \mathcal{F}_x \subset \prod_{a \in A} \mathcal{F}_a$$

for every $i \in I$ with $x \in A_i$. The only question is whether $s$ satisfies the local coherence condition. To see this, fix $a \in A$ and use local finiteness to find an open neighborhood $U$ of $a$ which meets only $A_{i_1} \ldots A_{i_n}$. The sections $s_{i_j}$ satisfy the local coherence condition at $a$, so there are open neighborhoods $U_{i_j}$ of $a$ and sections $t_{i_j} \in \mathcal{F}(U_{i_j})$ such that $t_{i_j}(x) = s_{i_j}(x) = s(x)$ for all $x \in U_{i_j} \cap A_{i_j}$. Let $V$ be the intersection of $U$ and the $U_{i_j}$, so $V$ is an open neighborhood of $a$ since there are only finitely many $i_j$. The set $W$ where the restrictions $t_{i_j}|_V$ all agree is an open subset of $V$ containing $a$. Let $t \in \mathcal{F}(W)$ be the common value. Then $t$ witnesses the local coherence of $s$ at $a$. □

Theorem 10.3. Let $X$ be a topological space, $\Phi$ a family of supports on $X$. The class of flasque sheaves satisfies the following properties:

1. For any short exact sequence of sheaves

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

on $X$ with $\mathcal{F}'$ flasque, the map $\Gamma_{\Phi}(X, \mathcal{F}) \to \Gamma_{\Phi}(X, \mathcal{F}'')$ is surjective.

2. For any short exact sequence of sheaves

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

on $X$ with $\mathcal{F}'$ and $\mathcal{F}$ flasque, $\mathcal{F}''$ is flasque.

3. A direct summand of a flasque sheaf is flasque.

4. Any sheaf admits a monomorphism into a flasque sheaf.

If $\Phi$ is paracompactifying, then the class of $\Phi$ soft sheaves satisfies the same properties.

Proof. The fact that summands of flasque or $\Phi$ soft sheaves are again flasque or $\Phi$-soft follows from the fact that $(\mathcal{F} \oplus \mathcal{G})(U) = \mathcal{F}(U) \oplus \mathcal{G}(U)$ for any open $U \subseteq X$. Any sheaf embeds in a flasque and $\Phi$-soft sheaf by taking the sheaf of discontinuous sections (see ).
Now assume
\[ 0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0 \]
is a short exact sequence of sheaves on \( X \) with \( \mathcal{F}' \) flasque. We will first show that \( \Gamma(X, \mathcal{F}) \to \Gamma(X, \mathcal{F}'') \) is surjective. Let \( s \in \Gamma(X, \mathcal{F}'') \). Consider the set \( A \) of all pairs \((U, t)\) where \( U \) is an open subset of \( X \), and \( t \in \mathcal{F}(U) \) maps to \( s|_U \) under \( \mathcal{F}(U) \to \mathcal{F}''(U) \). Order \( A \) by declaring \((U, t) \leq (U', t')\) iff \( U \subseteq U' \) and \( t'|_U = t \). By the sheaf properties for \( \mathcal{F} \) and \( \mathcal{F}' \), any chain in \( A \) has an upper bound, so by Zorn’s Lemma, \( A \) has a maximal element \((U, t)\). If \( U = X \) we’re done. If not, there is some \( x \in X \setminus U \). Since \( \mathcal{F} \to \mathcal{F}'' \) is surjective on stalks, we can find a neighborhood \( V \) of \( x \) in \( X \) and a section \( t' \in \mathcal{F}(V) \) such that \( t' \mapsto s|_V \) under \( \mathcal{F}(V) \to \mathcal{F}''(V) \).

Note \( t|_{U \cap V} - t'|_{U \cap V} \in \mathcal{F}(U \cap V) \) actually lies in \( \mathcal{F}(U \cap V) \) because both \( t|_{U \cap V} \) and \( t'|_{U \cap V} \) map to \( s|_{U \cap V} \) under \( \mathcal{F}(U \cap V) \to \mathcal{F}''(U \cap V) \). Since \( \mathcal{F}' \) is flasque, there is a \( w \in \mathcal{F}'(V) \) restricting to \( t|_{U \cap V} - t'|_{U \cap V} \) on \( U \cap V \). It follows that \( t \in \mathcal{F}(U) \) and \( t' + w \in \mathcal{F}(V) \) agree on \( U \cap V \), hence glue to a section \( \tilde{t} \in \mathcal{F}(U \cup V) \). The section \( \tilde{t} \) maps to \( s|_{U \cap V} \) because this can be checked on \( U \) and \( V \) by separation for \( \mathcal{F}'' \) and adding \( w \in \mathcal{F}'(V) \) to \( t' \) doesn’t change its image in \( \mathcal{F}''(V) \). We conclude that \((U \cup V, \tilde{t})\) contradicts maximality of \((U, t)\).

Now we show that \( \Gamma_\Phi(X, \mathcal{F}) \to \Gamma_\Phi(X, \mathcal{F}'') \) is also surjective. Let \( s \in \Gamma_\Phi(X, \mathcal{F}'') \). By what we just proved, there is some \( t \in \mathcal{F}(X) \) mapping to \( s \), but \( t \) may not have support in \( \Phi \), so we wish to find a \( t' \in \mathcal{F}'(X) \) such that \( t - t' \) has support in \( \Phi \)—subtracting off \( t' \) from \( t \) doesn’t change its image in \( \mathcal{F}''(X) \), so this will complete the proof. Note \( t|_{X \setminus \text{Supp } s} \in \mathcal{F}(X \setminus \text{Supp } s) \) is actually in \( \mathcal{F}'(X \setminus \text{Supp } s) \). Take \( t' \) to be any lift of this to \( \mathcal{F}'(X) \) (such a \( t' \) exists since \( \mathcal{F}' \) is flasque).

**Corollary 10.4.** A flasque sheaf is \( \Gamma_\Phi \)-acyclic for any family of supports \( \Phi \) and a \( \Phi \)-soft sheaf is acyclic for any paracompactifying \( \Phi \). Any injective sheaf is flasque.

**Proof.** Follows from the theorem by standard homological algebra.

**Lemma 10.5.** Let \( \mathcal{F} \to \mathcal{G}^\bullet \) be a resolution of \( \mathcal{F} \in \mathcal{Ab}(X) \) by \( \Gamma_\Phi \)-acyclic sheaves. Then for all \( p \geq 0 \) we have
\[ H^p(\Gamma_\Phi \mathcal{G}^\bullet) = H^p_\Phi(X, \mathcal{F}). \]

**Proof.** By standard homological algebra, there is a spectral sequence with
\[ E_2^{p,q} = H^p_\Phi(X, \mathcal{G}^q) \]
converging to \( H^{p+q}_\Phi(X, \mathcal{F}) \). If every \( \mathcal{G}^q \) is acyclic, this degenerates to yield the desired isomorphism.

According to the corollary and the lemma, sheaf cohomology (with supports) can be computed using resolutions by flasque sheaves (or \( \Phi \) soft sheaves when \( \Phi \) is paracompactifying).

**Lemma 10.6. (Flasque is local)** Let \( \mathcal{F} \) be a sheaf on \( X \), \( \mathcal{U} \) an open cover of \( X \). Then \( \mathcal{F} \) is flasque if and only if \( \mathcal{F}|_U \) is a flasque sheaf on \( U \) for every \( U \in \mathcal{U} \).
**Proof.** The implication \( \implies \) is obvious since \((\mathcal{F}|_U)(V) = \mathcal{F}(V)\). For the other implication, suppose \( s \in \mathcal{F}(V) \) and we want to show there is a lift of \( s \) to \( \mathcal{F}(X) \). Consider the set of pairs \((U, t)\) where \( U \) is a neighborhood of \( V \) and \( t \in \mathcal{F}(V) \) restricts to \( s \) on \( V \). Argue as in the proof of the above theorem, using Zorn’s Lemma.

**Lemma 10.7.** A flasque sheaf is \( \Phi \) soft for any paracompactifying \( \Phi \).

**Proof.** Follows immediately from (1.7).

---

### 11. Godement resolution

Given a sheaf \( \mathcal{F} \) on a space \( X \), let \( C^0(\mathcal{F}) \) be the sheaf of discontinuous sections

\[
U \mapsto \prod_{x \in U} \mathcal{F}_x,
\]

with the obvious restriction maps. There is a natural morphism of sheaves \( \varepsilon : \mathcal{F} \to C^0(\mathcal{F}) \) taking \( t \in \mathcal{F}(U) \) to the map \((t_x)_{x \in U} \in C^0(\mathcal{F})(U)\). This map is injective because if \( t_x = t'_x \) for all \( x \in U \) then \( t = t' \) because \( \mathcal{F} \) is separated. Taking \( C^1(\mathcal{F}) := C^0(\text{Cok} \varepsilon) \) and continuing yields a resolution \( \varepsilon : \mathcal{F} \to C^\ast(\mathcal{F}) \) called the **Godement resolution** or (by Godement) canonical resolution of \( \mathcal{F} \). It is clear that from the formula for sections that, for any sheaf \( \mathcal{F} \) and any \( n \geq 0 \), the sheaf \( C^n(\mathcal{F}) \) is flasque. A morphism of sheaves \( f : \mathcal{F} \to \mathcal{G} \) induces a natural morphism of canonical resolutions.

**Theorem 11.1.** For a sheaf \( \mathcal{F} \) on a topological space \( X \), and any family of supports \( \Phi \), there is a canonical isomorphism \( H^p(\Gamma_{\Phi}C^\ast(\mathcal{F})) \cong H^p_{\Phi}(X, \mathcal{F}) \).

**Proof.** The Godement resolution is, in particular, a flasque resolution, and a flasque resolution is \( \Gamma_{\Phi} \) acyclic, so this follows from standard homological algebra.

---

### 12. Soft sheaves

A sheaf \( \mathcal{F} \) on a space \( X \) is called **soft** iff the natural map

\[
\Gamma(X, \mathcal{F}) \to \Gamma(Y, i^{-1}\mathcal{F})
\]

is surjective for every closed subset \( i : Y \hookrightarrow X \). More generally, if \( \Phi \) is a family of supports on \( X \), then \( \mathcal{F} \) is called \( \Phi \)-soft iff, for every closed embedding \( i : Y \hookrightarrow X \) with \( Y \in \Phi \), the natural map \( \mathcal{F}(X) \to \Gamma(Y, i^{-1}\mathcal{F}) \) is surjective.

### 13. Filtered direct limits

The formation of direct limits of sheaves on a space \( X \), which involves sheafification, can often be difficult to understand. However, for **filtered** direct limits, the situation simplifies greatly when \( X \) is compact or noetherian. In this section, we...
begin by showing that global sections commute with filtered direct limits on a compact or noetherian space. We use these results to prove that appropriate classes of \(\Gamma\)-acyclic sheaves are also preserved under filtered direct limits on such spaces. Combining this with some general nonsense, we find that sheaf cohomology commutes with filtered direct limits on a compact or noetherian space.

**Theorem 13.1.** Let \(X\) be a topological space, \(c \mapsto F_c\) a direct limit system of sheaves on \(X\) indexed by a filtered category \(C\). If \(X\) is quasi-compact, then the natural map

\[
\lim_{\to} \Gamma(X, F_c) \to \Gamma(X, \lim_{\to} F_c)
\]

(13.1)
is injective. If, furthermore, \(X\) is Hausdorff (i.e. compact), or noetherian, then it is an isomorphism.

**Proof.** Recall that the direct limit sheaf \(\mathcal{F} := \lim_{\to} F_c\) is given by the sheafification of the presheaf direct limit \(\mathcal{F}_{\text{pre}}\), whose sections are given by

\[
\mathcal{F}_{\text{pre}}(U) = \lim_{\to} F_c(U).
\]

The map (13.1) is the map on global sections induced by the sheafification map \(\mathcal{F}_{\text{pre}} \to \mathcal{F}\).

The injectivity when \(X\) is quasi-compact is not hard: Recall from the usual construction of filtered direct limits that an element \([s] \in \lim_{\to} F_c(X)\) is represented by an index \(c \in C\) and an element \(s \in F_c(X)\). Two such pairs \([s] = (c, s), [t] = (d, t)\) are equivalent iff there are \(C\)-morphisms \(f : c \to e\) and \(g : d \to e\) so that

\[
\mathcal{F}(f)(s) = \mathcal{F}(g)(t) \in F_e(X).
\]

Unwinding the sheafification construction, we see that \([s]\) and \([t]\) map to the same element of \(\Gamma(X, \mathcal{F})\) iff there is a cover \(\{U_i : i \in I\}\) of \(X\), a function \(e : I \to C\), and \(C\)-morphisms \(f_i : c \to e(i), g_i : d \to e(i)\) for each \(i \in I\) so that

\[
\mathcal{F}(f_i)(s|U_i) = \mathcal{F}(g_i)(t|U_i) \in F_{e(i)}(U_i).
\]

If \(X\) is quasi-compact, we can assume that \(I\) is finite, hence, since \(C\) is filtered, we can find a single index \(e \in C\) and maps \(h_i : e(i) \to e\) so that \(h_if_i = h_jf_j\) and \(h_ig_i = h_ig_j\) for every \(i, j \in I\). Call the respective common maps \(f : c \to e\) and \(g : d \to e\). From the equality (13.2) we see that

\[
\mathcal{F}(f)(s)|U_i = \mathcal{F}(f)(s|U_i)
= \mathcal{F}(h_if_i)(s|U_i)
= \mathcal{F}(h_i)(\mathcal{F}(f_i)(s|U_i))
= \mathcal{F}(h_i)(\mathcal{F}(g_i)(t|U_i))
= \mathcal{F}(h_ig_i)(t|U_i)
= \mathcal{F}(g)(t|U_i)
= \mathcal{F}(g)(t)|U_i.
\]

Since \(\mathcal{F}\) is a sheaf, we conclude \(\mathcal{F}(f)(s) = \mathcal{F}(g)(t)\), hence \([s] = [t]\).
We now prove surjectivity of (13.1) when \( X \) is compact. In general, a section 
\([s] \in \Gamma(X, \mathcal{F})\) is represented by an open cover \( \{U_i : i \in I\} \), a function \( c : I \to \mathbb{C} \), and sections \( s_i \in \mathcal{F}(U_i) \) which satisfy the following compatibility condition on each pairwise intersection \( U_{ij} = U_i \cap U_j \): For every \( x \in U_{ij} \), there are: an index \( c(i,j,x) \in \mathbb{C} \), \( \mathbb{C} \) morphisms \( f = f(i,j,x) : c(i) \to c(i,j,x) \), \( g = g(i,j,x) : c(j) \to c(i,j,x) \), and a neighborhood \( W = W(i,j,x) \) of \( x \) in \( U_{ij} \) such that
\[
(13.3) \quad \mathcal{F}(f)(s_i|W) = \mathcal{F}(g)(s_j|W) \in \mathcal{F}(c(i,j,x))(W).
\]
The reader can spell out the details of when two such collections of data represent the same section—we will implicitly replace one such data set with an equivalent one in what follows.

First, since \( X \) is compact, we can assume \( I \) is finite, hence, since \( \mathbb{C} \) is filtered, we can assume that \( c := c(i) \) is independent of \( i \in I \). (Choose an index \( c \) so that there are \( \mathbb{C} \) morphisms \( f_i : c(i) \to C \) for each \( i \in I \) and replace \( s_i \) by \( \mathcal{F}(f_i)(s_i) \).) Now, by a standard topological argument\(^1\) we can find a cover \( \{V_i : i \in I\} \) (same indexing set \( I \)) of \( \overline{V}_i \subseteq U_i \) for all \( i \in I \). Now, using compactness of \( \overline{V}_{ij} \) and the fact that \( \mathbb{C} \) is filtered, we can find an index \( c(i,j,x) \in \mathbb{C} \), a neighborhood \( W(i,j,x) \) of \( \overline{V}_{ij} \) in \( U_{ij} \), and a map \( f(i,j) : c \to c(i,j) \) so that \( c(i,j) \) (resp. \( f(i,j) \), \( f(i,j), \overline{V}_{ij} \)) can serve as \( c(i,j,x) \) (resp. \( f(i,j,x), g(i,j,x), W(i,j,x) \)) in the compatibility condition for each \( x \in \overline{V}_{ij} \). In fact, since there are only finitely many pairs \((i,j)\), we can (using \( \mathbb{C} \) filtered) even arrange that \( d := c(i,j) \) and the map \( f := f(i,j) : c \to d \) are independent of the pair \((i,j)\). By the same kind of calculation we made in the injectivity argument, the compatibility condition (13.3) now implies that
\[
\mathcal{F}(f)(s_i)|V_{ij} = \mathcal{F}(f)(s_j)|V_{ij}.
\]
Hence the local sections \( \mathcal{F}(f)(s_i)|V_i \in \mathcal{F}(V_i) \) glue to a global section \( s \in \mathcal{F}(X) \). The image of \( s \) in \( \lim_{\rightarrow} \mathcal{F}_c(X) \) clearly maps to our \([s]\) under (13.1).

The argument in the case where \( X \) is noetherian is similar, but easier. In this case, since an open subset of a noetherian space is quasi-compact, the intersections \( U_{ij} \) are quasi-compact. Using this quasi-compactness and the fact that \( \mathbb{C} \) is filtered, we find an index \( c(i,j,x) \) and a map \( f(i,j) : c \to c(i,j) \) so that \( c(i,j) \) (resp. \( f(i,j), W(i,j,x) \)) can serve as \( c(i,j,x) \) (resp. \( f(i,j,x), g(i,j,x), W(i,j,x) \)) in the compatibility condition for each \( x \in U_{ij} \). The rest of the argument is identical (replace \( U \) with \( V \) everywhere). \( \square \)

**Corollary 13.2.** Let \( X \) be a noetherian topological space, \( c \to \mathcal{F}_c \) a filtered direct limit system of sheaves on \( X \). The direct limit sheaf \( \mathcal{F} := \lim_{\rightarrow} \mathcal{F}_c \) coincides with the direct limit presheaf:
\[
\mathcal{F}(U) = \lim_{\rightarrow} \mathcal{F}_c(U)
\]
for every open subset \( U \subseteq X \).

\(^1\)This uses in an essential way the fact that \( X \) is Hausdorff. The statement we want is an easy exercise in our case, but it is also a very special case of a general “Shrinking Lemma” (c.f. [Eng, 1.5.18]).
13. FILTERED DIRECT LIMITS

Proof. An open subset of a noetherian topological space is noetherian, so this follows from the previous Theorem.

Alternatively, one can just directly prove that the direct limit presheaf is a sheaf by using Corollary 5.3 and the fact that a filtered direct limit of equalizer diagrams of sets is an equalizer diagram. (Filtered direct limits commute with finite inverse limits of sets.) □

Corollary 13.3. A filtered direct limit of flasque sheaves on a noetherian topological space is flasque. A filtered direct limit of soft sheaves on a compact topological space is soft.

Proof. Since a filtered direct limit of surjective maps (of sets) is again surjective, the first statement is immediate from Corollary 13.2, and the second is immediate from Theorem 13.1 (note \(Z\) is compact when \(X\) is compact). □

Theorem 13.4. Sheaf cohomology commutes with filtered direct limits on a compact or noetherian space \(X\). That is: Let \(c \mapsto \mathcal{F}_c\) be a filtered direct limit system of sheaves of abelian groups on a compact or noetherian space \(X\). Then the natural map

\[
\lim_{\rightarrow} H^p(X, \mathcal{F}_c) \to H^p(X, \lim_{\rightarrow} \mathcal{F}_c)
\]

is an isomorphism for all \(p \geq 0\).

Proof. Let \(C\) be the filtered category indexing the direct limit system \(c \mapsto \mathcal{F}_c\). Let \(A := \text{Hom}(C, \text{Ab}(X))\) be the abelian category of \(C\)-indexed (hence filtered) direct limit systems of sheaves of abelian groups on \(X\). For a fixed \(p\), both sides of (13.4) are natural in \((c \mapsto \mathcal{F}_c)\) and may viewed as functors \(A \to \text{Ab}\). These functors agree when \(p = 0\) by Theorem 13.1, so, by general nonsense, they will agree for all \(p\) once we prove that both sides are effaceable \(\delta\)-functors \(A \to \text{Ab}\). To see that both sides are \(\delta\)-functors, first note that the formation of filtered direct limits of abelian groups

\[
\lim_{\rightarrow} : \text{Hom}(C, \text{Ab}) \to \text{Ab}
\]

is an exact functor (this is an easy exercise and is well-known). It follows easily from this fact and the fact that sheafification is exact that

\[
\lim_{\rightarrow} : \text{Hom}(C, \text{Ab}(X)) \to \text{Ab}(X)
\]

is also exact. Now the fact that both sides are \(\delta\)-functors follows from the fact that cohomology \(H^*\) is a \(\delta\)-functor.

It remains only to prove that both sides are effaceable. For any \(\mathcal{F} \in \text{Ab}(X)\), we can find an injection \(\mathcal{F} \hookrightarrow \mathcal{F}'\) from \(\mathcal{F}\) into a flasque sheaf, functorial in \(\mathcal{F}\) (use the first step of the Godement resolution, for example), so we can find an injection from any limit system \((c \mapsto \mathcal{F}_c)\) into a limit system \((c \mapsto \mathcal{F}'_c)\) where each \(\mathcal{F}'_c\) is flasque. It now remains only to prove that, for \(p > 0\), both sides of (13.4) vanish when the \(\mathcal{F}_c\) are flasque. The left side vanishes (with no hypotheses on \(X\)) because a flasque sheaf...
IV. COHOMOLOGY

has no higher cohomology. In the noetherian case, the right side vanishes because a filtered direct limit of flasques is flasque (Corollary 13.3). In the compact case, the right side vanishes because a flasque sheaf is soft (Lemma ??), a direct limit of soft sheaves is soft (Corollary 13.3), and a soft sheaf on a (para)compact space has no higher cohomology.

14. Vanishing theorems

Recall that a closed subset $Z$ of a topological space $X$ is called irreducible iff it cannot be written as a union of two proper closed subsets. The dimension of a topological space $X$ (denoted $\dim X$) is the maximal length $n$ of a strictly increasing chain

$$\emptyset \neq Z_0 \subseteq Z_1 \subseteq \cdots \subseteq Z_n$$

of non-empty irreducible closed subsets of $X$. We set $\dim X = \infty$ if $X$ contains arbitrarily long such chains.

**Theorem 14.1. (Grothendieck’s Vanishing)** Let $X$ be a noetherian topological space of dimension $n$, $\Phi$ a family of supports on $X$. Then $H^p(X, \mathcal{F}) = 0$ for every $p > n$ and every abelian sheaf $\mathcal{F}$ on $X$.

**Proof.**

The following cohomology vanishing theorem sometimes comes in handy.

**Theorem 14.2.** Let $X$ be a zero dimensional, second countable topological space, $\mathcal{F}$ an abelian sheaf on $X$. Then $H^p(X, \mathcal{F}) = 0$ for every $p > 0$. Indeed, the global section functor is exact.

**Proof.** It is enough to prove that, for any surjection $f : \mathcal{F} \to \mathcal{G}$ of sheaves on $X$, the map $\Gamma(X, \mathcal{F}) \to \Gamma(X, \mathcal{G})$ is surjective. Let $s \in \Gamma(X, \mathcal{G})$. Since $\mathcal{F} \to \mathcal{G}$ is surjective, there are a cover $U = \{U_i : i \in I\}$ and sections $t_i$ with $f(U_i)(t_i) = s|_{U_i}$ for every $i \in I$. Since $X$ is zero dimensional and second countable, we may assume, after possibly passing to a refinement and restricting sections that $U = \{U_1, U_2, \ldots\}$ is countable and that every $U_i$ is open-and-closed in $X$. By replacing $U_i$ with the open set

$$\overline{U}_i := U_i \setminus (\cup_{j=1}^{i-1} U_j)$$

and restricting sections, we may even assume that the $U_i$ are pairwise disjoint. But then the $s_i$ automatically agree on overlaps, so by the sheaf property on $\mathcal{F}$ they lift to a section $t \in \Gamma(X, \mathcal{F})$ and $f(X)(t) = s$ by the sheaf property on $\mathcal{G}$, since this is true on each $U_i$.

15. Higher direct images

Let $f : X \to Y$ be a morphism of topological spaces. The direct image functor $f_* : \text{Ab}(X) \to \text{Ab}(Y)$ is easily seen to be left exact (this left exactness is inherited
from left exactness of the section functors). Since \( \text{Ab}(X) \) has enough injectives, we can form the right derived functors \( R^p f_* : \text{Ab}(X) \to \text{Ab}(Y) \).

**Lemma 15.1.** For a sheaf \( \mathcal{F} \in \text{Ab}(X) \), the sheaf \( R^p f_* \mathcal{F} \in \text{Ab}(Y) \) on \( Y \) is naturally isomorphic to the sheaf associated to the presheaf \( U \mapsto H^p(f^{-1}(U), \mathcal{F}|_{f^{-1}(U)}) \).

In particular, the stalks of \( R^p f_* \mathcal{F} \) are given by
\[
(R^p f_* \mathcal{F})_y = \lim_{\rightarrow} H^p(f^{-1}(U), \mathcal{F}|_{f^{-1}(U)}),
\]
where the filtered direct limit system is over neighborhoods \( U \) of \( y \) in \( Y \).

**Proof.** Certainly this is true for \( p = 0 \), so by general nonsense it suffices to prove that the indicated sheafifications form an effaceable \( \delta \)-functor from \( \text{Ab}(X) \) to \( \text{Ab}(Y) \). To see that they form a \( \delta \)-functor, first observe that before sheafifying, the presheaves defined by the above formula certain form a \( \delta \)-functor \( \text{Ab}(X) \to P\text{Ab}(Y) \). But sheafification is exact, so the sheafified versions also form a \( \delta \)-functor. To see that it is effaceable, it will suffice to show that the presheaves defined by the above formula are zero when \( p > 0 \) and \( \mathcal{F} \) is injective. This is clear from the fact that \( \mathcal{F}|_{f^{-1}(U)} \) is also injective in \( \text{Ab}(f^{-1}(U)) \) by (??), hence has no higher cohomology. \( \square \)

16. Leray spectral sequence

**Theorem 16.1. (Leray)** Let \( f : X \to Y \) be a continuous map of topological spaces, \( \mathcal{F} \) a sheaf on \( X \). There is a first quadrant spectral sequence
\[
E_2^{p,q} = H^p(Y, R^q f_* \mathcal{F}) \implies H^{p+q}(X, \mathcal{F}).
\]

**Proof.** We have \( \Gamma(Y, f_* \mathcal{F}) = \Gamma(X, \mathcal{F}) \) for any sheaf \( \mathcal{F} \) on \( X \), and by (10.1) \( f_* \) takes injective sheaves on \( X \) (which are flasque) to flasque sheaves on \( Y \) (which are \( \Gamma(Y, -) \) acyclic by (10.4)), so this spectral sequence is a special case of the Grothendieck spectral sequence. \( \square \)

One consequence of the Leray spectral sequence is the following celebrated theorem.

**Corollary 16.2. (Vietoris-Begle)** Let \( f : X \to Y \) be a proper map with acyclic fibers:
\[
H^p(f^{-1}(y), \mathbb{Z}) = \begin{cases} \mathbb{Z}, & p = 0 \\ 0, & p > 0. \end{cases}
\]

Then \( H^p(Y, \mathcal{F}) \to H^p(X, f^{-1} \mathcal{F}) \) is an isomorphism for every \( p \).

**Proof.** First of all, by a “universal coefficients” argument, the “acyclic fibers” assumption implies an analogous formula for cohomology with coefficients in any constant sheaf. We will show that the Leray spectral sequence (??) degenerates by
showing that \( f_* f^{-1} \mathcal{F} = \mathcal{F} \) and \( R^p f_* f^{-1} \mathcal{F} = 0 \) for \( p > 0 \). This follows from the formula
\[
(R^p f_* \mathbb{Z}_X)_y = H^p(f^{-1}(y), (f^{-1} \mathcal{F})|_{f^{-1}(y)})
\]
of (15.1) and the fact that \( (f^{-1} \mathcal{F})|_{f^{-1}(y)} \) is just the constant sheaf on \( f^{-1}(y) \) associated to the abelian group \( \mathcal{F}_y \), as is clear from functoriality of inverse images and commutativity of the cartesian diagram
\[
\begin{array}{ccc}
f^{-1}(y) & \longrightarrow & X \\
\downarrow & & \downarrow f \\
\{y\} & \longrightarrow & Y.
\end{array}
\]

\[\square\]

17. Group quotients

18. Local systems

For an abelian group \( A \), an \( A \)-local system or an \( A \)-locally constant sheaf on a space \( X \) is a sheaf \( \mathcal{F} \in \text{Ab}(X) \), which, locally on \( X \), is isomorphic to the constant sheaf \( A_X \). The Leray spectral sequence is most easily understood when \( f : X \to Y \) is a fiber bundle with fiber \( F \) (i.e. \( Y \) can be covered with open sets \( U_i \) so that \( f^{-1}(U_i) \) is isomorphic, over \( U_i \), to \( U_i \times F \)). In this case, the sheaf \( R^p f_* \mathbb{Z}_X \) will be an \( H^p(F, \mathbb{Z}) \)-local system on \( Y \).

19. Examples

20. Exercises

The exercises below are intended to show that one cannot prove (1.7) without some assumptions along the lines of (2) or (1).

**EXERCISE 20.1.** Construct an example where \( f : X \to Y \) is a closed embedding of finite topological spaces, but the natural monomorphism of (1.7) is not surjective.

**EXERCISE 20.2.** Let \( S \) be the Sorgenfrey line (\( \mathbb{R} \) with basic open sets \([a, b)\) for \( a < b \)), let \( Y = S \times S \), and let \( \bar{f} : X \to Y \) be the inclusion of the skew diagonal \( \{(-x, x) : x \in \mathbb{R}\} \) in the subspace topology it inherits from \( Y \). Let \( \mathcal{F} \) be the constant sheaf associated to the two element set \( \{0, 1\} \). Show that \( X \) has the discrete topology, hence
\[
\Gamma(X, f^{-1} \mathcal{F}) = \prod_{x \in X} \mathcal{F}_x = \{0, 1\}^\mathbb{R}.
\]
Show that the characteristic function of the irrational numbers \( f \in \{0, 1\}^\mathbb{R} \) is not in the image of the natural map of (1.7). **Hint:** Baire category theorem.
Exercise 20.3. Let $X$ be a locally compact Hausdorff space. Show that the family of supports $\Phi$ consisting of compact subsets of $X$ is paracompactifying. Cohomology with support in $\Phi$ in this situation is usually called compactly supported cohomology.

Exercise 20.4. Let $A$ be an abelian group, $X$ a topological space, $F$ a subsheaf of $A_X$. Show that for any $a \in A$, the set $U_a := \{ x \in X : a \in F_x \subseteq A \}$ is open in $X$. Now suppose that $A$ is a ring and $I$ is an ideal sheaf of $A_X$ (a subobject of $A_X$ in the category of $A_X$-modules). Show that, for a finitely generated ideal $I$ of $A$, the set $U_I := \{ x \in X : I \subseteq F_x \}$ is open in $X$.

Exercise 20.5. Let $F$ be a subsheaf of $\mathbb{Z}_X$ for a quasi-compact space $X$. Show that there are only finitely many $n \in \mathbb{Z}_{>0}$ such that $U_n$ (notation from the previous exercise) is nonempty. For $x \in X$, let $n(x)$ be the smallest non-negative integer such that $n(x) \in \mathbb{F}_x$. Show that the set $V := \{ x \in X : n(x) > 0 \}$ is open in $X$. Show that the sets $V_s := \{ x \in V : n(x) \leq s \}$ are open in $V$ and that $V_s = V$ for $s$ sufficiently large. Then $Z_s := V_s \setminus V_{s-1}$ is locally closed subspace of $X$. Show that the sheaves $\mathcal{F}_i := (\mathcal{F}|_{V_i})^X$ define a finite filtration of $\mathcal{F}$ with $\mathcal{F}_s/\mathcal{F}_{s-1} \cong \mathbb{Z}_{X_s}$.

Exercise 20.6. Let $X$ be an infinite set with the cofinite topology, and let $x \in X$ be a point. Show directly that $\text{Hom}_X(x, \mathbb{Z}, \mathbb{Z}_X) = 0$. Show that $x^*\mathbb{Z}_X = 0$ (c.f. Theorem 1.4). Conclude that $\mathcal{H}om(x, \mathbb{Z}, \mathbb{Z}_X)_x$ is not equal to $\text{Hom}_{\text{Ab}}((x, \mathbb{Z})_x, (\mathbb{Z}_X)_x)$ so “sheaf Hom doesn’t commute with stalks.”

Exercise 20.7. Recall that the family of constructible subsets of a topological space $X$ is the smallest family of subsets of $X$ containing the open subsets, and closed under finite intersections and complements (hence also containing the closed subsets and closed under finite unions). If $X$ is noetherian, show that a union of two locally closed subsets is locally closed, and hence the complement of a locally closed subset is locally closed. Conclude that the locally closed subsets coincide with the constructible subsets. On the other hand, if $X = \mathbb{R}$, show by example that the union of a closed subset and a locally closed subset need not be locally closed. Hint: Take a sequence of positive real numbers $x_1 > x_2 > x_3 > \cdots$ with $x_i \to 0$ as $i \to \infty$. Let $A := [x_1, x_2) \cup [x_3, x_4) \cup \cdots$ and let $B = \{0\}$. Note that $A$ is locally closed, but $A \cup B$ is not.

Exercise 20.8. Show that a direct limit of flasque sheaves need not be flasque in general.

Exercise 20.9. Let $i : U \hookrightarrow X$ be an open set. Show that the sheaf $\oplus_{i_U} \mathbb{Z}$ on $X$ considered in Section 4 coincides with the sheaf $i^*\mathbb{Z}_U$. 
EXERCISE 20.10. Give an example of a sheaf $\mathcal{F}$ on $\mathbb{R}$ such that the set $\{x \in \mathbb{R} : \mathcal{F}_x \neq 0\}$ is not locally closed in $\mathbb{R}$. Hint: Recall that a subspace of a locally compact space is locally closed if it is locally compact. Let $\alpha$ be the smallest ordinal with infinitely many limit ordinals less than it. Show that the space $\alpha+1$ (with the order topology) embeds in $\mathbb{R}$ (as does any countable ordinal with its order topology), but is not locally compact. One can arrange that $\mathcal{F}_x \neq 0$ if and only if $x$ is in the image of this embedding.

EXERCISE 20.11. If $f : \mathcal{F} \to \mathcal{G}$ is a surjective morphism of flasque sheaves then the induced maps $f^p : C^p(\mathcal{F}) \to C^p(\mathcal{G})$ are surjective for every $p \geq 0$. Hint: Induct on $p$ using 10.3.

EXERCISE 20.12. $\text{Supp} \mathcal{F} \subseteq \text{Supp} C^p(\mathcal{F})$ for every $p$.

EXERCISE 20.13. For any nonnegative integer $n$, the functor $\text{Ab}(X) \to \text{Ab}(X)$ given by $\mathcal{F} \mapsto C^n(\mathcal{F})$ is exact. Hint. Check on stalks for $p = 0$ and induct using the 3x3 Lemma.

EXERCISE 20.14. Construct an example of a topological space $X$, a basis $\mathcal{B} \subseteq \text{Ouv}(X)$ and a sheaf $\mathcal{F} \in \text{Ab}(X)$ such that $\mathcal{F}(U) \to \mathcal{F}(V)$ is surjective for every $\mathcal{B}$ morphism $V \hookrightarrow U$, but $\mathcal{F}$ is not flasque.

EXERCISE 20.15. Show that the Sorgenfrey line $S$ is zero dimensional, but not second countable. Is there a sheaf of abelian groups on $S$ (or $S \times S$) with non-vanishing higher cohomology? c.f. (14.2).
CHAPTER V

Base Change

1. Proper maps

Some of the most topologically intricate arguments in sheaf theory occur in the study of the “compatibility” of direct and inverse images with base change—the subject of this section and the next.

**Definition 1.1.** Let \( X \) be a topological space, \( Y \subseteq X \) a subspace, \( \{A_i : i \in I\} \) a family of subsets of \( X \). We say that \( Y \) is relatively Hausdorff in \( X \) iff any two distinct points of \( Y \) have disjoint neighborhoods in \( X \). We say that \( \{A_i\} \) is locally finite iff any point \( x \in X \) has a neighborhood \( U \) in \( X \) such that \( \{i \in I : U \cap A_i \neq \emptyset\} \) is finite. We say that \( X \) is paracompact iff every open cover of \( X \) has a locally finite open refinement.

**Definition 1.2.** We say that a map \( f : X \to Y \) of topological spaces is proper iff it is closed and each fiber \( f^{-1}(y) \) is compact and relatively Hausdorff in \( X \).

The condition that the fibers of \( f \) be relatively Hausdorff is equivalent to saying that the diagonal \( X \subseteq X \times Y \) is closed (i.e. \( f \) is separated). It is easy to see that this condition is closed under base change. The property of being a closed map is certainly not preserved under base change. For example, \( \pi_1 : \mathbb{R}^2 \to \mathbb{R} \) is not closed even though it is a base change of the closed map from \( \mathbb{R} \) to a point. Never-the-less, we have:

**Lemma 1.3.** Proper maps of topological spaces are closed under base change and composition. Being proper is local on the base.

**Proof.** Everything is straightforward except perhaps the statement that the base change \( \pi_1 : Z \times_Y X \to Z \) of a proper map \( f : X \to Y \) along an arbitrary map \( g : Z \to Y \) is a closed map. To see this, suppose \( W \subseteq Z \times_Y X \) is closed, and \( z \notin \pi_1(W) \). We want to see that \( z \) has a neighborhood disjoint from \( \pi_1(W) \), so that \( \pi_1(W) \) is closed. Since \( z \notin \pi_1(W) \), we know that, for any \( x \in f^{-1}(g(z)) \), \( (z,x) \notin W \). Since \( W \) is closed we can find a basic open neighborhood \( U_x \times_Y V_z \) of \( (z,x) \) disjoint from \( W \). Since \( f^{-1}(g(z)) \) is compact, we only need finitely many such \( U_x \times_Y V_z \) to cover \( \pi_1^{-1}(z) \), so we can intersect the finitely many \( U_x \) that we need to find a single open neighborhood \( U \) of \( z \in Z \) and a neighborhood \( V \) of \( f^{-1}(g(z)) \) in \( X \) so that \( U \times_Y V \) is a neighborhood of \( \pi_1^{-1}(z) \) in \( Z \times_Y X \). The set \( T := Y \setminus f(X \setminus V) \) is open in \( Y \) since \( f \) is closed, and \( T \) contains \( g(z) \) because \( V \) contains \( f^{-1}(g(z)) \), so \( g^{-1}(T) \) is an open neighborhood of \( z \) in \( Z \). I claim that \( U \cap g^{-1}(T) \) is disjoint from
\[ \pi_1(W). \] Indeed, if \( z' \in U \cap g^{-1}(T) \), then \( g(z') \in T \), so \( f^{-1}(g(z')) \subseteq V \), hence no point \((z', x) \in Z \times Y \times X \) can be in \( W \) because \( U \times_Y V \) is disjoint from \( W \).

**Lemma 1.4.** A map \( f : X \to Y \) of locally compact Hausdorff spaces is proper iff \( f^{-1}(K) \) is compact for every compact subspace \( K \subseteq Y \). If \( f : X \to Y \) and \( g : Y \to Z \) are maps of locally compact Hausdorff spaces such that \( gf \) and \( g \) are proper, then \( f \) is proper.

**Proof.** \((\implies)\) Suppose \( \{U_i : i \in I\} \) is a family of open subsets of \( X \) covering \( Z := f^{-1}(K) \). Since \( f \) is proper, each fiber of \( f \) is compact, so for each \( x \in K \), there is a finite subset \( I_x \subseteq I \) such that \( U_x := \bigcup_{i \in I_x} U_i \) covers \( f^{-1}(x) \). Since the proper map \( f \) is closed, \( V_x := Y \setminus f(Z \setminus U_x) \) is an open neighborhood of \( x \) in \( Y \), and it is clear that \( f^{-1}(y) \subseteq U_x \) for every \( y \in V_x \cap K \). Since \( K \) is compact, we can find a finite subset \( S \subseteq K \) so that \( \{V_x : x \in S\} \) covers \( K \). Then \( J := \bigcup_{x \in S} I_x \) is a finite subset of \( I \) and such that the \( U_j \) cover \( Z \).

For \((\iff)\), the only issue is to prove that \( f \) is closed. Suppose \( Z \subseteq X \) is closed and \( y \notin f(Z) \), but \( y \) is in the closure of \( f(Z) \). Choose a neighborhood \( W \) of \( y \) in \( Y \) with compact closure \( \overline{W} \). Then \( y \) is still in the closure of \( \overline{W} \cap f(Z) \). Since \( f^{-1}(\overline{W}) \) is compact and \( Z \) is closed, \( Z \cap f^{-1}(\overline{W}) \) is compact, so \( K := f(Z \cap f^{-1}(\overline{W})) \) is also compact, hence closed. But this \( K \) is a closed subset of \( Y \) containing \( \overline{W} \cap f(Z) \) with \( y \notin K \), which contradicts \( y \) being in the closure of \( \overline{W} \cap f(Z) \).

For the final statement, note that by the first part of the lemma, it suffices to prove that \( f^{-1}(K) \) is compact for any compact \( K \subseteq Y \). Certainly \( g(K) \) is compact in \( Z \), so \( (gf)^{-1}(g(K)) \) is compact because \( gf \) is proper. But \( f^{-1}(K) \) is a closed subspace of \((gf)^{-1}(g(K))\), so it is also compact.

The following technical lemmas are the main topological input for the proper base change theorem:

**Lemma 1.5.** Let \( X \) be a topological space, \( Y \subseteq X \) a compact, relatively Hausdorff subspace, \( \{U_i : i \in I\} \) a family of opens in \( X \) covering \( Y \). Then there is a finite family \( \{V_j : j \in J\} \) of opens in \( X \) whose union \( V \) contains \( Y \) and a map \( \phi : J \to I \) such that \( (V \cap V_j) \subseteq U_{\phi(j)} \) for all \( j \in J \).

**Proof.** Fix \( y \in Y \). Choose \( \phi(y) \in I \) so that \( y \in U_{\phi(y)} \). By the relatively Hausdorff assumption, we can choose, for each \( y' \in Y \setminus U_{\phi(y)} \), disjoint open neighborhoods \( S_{y,y'} \subseteq U_{\phi(y)} \) and \( W_{y,y'} \) of \( y \) and \( y' \) respectively. By compactness of \( Y \), there is a finite subset \( I_y \subseteq Y \setminus U_{\phi(y)} \) so that

\[ T_y := U_{\phi(y)} \cup \left( \bigcup_{y' \in I_y} W_{y,y'} \right) \]

is an open neighborhood of \( Y \) in \( X \). Set \( S_y := \bigcap_{y \in I} S_{y,y'} \). Then it is easy to see that \( (\overline{S_y} \cap T_y) \subseteq U_{\phi(y)} \). Again using compactness of \( Y \), find a finite \( J \subseteq Y \) so that \( \bigcup_{y \in J} S_y \) contains \( Y \). Set \( T := \bigcap_{y \in J} T_y \). Then it is easy to see that \( \{V_j := S_j \cap T\} \) is as desired. \(\square\)
Lemma 1.6. Let $X$ be a paracompact topological space, $\{U_i : i \in I\}$ an open cover of $X$. Then there exists a locally finite open cover $\{V_j : j \in J\}$ of $X$ and a map $\phi : J \to I$ so that $V_j \subseteq U_{\phi(j)}$.

Proof. Fix $x \in X$ and choose $\mu(x) \in I$ so that $x \in U_{\mu(x)}$. For each $y \in X \setminus U_{\mu(x)}$, choose disjoint open neighborhoods $S_{x,y} \subseteq U_{\mu(x)}$ and $W_{x,y}$ of $x$ and $y$ respectively. By paracompactness of $X$, we can find a locally finite family of opens $\{T_k : k \in K\}$ and a map $\psi : K \to X \setminus U_{\mu(x)}$ so that $T_k \subseteq W_{x,\psi(k)}$ and so that the $T_k$ and $U_{\mu(x)}$ cover $X$. Find an open neighborhood $R_x \subseteq U_{\mu(x)}$ so that

$$K(x) := \{k \in K : T_k \cap R_x \neq \emptyset\}$$

is finite and let $S_x := \bigcap_{k \in K(x)} S_{x,\psi(k)}$. Then it is easy to see that $S_x \subseteq U_{\mu(x)}$. Choose a locally finite refinement $\{V_j : j \in J\}$ of the cover $\{S_x : x \in X\}$, so we have a map $\nu : J \to X$ with $V_j \subseteq S_{\nu(j)}$. Then this $\{V_j\}$ is as desired if we set $\phi := \nu \circ \psi$. □

Lemma 1.7. Suppose $X$ is a topological space and $Y \subseteq X$ is a subspace. Then the natural map

$$\lim_{U \supseteq Y} \mathcal{F}(U) \to \Gamma(Y, \mathcal{F}|_Y)$$

is monic (the direct limit is over open neighborhoods $U$ of $Y$ in $X$). Suppose furthermore that at least one of the following assumptions holds:

1. $Y$ is compact and relatively Hausdorff in $X$.
2. There is a paracompact neighborhood $U$ of $Y$ in $X$ so that $Y$ is closed in $U$.

Then the natural map above is an isomorphism.

Remark 1.8. The “natural map” is $\Gamma$ of the sheafification morphism $f_{pre}^{-1}\mathcal{F} \to f^{-1}\mathcal{F}$.

Proof. To prove the first statement, suppose $U, V \supseteq Y$ are open neighborhoods and $s \in \mathcal{F}(U)$ and $t \in \mathcal{F}(V)$ agree in $\Gamma(Y, \mathcal{F}|_Y)$. Then $s_y = t_y \in \mathcal{F}_y$ for any $y \in Y$, so we can find a neighborhood $U_y$ of $y$ in $X$ (contained in $U \cap V$) with $s|U_y = t|U_y$. Then $W := \cup_{y \in Y} U_y$ is a neighborhood of $y$ in $X$ on which $s$ and $t$ agree, hence they are equal in the (filtered) direct limit.

For the second statement it remains only to prove surjectivity. By definition of the inverse image $\mathcal{F}|_Y$, a section $s \in \Gamma(Y, \mathcal{F}|_Y)$ can be viewed as an element

$$s = (s(y)) \in \prod_{y \in Y} \mathcal{F}_y$$

so that there is a family of opens $\{U_i : i \in I\}$ in $X$ with $Y \subseteq U := \cup U_i$ and sections $s_i \in \mathcal{F}(U_i)$ so that $s(y) = s_i^y$ for all $y \in U_i \cap Y$. We can always pass to a smaller neighborhood of $Y$ in $U$ and restrict our cover $U_i$ and our sections $s_i$, so under either assumption, we can assume that $\{U_i\}$ is a locally finite cover of $U$. When the second assumption holds, we can (and will) assume without loss of generality that $X$ itself is paracompact and $Y$ is closed in $X$. Our goal is to produce (after
possibly shrinking $U$ to a smaller neighborhood of $Y$ in $X$) a section $t \in \mathcal{F}(U)$ so that $t_y = s(y)$ for all $y \in Y$.

I next claim that we can further assume that there is a locally finite open cover \( \{V_j : j \in J\} \) of $U$ and a map $\phi : J \to I$ with $(\overline{V_j} \cap U) \subseteq U_{\phi(j)}$. To see this under the first assumption, apply Lemma 1.5 and then make the replacements $U \mapsto U \cap V$, $U_i \mapsto U_i \cap V$, $V_j \mapsto V_j \cap U$, etc. to assume $V = U$. (We can even assume that $J$ is finite in this case.) To see this under the second assumption, apply Lemma 1.6 to the cover of $X$ given by the $U_i$ and $X \setminus Y$.

Now let \( \{T_x : x \in U\} \) be neighborhoods in $U$ witnessing the local finiteness of \( \{U_i\} \) and \( \{V_j\} \), so the sets
\[
I(x) := \{i \in I : T_x \cap U_i \neq \emptyset\}
\]
\[
I'(x) := \{i \in I : x \in U_i\}
\]
\[
J(x) := \{j \in J : x \in V_j\}
\]
are all finite and $\phi(J(x)) \subseteq I'(x) \subseteq I(x)$. For any $x \in U$, the sections \( \{s^i : i \in I'(x)\} \) all have the same stalk (namely $s(x)$) at $x$, so, since $I'(x)$ is finite, we can find a neighborhood
\[
W_x'' \subseteq (\cap_{i \in I'(x)} U_i) \cap T_x
\]
of $x$ in $U$ so that the sections \( \{s^i : i \in I'(x)\} \) all agree on $W_x''$. Call this common section $t'(x) \in \mathcal{F}(W_x'')$. Set
\[
W_x' := W_x'' \cap (\cap_{j \in J(x)} V_j)
\]
\[
W_x := W_x' \cap (\cap_{j \in J(x); \phi(j) \in I(x)} J(x)'(X \setminus \overline{V_j}))
\]
\[
= W_x' \cap (\cup_{j \in J(x); \phi(j) \in I(x) \cap I'(x)} V_j)
\]
so that $W_x$ is a neighborhood of $x$ in $U$ because
\[
\phi(j) \in I(x) \setminus I'(x) \implies x \notin U_{\phi(j)} \implies x \notin \overline{V_j}.
\]
Set $t(x) := t'(x)|W_x$.

I claim that $t(x_1)|(W_{x_1} \cap W_{x_2}) = t(x_2)|(W_{x_1} \cap W_{x_2})$ for all $x_1, x_2 \in U$ so that the $t(x)$ glue to a global section over $U$ (which will then clearly map to $s$ as desired). We can check this on stalks at a typical point $u \in W_{x_1} \cap W_{x_2}$. Since $u \in W_{x_1}$, in particular $u \in T_{x_1}$. Since $u \in W_{x_2}$ there is $j_2 \in J(x_2)$ so that $u \in V_{j_2}$. Then $\phi(j_2)$ is in $I(x_1)$ because
\[
u \in T_{x_1} \cap V_{j_2} \subseteq T_{x_1} \cap U_{\phi(j_2)},
\]
but $\phi(j_2)$ must actually be in $I'(x_1) \subseteq I(x_1)$, because otherwise $u$ wouldn’t be in $W_{x_1}$ since it is in $\overline{V_{j_2}}$. Since $\phi(j_2) \in I'(x_1) \cap I'(x_2)$, we have $t(x_1)_y = s^j_y = t(x_2)_y$ by definition of $t(x_1), t(x_2)$.

\[\square\]
Now we begin our discussion of base change. Consider a commutative diagram

\[
\begin{array}{ccc}
X' & \xrightarrow{g} & X \\
\downarrow f' & & \downarrow f \\
Y' & \xrightarrow{h} & Y
\end{array}
\]

of topological spaces. For any sheaf \( \mathcal{F} \) on \( X \), there is a natural map

\[ h^{-1}f_*\mathcal{F} \to (f')_*g^{-1}\mathcal{F} \tag{2.1} \]

of sheaves on \( Y' \), defined as follows. First of all, by the adjointness \((h^{-1}, h_*)\), it is equivalent to give a map

\[ f_*\mathcal{F} \to h_*(f')_*g^{-1}\mathcal{F} \tag{2.2} \]

of sheaves on \( Y \). For any open set \( U \subseteq Y \), we have \((f_*\mathcal{F})(U) = \mathcal{F}(f^{-1}U)\) and

\[
(h_*(f')_*g^{-1}\mathcal{F})(U) = (g^{-1}\mathcal{F})(((f')^{-1}h^{-1}U) = \lim_{V \supseteq g((f')^{-1}h^{-1}U)} \mathcal{F}(V).
\]

But commutativity implies

\[ g((f')^{-1}h^{-1}U) = g(g^{-1}f^{-1}U) \subseteq f^{-1}(U), \]

so \( f^{-1}(U) \) is one of the \( V \)'s in the direct limit system, hence we have a natural map

\[(f_*\mathcal{F})(U) \to (h_*(f')_*g^{-1}\mathcal{F})(U)\]

given by the structure map to the direct limit. This is clearly natural in \( U \) since the restriction maps are induced by the restriction maps for \( \mathcal{F} \) on both sides, hence we obtain a map \( f_*\mathcal{F} \to h_*(f')_*g^{-1}\mathcal{F} \). Composing with \( h_*(f')_* \) of the sheafification map \( g^{-1}\mathcal{F} \to g^{-1}\mathcal{F} \) yields the map (2.2).

There are variants for pushforward with proper support, as follows: If the diagram is cartesian, then there is a natural map

\[ h^{-1}f_*\mathcal{F} \to (f')_*g^{-1}\mathcal{F} \tag{2.3} \]

defined in the same way. The key point is that, if \( s \in \mathcal{F}(U) \) has proper support \( Z \) over \( Y \), then \( g^{-1}s \in \mathcal{F}(g^{-1}(U)) \) has proper support over \( Y' \), because

\[ \text{Supp}(g^{-1}s) = g^{-1}\text{Supp }s = g^{-1}Z, \]
and we have a cartesian diagram

\[
\begin{array}{ccc}
g^{-1}Z & \longrightarrow & Z \\
\downarrow & & \downarrow \\
X' & \longrightarrow & X \\
\downarrow & & \downarrow \\
Y' & \longrightarrow & Y
\end{array}
\]

where \( Z \rightarrow Y \) is proper, hence \( g^{-1}Z \rightarrow Y' \) is also proper (Lemma 1.3).

One also has the map (2.3) when the spaces \( X, X', Y, Y' \) in the diagram are locally compact (and Hausdorff) and \( g \) and \( h \) are proper. In this case the point is that \( f'|g^{-1}Z : g^{-1}Z \rightarrow Y' \) is proper when \( f|Z : Z \rightarrow Y \) is proper. To see this, first note that \( g|g^{-1}Z \rightarrow Z \) is proper (since it is a base change of \( g \)), hence the composition \( (f|Z)(g|g^{-1}Z) \) is proper (Lemma 1.3) and we have

\[
h(f'|g^{-1}Z) = (f|Z)(g|g^{-1}Z)
\]

by commutativity, hence \( f'|g^{-1}Z \) is proper by the 2-out-of-3 property of proper maps from Lemma 1.4.

An important special case of the map (2.5) occurs when \( h : Y' \rightarrow Y \) is the inclusion of a point \( y \in Y \) and the diagram is cartesian so that \( X' = X_y \) is the fiber of \( f \) over \( y \in Y \). In this case, \( h^{-1} \) is the stalk functor \( \mathcal{F} \mapsto \mathcal{F}_y \) and \( f' \) is the functor of global sections over \( X_y \), so (2.5) can be viewed as a map

\[
(2.4) \quad (f_*\mathcal{F})_y \rightarrow \Gamma(X_y, \mathcal{F}|X_y).
\]

The next result is a special case of the general base change theorem (Theorem ??), but it actually turns out that the more general base change theorem reduces to this special case rather easily.

**Theorem 2.1.** Suppose \( f : X \rightarrow Y \) is a proper map. Then the natural map (2.4) is an isomorphism for every \( y \in Y \) and every sheaf \( \mathcal{F} \) on \( X \). For any \( \mathcal{F} \in \text{Ab}(X) \), the natural map \( (R^i f_*\mathcal{F})_y \rightarrow H^i(X_y, \mathcal{F}|X_y) \) is an isomorphism for all \( i \).

**Proof.** The natural map in question is the natural map

\[
\lim_{U} \mathcal{F}(f^{-1}(U)) \rightarrow \Gamma(X_y, \mathcal{F}|X_y),
\]

where the direct limit runs over all neighborhoods \( U \) of \( y \) in \( Y \). But the fact that \( f \) is closed implies that the \( f^{-1}(U) \) are cofinal in the set of all neighborhoods of the fiber \( X_y \), so the map in question is the same as the map

\[
\lim_{V} \mathcal{F}(V) \rightarrow \Gamma(X_y, \mathcal{F}|X_y),
\]

where \( V \) runs over all neighborhoods of \( X_y \) in \( X \). Since the fiber \( X_y \) is compact and relatively Hausdorff in \( X \), this map is an isomorphism by Lemma 1.7(1).
The second statement follows from the first by general “universal $\delta$-functors” nonsense.

**Theorem 2.2.** Suppose

\[
\begin{array}{ccc}
X' & \xrightarrow{g} & X \\
\downarrow{f'} & & \downarrow{f} \\
Y' & \xrightarrow{h} & Y
\end{array}
\]

is a cartesian diagram of topological spaces with $f$ proper. Then the natural map

\[
h^{-1}f_* \mathcal{F} \to (f')_*g^{-1}\mathcal{F}
\]

is an isomorphism for every sheaf $\mathcal{F}$ on $X$.

**Proof.** We can check this on stalks at any $y' \in Y'$. But $f'$ is also proper (Lemma 1.3) and $f$ and $f'$ have the same fibers, so we reduce formally to the situation of the previous theorem. \qed

3. Homotopy invariance

Let $I := [0, 1]$ denote the unit interval. For a continuous map $H : X \times I \to Y$ and $t \in I$, let $t_t : X \leftrightarrow X \times I$ denote the map $x \mapsto (x, t)$ and set $H_t := H_{t_t} : X \to Y$. Maps $f, g : X \to Y$ are called homotopic if there is a homotopy $f$ to $g$: a map $H : X \times I \to Y$ such that $H_0 = f$ and $H_1 = g$. Being homotopic is an equivalence relation on maps preserved by compositions.

Sheaf cohomology enjoys the following homotopy invariance property:

**Theorem 3.1.** Let $f, g : X \to Y$ be homotopic maps of topological spaces. For any abelian group $A$, the maps $f^*, g^* : \text{H}^p(Y, A) \to \text{H}^p(X, A)$ coincide.

**Proof.** Let $H : X \times I \to Y$ be a homotopy from $f$ to $g$. The maps $t_0, t_1$ induce isomorphisms on cohomology by the “2-out-of-3” property of isomorphisms, the fact that $\pi_{1,t} = \text{Id} : X \to X$, and the fact that $\pi_1 : X \times I \to X$ induces isomorphisms on all constant sheaf cohomology by Vietoris-Begle (use the fact that the interval is acyclic—see the computations in (??)). The result is now immediate since $H_{t_0} = f$ and $H_{t_1} = g$. \qed

**Warning:** Do not read too much into the “homotopy invariance” property above. Many people falsely believe that a sheaf of abelian groups on a contractible topological space can have no higher cohomology groups (this is true for constant sheaves). In fact, the very existence of the Leray spectral sequence ensures that this cannot be the case:

**Example 3.2.** Let $S^1 \subseteq \mathbb{R}^2$ be the unit circle and let $f : S^1 \to I$ be the projection from $S^1$ onto the interval $I = [-1, 1]$ so that $f$ is a double cover branched over $\{\pm 1\} \subseteq I$. Certainly $f$ is a proper map by (3.4), so we easily conclude that
\( R^0 f_* = 0 \) by the stalk formula (3.5) and the fact that no sheaf has higher cohomology on a space with at most two points. The Leray spectral sequence for \( f \) therefore yields an isomorphism

\[
H^1(S^1, \mathbb{Z}) = H^1(I, f_* \mathbb{Z}_{S^1}).
\]

Note that this group is \( \mathbb{Z} \) by the computations in (??). To get a feel for the sheaf \( \mathcal{F} := f_* \mathbb{Z}_{S^1} \), note that

\[
\mathcal{F}([-1, 1)) = \mathbb{Z} \\
\mathcal{F}([-1, 1]) = \mathbb{Z} \\
\mathcal{F}((-1, 1)) = \mathbb{Z}^2
\]

and both restriction maps

\[
\mathcal{F}([-1, 1)), \mathcal{F}([-1, 1]) \to \mathcal{F}((-1, 1))
\]

are the diagonal \( \Delta : \mathbb{Z} \to \mathbb{Z}^2 \).
CHAPTER VI

Duality

1. Adjoint for immersions

Let \( f : X \to Y \) be a morphism of topological spaces. We always have a left adjoint \( f^{-1} : \text{Ab}(Y) \to \text{Ab}(X) \) to \( f_* : \text{Ab}(X) \to \text{Ab}(Y) \). It would also be desirable, for many reasons, to have a right adjoint \( f^! : \text{Ab}(Y) \to \text{Ab}(X) \) to \( f_* \). In general, we know this is impossible because the existence of a right adjoint \( f^! \) to \( f_* \) would imply that \( f_* \) commutes with all direct limits, which it generally does not (even when \( Y \) is a point). In the general case then, one is forced to look for the right adjoint \( f^! \) only in an appropriate derived category sense. However, in the case where \( f \) is a closed embedding (so \( f_* \) does preserve direct limits) we will construct the desired right adjoint \( f^! \) in this section by elementary means.

In fact, it is natural to work not only with closed embeddings, but with immersions. Recall that \( f : X \to Y \) is an immersion iff \( f \) can be factored \( f = ji \) as an open embedding \( i \) followed by a closed embedding \( j \). In other words, \( f \) is an immersion iff it is an isomorphism onto a subset \( A \subseteq X \) which can be written \( A = Z \cap U \) where \( Z \) is closed in \( X \) and \( U \) is open in \( X \). Such a subset \( A \) is called locally closed, and an immersion is hence sometimes called a locally closed embedding. The term “locally closed” is used because of the following:

**Lemma 1.1.** \( A \subseteq X \) is locally closed iff, for each \( x \in A \) there is a neighborhood \( U_x \) of \( x \) in \( X \) such that \( A \cap U_x \) is closed in \( U_x \).

**Proof.** The implication \( \Rightarrow \) is clear. If \( A \) satisfies the second condition, then \( A \) will be closed in \( U := \bigcup_{x \in A} U_x \) because being closed is local, hence \( A = U \cap Z \) for some \( Z \) closed in \( X \), i.e. \( A \) is locally closed in our original sense. \( \square \)

Here are the basic facts about immersions, all of which follow easily from the lemma and discussion above.

**Proposition 1.2.** An open embedding is an immersion, and a closed embedding is an immersion. Immersions are the smallest class of morphisms containing open embeddings and closed embeddings and closed under composition. Immersions are closed under base change, and the property of being an immersion is local on the codomain (base).
In a noetherian topological space, locally closed subsets are the same as constructible subsets, but this is not true in general (Exercise 20.7).

**Lemma 1.3.** Let \( j : Z \hookrightarrow X \) be the inclusion of any subspace. Then \( j_* : \text{PAb}(Z) \to \text{PAb}(X) \) is fully faithful, hence \( j_* : \text{Ab}(Z) \to \text{Ab}(X) \) is also fully faithful.

**Proof.** We want to show that for any \( \mathcal{F}, \mathcal{G} \in \text{PAb}(Z) \), the map
\[
j_* : \text{Hom}_{\text{PAb}(Z)}(\mathcal{F}, \mathcal{G}) \to \text{Hom}_{\text{PAb}(X)}(j_* \mathcal{F}, j_* \mathcal{G})
\]
is bijective. Given any morphism \( \text{PAb}(X) \) morphism \( f : j_* \mathcal{F} \to j_* \mathcal{G} \) and any \( U \in \text{Ouv}(X) \), the fact that the \( f(U) \) respect the restriction maps implies that the morphism \( f(U) : \mathcal{F}(U \cap Z) \to \mathcal{G}(U \cap Z) \) depends only on \( U \cap Z \): Indeed, if \( V \cap Z = U \cap Z \), then both are also equal to \( (V \cap U) \cap Z \) and the restriction maps for the inclusions \( (V \cap U) \hookrightarrow V \) and \( (V \cap U) \hookrightarrow U \) are the identity for both presheaves \( j_* \mathcal{F}, j_* \mathcal{G} \). In other words, for \( U \in \text{Ouv}(Z) \), we can define a map \( \tilde{f}(U) : \mathcal{F}(U) \to \mathcal{G}(U) \) unambiguously by choosing any \( U' \in \text{Ouv}(X) \) with \( U = U' \cap Z \) and setting \( \tilde{f}(U) := f(U') \). The maps \( \tilde{f}(U) \) clearly respect the restriction maps since this is true for the maps \( f(U') \) and any inclusion of opens in \( Z \) lifts to an inclusion of opens in \( X \). The function \( f \mapsto \tilde{f} \) hence defines an inverse to our \( j_* \) above. \( \square \)

**Theorem 1.4.** Let \( j : Z \hookrightarrow X \) be a closed subspace of a topological space \( X \).

(1) The direct image functor \( j_* : \text{Ab}(Z) \to \text{Ab}(X) \) commutes with all direct and inverse limits and is fully faithful. Its essential image consists of those \( \mathcal{F} \in \text{Ab}(X) \) where \( \text{Supp} \mathcal{F} \subseteq Z \).

(2) For \( \mathcal{F} \in \text{Ab}(X) \), define \( j^! \mathcal{F} \in \text{Ab}(Z) \) by
\[
U \mapsto \Gamma_{Y \cap U}(U, \mathcal{F})
\]

\[
= \{ s \in \mathcal{F}(U) : \text{Supp } s \subseteq Z \cap U \}
\]
on \( X \) (with the obvious restriction maps inherited from those of \( \mathcal{F} \)). The functor \( j^! : \text{Ab}(X) \to \text{Ab}(Z) \) is right adjoint to \( j_* \) and the adjunction morphism \( \mathcal{F} \to j^! j_* \mathcal{F} \) is an isomorphism for any \( \mathcal{F} \in \text{Ab}(X) \).

**Proof.** First of all, any section \( s \) of \( j_* \mathcal{F} \) over \( U \in \text{Ouv}(X) \) trivially satisfies \( \text{Supp } s \subseteq Z \cap U \) since the stalks of \( j_* \mathcal{F} \) at any \( x \in X \setminus Z \) are zero, so the final statement of the theorem is clear. Also, if \( f : j_* \mathcal{F} \to \mathcal{G} \) is any \( \text{Ab}(X) \) morphism, \( U \in \text{Ouv}(X) \), \( s \in (j_* \mathcal{F})(U) \), then obviously
\[
\text{Supp } f(U)(s) \subseteq \text{Supp } s \subseteq Z \cap U,
\]
so \( f \) factors uniquely through the subsheaf \( j_* j^{-1} \mathcal{G} \subseteq \mathcal{G} \), hence we have
\[
\text{Hom}_{\text{Ab}(X)}(j_* \mathcal{F}, \mathcal{G}) = \text{Hom}_{\text{Ab}(X)}(j_* \mathcal{F}, j_* j^{-1} \mathcal{G}).
\]
But by Lemma 1.3, \( j_* \) is fully faithful, so
\[
\text{Hom}_{\text{Ab}(X)}(j_* \mathcal{F}, j_* j^{-1} \mathcal{G}) = \text{Hom}_{\text{Ab}(Z)}(\mathcal{F}, f^{-1} \mathcal{G})
\]
and putting the two together yields the desired adjunction. The functor $j_*$ commutes with all limits because it is both a left adjoint (to $j^!$) and a right adjoint (to $j^{-1}$).

**Theorem 1.5.** Let $i : U \to X$ be an open subspace of a topological space $X$. For a $\mathcal{F} \in \textbf{Ab}(U)$, let $i_!\mathcal{F}$ be the sheaf associated to the presheaf $i_{!\text{pre}}\mathcal{F}$ defined by

$$V \mapsto \begin{cases} \mathcal{F}(V), & V \subseteq U \\ \{0\}, & \text{otherwise.} \end{cases}$$

The functor $i_! : \textbf{Ab}(U) \to \textbf{Ab}(X)$ is left adjoint to $i^{-1}$ (i.e. the restriction functor $\mathcal{F} \mapsto \mathcal{F}|U$).

**Proof.** The functor $i_{!\text{pre}} : \textbf{PAb}(U) \to \textbf{PAb}(X)$ is clearly left adjoint to $i^{-1} : \textbf{PAb}(X) \to \textbf{PAb}(U)$, so the desired adjointness follows formally from the adjointness property of sheafification.

**Remark 1.6.** Unravelling the local coherence condition in the sheafification construction (7), we see that the sheaf $i_!\mathcal{F}$ can be explicitly described as follows: A section $s \in (i_!\mathcal{F})(V)$ is an element $s = (s(x)) \in \prod_{x \in V \cap U} \mathcal{F}_x$ satisfying the two conditions:

1. For every $x \in V \setminus U$ there is a neighborhood $W$ of $x$ in $V$ such that $s(x') = 0$ for all $x' \in W \cap U$.
2. For every $x \in V \cap U$, there is a neighborhood $W$ of $x$ in $V \cap U$ and a section $t \in \mathcal{F}(W)$ such that $t_{x'} = s(x')$ for $x' \in W$.

2. Extension by zero

Extension by zero will allow us to construct a kind of “wrong way” map which is the basis for Poincaré and Verdier duality.

**Lemma 2.1.** Let $i : U \to X$ be the inclusion of an open subset, $\mathcal{F}$ a sheaf on $U$. The sheaf $i_!\mathcal{F}$ associated to the presheaf

$$i_{!\text{pre}}\mathcal{F} : V \mapsto \begin{cases} \mathcal{F}(V), & V \subseteq U \\ 0, & \text{otherwise.} \end{cases}$$

is given by $(i_!\mathcal{F})(V) = \{ s \in \mathcal{F}(V \cap U) : \text{Supp } s \text{ is closed in } V \}$. The stalk of $i_!\mathcal{F}$ at $x \in X$ is $\mathcal{F}_x$ if $x \in U$ and zero otherwise.

**Proof.** First note that $i_!\mathcal{F}$ is a sheaf because being closed is a local property. If $V \subseteq U$, then $U \cap V = V$ and the support of any $s \in \mathcal{F}(V)$ is certainly closed in $V$, so there is a natural morphism $i_{!\text{pre}}\mathcal{F} \to i_!\mathcal{F}$. To see that this induces an isomorphism $(i_{!\text{pre}}\mathcal{F})^+ \to i_!\mathcal{F}$, it suffices to check that it is an isomorphism on stalks. Certainly if $x \in U \subseteq X$, then $(i_{!\text{pre}}\mathcal{F})_x \to (i_!\mathcal{F})_x$ is an isomorphism because both sheaves coincide on all sufficiently small neighborhoods of $x$ (namely those contained in $U$). It remains only to show that $(i_!\mathcal{F})_x = 0$ for $x \in X \setminus U$. Indeed, if $V$ is any neighborhood of $x$ and $s \in (i_!\mathcal{F})(V)$ any section, then $s$ restricts to zero on the neighborhood $V \setminus \text{Supp } s$ of $x$ and is hence zero in the stalk. □
Corollary 2.2. The functor \( \iota_! : \text{Ab}(U) \to \text{Ab}(X) \) is exact for any open set \( \iota : U \hookrightarrow X \).

Proof. If \( 0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0 \) is exact on \( U \), then one easily checks that
\[
0 \to \iota_! \mathcal{F}' \to \iota_! \mathcal{F} \to \iota_! \mathcal{F}'' \to 0
\]
is exact on \( X \) by checking exactness at stalks using the lemma. \( \square \)

The sheaf \( \iota_! \mathcal{F} \) is called the extension by zero of \( \mathcal{F} \) (it is sometimes denoted \( \mathcal{F}^X \)). It is manifestly a subsheaf of \( i_* \mathcal{F} \), though the two do not generally coincide. For example, if \( i : U \to X \) is the inclusion of the open interval \((0,1)\) in the closed interval \([0,1]\) and \( \mathcal{F} = \underline{A}_{U} \) is a constant sheaf on \( U \), then \( i_* \mathcal{F} \) is just the constant sheaf \( \underline{A}_{X} \), which has stalk \( A \) at every point of \( X \), while the stalk of \( \iota_! \mathcal{F} \) is zero at \( 0,1 \in X \).

Whenever \( i : U \hookrightarrow X \) and \( j : V \hookrightarrow X \) are open sets with \( U \subseteq V \) and \( \mathcal{F} \in \text{Ab}(X) \), we have a natural morphism \( \iota \iota^{-1} \mathcal{F} \to j j^{-1} \mathcal{F} \). On \( W \in \text{Ouv}(X) \), this map sends a section \( s \in \mathcal{F}(W \cap U) \) with \( \text{Supp} \, s \) closed in \( W \) (hence also in \( W \cap V \)) to the section of \( \mathcal{F}(W \cap V) \) obtained by gluing \( s \) and \( 0 \in \mathcal{F}((W \cap V) \setminus \text{Supp} \, s) \). Note that the support of this glued section is the same as the support of \( s \), hence it is still closed in \( W \) and this is a well-defined map
\[
(i \iota^{-1} \mathcal{F})(W) \to (j j^{-1} \mathcal{F})(W)
\]
which clearly respects restriction.

Theorem 2.3. (Mayer-Vietoris Sequences) Let \( i : U \hookrightarrow X \) be the inclusion of an open set, \( j : Z \hookrightarrow X \) its closed complement. There is a short exact sequence
\[
0 \to i i^{-1} \mathcal{F} \to \mathcal{F} \to j j^{-1} \mathcal{F} \to 0
\]
in \( \text{Ab}(X) \) natural in \( \mathcal{F} \in \text{Ab}(X) \). Let \( k : V \hookrightarrow X \) be another open set and let \( l : U \cup V \hookrightarrow X \) and \( m : U \cap V \hookrightarrow X \) be the inclusions. There is a short exact sequence
\[
0 \to m \iota \iota^{-1} \mathcal{F} \to i i^{-1} \mathcal{F} \oplus k k^{-1} \mathcal{F} \to l l^{-1} \mathcal{F} \to 0.
\]

Proof. Check exactness on stalks by using the previous lemma. \( \square \)

Theorem 2.4. (Duality for an open embedding) Let \( i : U \hookrightarrow X \) be the inclusion of an open subset of a topological space \( X \). The functor \( i^{-1} : \text{Ab}(X) \to \text{Ab}(U) \) is right adjoint to the exact functor \( i_! : \text{Ab}(U) \to \text{Ab}(X) \) and \( i^{-1} i_! = \text{Id} \).

The functor \( i^{-1} : \text{Ab}(X) \to \text{Ab}(U) \) takes injectives to injectives.

Proof. Checking the adjointness is a straightforward exercise and the equality \( i^{-1} i_! = \text{Id} \) is clear from the definitions. The final statement then follows formally: Suppose \( \mathcal{F}' \in \text{Ab}(X) \) is injective and
\[
0 \to \iota^{-1} \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0
\]
is an exact sequence in $\text{Ab}(U)$. Applying $\iota$ and (2.2), we obtain a diagram of solid arrows with exact row

$$0 \to \iota_{1}^{-1}F' \to \iota_{1}s \to \iota_{1}s'' \to 0$$

in $\text{Ab}(X)$, which completes as indicated by injectivity of $F' \in \text{Ab}(X)$. Applying the functor $\iota^{-1}$ and using $\iota^{-1}\iota = \text{Id}$, the vertical arrow above becomes an isomorphism and $\iota^{-1}s$ provides a splitting of the original sequence. □

3. Proper maps - Second treatment

To better understand the stalk formula in (15.1), we will need to prove some rather technical lemmas. It turns out that the stalk formula behaves very well for proper maps, which we now introduce.

**Definition 3.1.** A subspace $Z \subseteq X$ is called relative Hausdorff if any two points of $Z$ have disjoint neighborhoods in $X$. A map of topological spaces $f : X \to Y$ is called proper if $f$ is closed (the image of any closed set is closed) and the fibers of $f$ are compact and relative Hausdorff in $X$.

For example, the inclusion of a closed subspace $Z \hookrightarrow X$ is always a proper map. A space $X$ is proper over a point iff $X$ is compact Hausdorff. The basic properties of proper maps are summed up in the following theorem.

**Theorem 3.2.** The composition of proper maps is proper. Proper maps are preserved under arbitrary base change. A map $f : X \to Y$ is proper iff for some (equivalently any) cover $\{U_i\}$ of $Y$, the maps $f : f^{-1}(U_i) \to U_i$ are proper. If $f : X \to Y$ is proper and $y \in Y$, then the neighborhoods $f^{-1}(U)$ of $f^{-1}(y)$ (as $U$ ranges over all neighborhoods of $y \in Y$) are cofinal in the set of neighborhoods of $f^{-1}(y)$ in $X$.

**Proof.** We first prove that proper maps are closed under base change. Let $g : Z \to Y$ be arbitrary, and let $Z \times_Y X$ be the fibered product. To check that $\pi_1 : Z \times_Y X \to Z$ is closed, it is enough to check that $\pi_1(A \times_Y B)$ is closed whenever $A \subseteq Z$ and $B \subseteq X$ are closed, since closed sets of this form form a basis for all closed sets. On the other hand, we have $\pi_1(A \times_Y B) = A \cap g^{-1}(f[B])$, which is closed when $f$ is closed. Similarly, $\pi_1(z) = \{z\} \times f^{-1}(g(z))$ is compact if $f^{-1}(g(z))$ is compact and if $p, q \in f^{-1}(g(z))$ have disjoint neighborhoods $U, V$ in $X$, then $Z \times_Y U$ and $Z \times_Y V$ are disjoint neighborhoods of $(z, p)$ and $(z, q)$ in $Z \times_Y X$.

The fact that being proper is local on the codomain follows from the fact that being closed is local; the “compact relative Hausdorff fibers” condition is clearly local on the codomain.
The last statement is immediate from the fact that \( f \) is closed: if \( V \) is an open neighborhood of \( f^{-1}(y) \) in \( X \), then \( f[X \setminus V] \) is closed in \( X \) since \( f \) is closed, so \( Y \setminus f[X \setminus V] \) is an open neighborhood of \( y \) in \( Y \) and we have \( f^{-1}(Y \setminus f[X \setminus V]) \subseteq V \).

**Lemma 3.3.** A map \( f : X \to Y \) of locally compact Hausdorff spaces is compact \( \iff \) \( f^{-1}(K) \) is compact for every compact subset \( K \subseteq Y \).

**Proof.** Exercise ??.

**Corollary 3.4.** If \( f : X \to Y \) and \( g : Y \to Z \) are maps of locally compact Hausdorff spaces such that \( gf \) is proper, then \( f \) is proper. It follows that any map between compact Hausdorff spaces is proper.

**Proof.** By the lemma, it suffices to prove that \( f^{-1}(K) \) is compact for any compact \( K \subseteq Y \). Certainly \( g[K] \) is compact in \( Z \), so \( (gf)^{-1}(g[K]) \) is compact because \( gf \) is proper. But \( f^{-1}(K) \) is a closed subspace of \( (gf)^{-1}(g[K]) \), so it is also compact.

**Lemma 3.5.** Let \( f : X \to Y \) be a proper map, \( \mathcal{F} \) a sheaf on \( X \). For every \( y \in Y \), the natural map \((R^p f_!, \mathcal{F})_y \to H^p(f^{-1}(y), \mathcal{F}|_{f^{-1}(y)})\) is an isomorphism.

**Proof.** Apply (1.7) to the inclusion of the subspace \( f^{-1}(y) \subseteq X \) using the assumption (1) and use (3.2) to note that the \( f^{-1}(U) \) are cofinal in the neighborhoods of \( f^{-1}(y) \).
CHAPTER VII

Finite Spaces

This section is devoted to sheaf theory on finite, typically non-Hausdorff, topological spaces. We begin with some preliminaries on the topology of a finite topological space $X$.

1. Supports

For any family of supports $\Phi$ on $X$, there is a closed subspace $Y$ such that $\Phi$ consists of all closed subsets of $Y$. Indeed, just take $Y$ to be the union of every closed set in $\Phi$: then $Y$ is in $\Phi$ because $\Phi$ is closed under finite unions and every closed subset of $Y$ is in $\Phi$ because $\Phi$ is closed under passage to smaller closed subsets. As usual, we will write $\Gamma_Y(X, \mathcal{F})$ instead of $\Gamma_{\Phi}(X, \mathcal{F})$.

2. Specialization

The points of any topological space $X$ are quasi-ordered by specialization: $x \leq y$ iff $x \in \{y\}^-$. We are only interesting in making statements about $\text{Sh}(X)$, so there would be no harm in replacing $X$ with its sobrification, thereby ensuring that $\leq$ is, in fact, a partial order (equivalently: $X$ is $T_0$). The specialization ordering is particularly important when $X$ is finite, as it completely determines the topology of $X$. A set is closed iff it is closed under passage to smaller elements ("specializations"): this is because any point in the closure of a finite set $S$ must be in the closure of $\{s\}$ for some $s \in S$. A subset of $X$ is open iff it is closed under passage to larger elements.

In particular, the set $U_1$ consisting of $\leq$ maximal points of (finite) $X$ is open. Let $Z_1$ be its closed complement. Similarly, the set $U_2$ of $\leq$ maximal points of $Z_1$ is open in $Z_1$ and $Z_2 := Z_1 \setminus U_2$ is hence closed in $X$. Continuing in this manner, we obtain a strictly decreasing sequence

$$X = Z_0 \supseteq Z_1 \supseteq \cdots \supseteq Z_n \supseteq \emptyset$$

of closed subspaces of $X$.

**Lemma 2.1.** Let $X$ be a sober noetherian space. The noetherian dimension $n$ of $X$ is equal to the length of the longest chain $x_0 < x_1 < \cdots < x_n$ in $(X, \leq)$ ($n = \infty$ iff there are arbitrarily long such chains).

**Proof.** Suppose $\emptyset \subsetneq Z_0 \subseteq \cdots \subseteq Z_n$ is a chain of irreducible closed subspaces in $X$. Since $X$ is sober, each $Z_i$ has a generic point $x_i$. We have $x_0 < x_1 < \cdots < x_n$
in \((X, \leq)\). Conversely, given such a chain in \((X, \leq)\), we get a chain of irreducible closed subspaces \(\emptyset \subsetneq \{x_0\}^- \subsetneq \cdots \subsetneq \{x_n\}^-\). \[\square\]

3. Coniveau

Let \(X\) be any ringed space, not necessarily finite, and let \(F \in \text{Mod}(X)\). For any specialization \(x \leq y\), we have a natural map \(f_{x,y} : F_x \to F_y\) which is \(O_{X,x} \to O_{Y,y}\) linear. It is nothing but the direct limit of the maps \(F(U) \to F_y\) over neighborhoods \(U\) of \(x\), all of which contain \(y\) since \(x \in \{y\}^-\). We view \(f_{x,y}\) as an \(O_{X,x}\) linear map by regarding \(F_y\) as an \(O_{X,x}\) module by restriction of scalars along \(O_{X,x} \to O_{Y,y}\).

Applying \(x^*\), we obtain a \(\text{Mod}(X)\) morphism

\[x^*f_{x,y} : x^*F_x \to x^*F_y.\]

If we have a sequence of specializations \(x \leq y \leq z\), then it is clear that

\[f_{x,z} = f_{y,z}f_{x,y},\]

so we may in fact view the stalk functor as a functor

\[\text{Ab}(X) \to \text{Hom}_{\text{Cat}}((X, \leq), \text{Ab}).\]

The coniveau complex is the complex

\[0 \to F \to \prod_{x_0} x_0^*F_{x_0} \to \prod_{x_0 < x_1} x_0^*F_{x_1} \to \prod_{x_0 < x_1 < x_2} x_0^*F_{x_2} \to \cdots\]

in \(\text{Mod}(X)\), where the products are over strictly increasing chains in \((X, \leq)\) and where the boundary maps are given by

\[(d^n s)(x_0, \ldots, x_{n+1}) := x_1^*f_{x_1, x_{n+1}}s(x_1, \ldots, x_{n+1}) + \sum_{i=1}^n (-1)^i x_0^*f_{x_0, x_{n+1}}s(x_0, \ldots, \hat{x}_i, \ldots, x_{n+1}) + (-1)^{n+1} x_0^*f_{x_0, x_n}s(x_0, \ldots, x_n).\]

By abuse of notation, we will often simply write

\[(d^n s)(x_0, \ldots, x_{n+1}) := \sum_{i=1}^n (-1)^i s(x_0, \ldots, \hat{x}_i, \ldots, x_{n+1})|_{x_{n+1}}.\]

Observe that this complex “depends on \(X\)” even when \(\text{Sh}(X)\) does not. For example, the coniveau complex computed on \(X\) is not the same as the coniveau complex computed on its sobrification \(X^{\text{sob}}\) even though \(\text{Sh}(X) = \text{Sh}(X^{\text{sob}})\). It is generally more natural to work with the coniveau complex computed on \(X^{\text{sob}}\), where the specialization ordering better reflects the topology of \(X\).
4. Equivalence

When \( X \) is finite, every \( x \in X \) has a smallest neighborhood \( U_x \) and \( \mathcal{F}_x = \mathcal{F}(U_x) \). We will write \( i_x : U_x \to X \) for the inclusion of this open set throughout. Note \( x \leq y \) iff \( U_y \subseteq U_x \). When \( x \leq y \), the restriction map \( \mathcal{F}(U_x) \to \mathcal{F}(U_y) \) may be viewed as a map \( f_{x,y} : \mathcal{F}_x \to \mathcal{F}_y \) (we will usually denote this \( s \mapsto s|_y \)). The restriction maps \( f_{x,y} \) completely determine the sheaf \( \mathcal{F} \), as we will now show. Note that \( U = \cup_{x \in U} U_x \) for any \( U \in \text{Ouv}(X) \) and

\[
U_x \cap U_y = \bigcup_{z \geq x,y} U_z,
\]

so, by using the sheaf property for \( \mathcal{F} \), we can recover \( \mathcal{F}(U) \) as the equalizer

\[
(4.1) \quad \mathcal{F}(U) = \lim_{\leftarrow} \left( \prod_{x \in U} \mathcal{F}_x \rightrightarrows \prod_{(x,y) \in U \times U} \prod_{z \geq x,y} \mathcal{F}_z \right),
\]

where the two maps are \((s_x) \mapsto (f_{x,z}(s_x))\) and \((s_x) \mapsto (f_{y,z}(s_x))\). There is an equivalence of categories

\[
\text{Sh}(X) = \text{Hom}_{\text{Cat}}((X, \leq), \text{Ens}) \quad \mathcal{F} \mapsto (x \mapsto \mathcal{F}_x)
\]

natural in \( X \), whose inverse is given by taking a functor \( f \) to the presheaf (in fact sheaf) whose sections are defined by the formula (4.1).

5. Stalk formulas

The formulae

\[
(x_*A)(y) = \begin{cases} A, & y \leq x \\ 0, & \text{otherwise} \end{cases}
\]

\[
(i_x!A)_y = \begin{cases} A, & x \leq y \\ 0, & \text{otherwise} \end{cases}
\]

are clear from the formulae for stalks (3), (2.1).

6. Hom formulas

Similarly, we have

\[
\text{Hom}_X(\mathcal{F}, x_*A) = \text{Hom}(\mathcal{F}_x, A)
\]

\[
\text{Hom}_X(i_x!A, \mathcal{F}) = \text{Hom}(A, \mathcal{F}_x).
\]

For example, the first isomorphism is given by mapping \( g : \mathcal{F} \to x_*A \) to \( g_x : \mathcal{F}_x \to A \). This is an isomorphism because \( g_y : \mathcal{F}_y \to (x_*A)_y \) must be given by \( g_x f_{x,y} \) if \( x \leq y \) (or zero otherwise).
There is a kind of duality for finite spaces obtained by inverting the specialization ordering (replacing \((X, \leq)\) with the opposite category). This exchanges \(x_*\) and \(i^*\), etc.

**Lemma 6.1.** Let \(X\) be a finite ringed space, \(x\) a point of \(X\). If \(I\) is an injective \(\mathcal{O}_{X,x}\) module, then \(x_*I \in \text{Mod}(X)\) is injective. If \(P\) is a projective \(\mathcal{O}_{X,x}\) module, then \(i^*P \in \text{Mod}(X)\) is projective.

**Proof.** Both statements follow from the formulas for \(\text{Hom}\) above and the exactness of the stalk functor. \(\square\)

**Theorem 6.2.** Let \(X\) be a finite ringed space of dimension \(n\). For every \(F \in \text{Mod}(X)\), the coniveau complex of (3) is a resolution

\[
0 \to F \to \bigoplus_{x_0} x_0 \cdot F_{x_0} \to \bigoplus_{x_0 < x_1} x_0 \cdot F_{x_1} \to \bigoplus_{x_0 < \cdots < x_n} x_0 \cdot \mathcal{F}_{x_n} \to 0
\]

by finite direct sums/products of skyscraper sheaves. There is a “dual” resolution by direct sums of sheaves of the form \(i^*F_x\).

**Proof.** We will show the the stalk of this complex at a point \(z \in X\) is a contractible complex. The key point is that we can commute the finite direct products in the definition of the coniveau complex with stalks (finite inverse limits commute with filtered direct limits). Therefore, by (3), this stalk complex is given by

\[
0 \to F_z \to \bigoplus_{z \leq x_0} F_{x_0} \to \bigoplus_{z \leq x_0 < x_1} F_{x_1} \to \cdots \to \bigoplus_{z \leq x_0 < \cdots < x_n} F_{x_n} \to 0.
\]

Define

\[
c_n : \bigoplus_{z \leq x_0 < \cdots < x_n < x_{n+1}} F_{x_{n+1}} \to \bigoplus_{z \leq x_0 < \cdots < x_n} F_{x_n}
\]

by

\[
(c_n s)(x_0, \ldots, x_n) := \begin{cases} 
  s(z, x_0, \ldots, x_n), & z < x_0 \\
  0, & \text{otherwise}
\end{cases}
\]

with the convention that \(c_{-1} : \bigoplus_x \mathcal{F}_x \to \mathcal{F}_z\) sends \(s\) to \(s(z)\). One can check that

\[
d^n_z c_n + c_{n+1} d^{n+1}_z = \text{Id}.
\]

Indeed, the computation is formally similar to that in the proof of Theorem 1.1. \(\square\)

The resolution of the theorem will be called the coniveau resolution.

### 7. Poincaré duality

We now prove an instance of the Poincaré Duality theorem. Let \(X\) be a finite topological space with the constant sheaf of rings \(k\) associated to a field \(k\). Let \(D^b(k)\) be the bounded derived category of \(\text{Mod}(k)\) and let \(D^b(k)\) be the bounded derived category of \(k\) vector spaces.
8. Examples

Theorem 7.1. There is a complex $\omega_X^\bullet \in D^b(k)$ and a $D^b(k)$ isomorphism
\[
R \text{Hom}^*_k(R \Gamma \mathcal{F}^\bullet, V^\bullet) = R \text{Hom}_k^*(\mathcal{F}^\bullet, \omega_X^\bullet \otimes V^\bullet)
\]
natural in $\mathcal{F}^\bullet \in D^b(k)$ and $V^\bullet \in D^b(k)$.

Proof. By (6.1), the coniveau resolution $\mathcal{F} \to \mathcal{C}(\mathcal{F})^\bullet$ is an injective resolution
of $X$ (every vector space $\mathcal{F}_x$ is injective). Hence, we have an isomorphism $R \Gamma \mathcal{F} = \Gamma \mathcal{C}(\mathcal{F})^\bullet$, natural in $\mathcal{F}$. For a $k$ vector space $V$, we have
\[
R \text{Hom}^*_k(R \Gamma \mathcal{F}, V) = \text{Hom}_k^*(\Gamma \mathcal{C}(\mathcal{F})^\bullet, V).
\]
This latter complex looks like
\[
\text{Hom}_k(\bigoplus_{x_0<\cdots<x_n} \mathcal{F}_{x_n}, V) \to \cdots \to \text{Hom}_k(\bigoplus \mathcal{F}_{x_0}, V),
\]
sitting in the interval $[-n, 0]$. On the other hand, by the Hom formulae (6), this is naturally isomorphic to the complex
\[
\bigoplus_{x_0<\cdots<x_n} \text{Hom}_k(\mathcal{F}, x_n, V) \to \cdots \to \bigoplus \text{Hom}_k(\mathcal{F}, x_0, V),
\]
which is nothing but the complex
\[
\text{Hom}_k^*(\mathcal{F}, \omega_X^\bullet \otimes V),
\]
where $\omega_X^\bullet$ is the complex
\[
\bigoplus_{x_0<\cdots<x_n} x_n, k \to \cdots \to \bigoplus x_0, k
\]
(sitting in the interval $[-n, 0]$). Furthermore, $\omega_X^\bullet \otimes V$ is a complex of injective $k$ modules, so this is also
\[
R \text{Hom}_k^*(\mathcal{F}, \omega_X^\bullet \otimes V).
\]
This discussion is easily extended to complexes $\mathcal{F}^\bullet, V^\bullet$ by applying the coniveau resolution termwise, and making the same arguments with the resulting double/triple complexes.

8. Examples

In this section we work out some sheaf cohomology groups on finite topological spaces. Finite (non-Hausdorff) topological spaces are a rich source of examples in the theory of sheaves.

Our main examples will involve the following spaces. The Sierpinski space $S$ is the unique topology on $\{1, 2\}$ where $\{1\}$ is closed, but $\{2\}$ is not. The space $X = \{1A, 2, 1B\}$ has three points with basic open sets
\[
U_{1A} := \{1A, 2\}
\]
\[
U_2 := \{2\}
\]
\[
U_{1B} := \{2, 1B\}.
\]
The space $Y = \{0A, 1A, 2, 1B, 0B\}$ has basic open sets those of $X$, plus the sets

$$U_{0A} := \{0A, 1A, 2, 1B\}$$

$$U_{0B} := \{1A, 2, 1B, 0B\}.$$ 

Let us start with the Sierpinski space $S$. Here every presheaf $\mathcal{F} \in \mathbf{PAb}(S)$ is a sheaf as long as $\mathcal{F}(\emptyset) = 0$ (the sheafification functor does nothing but set $\mathcal{F}(\emptyset) = 0$) because $S$ has only two interesting open sets $S, \{2\}$ and neither of these can be covered nontrivially. A sheaf $\mathcal{F} \in \mathbf{Ab}(S)$ is nothing but the data of the restriction map $\mathcal{F}(S) \to \mathcal{F}(\{2\})$, which is also the data of the specialization map $\mathcal{F}_1 \to \mathcal{F}_2$ (note $0 < 1$ is the specialization ordering of $S$), so $\mathbf{Ab}(S)$ is nothing but the arrow category $\mathbf{Ab}^{\to\to}$ of $\mathbf{Ab}$. The global section functor is identified with the stalk functor $\mathcal{F} \mapsto \mathcal{F}_1$, hence it is exact. However, sections $\Gamma_1$ with support in the closed set $\{1\}$ is identified with the kernel functor

$$\Gamma_1(S, \mathcal{F}) = \text{Ker}(\mathcal{F}_1 \to \mathcal{F}_2).$$

This is not exact. Indeed, an exact sequence in $\mathbf{Ab}(S)$ is just a morphism of exact sequences in $\mathbf{Ab}$, and the sequence of kernels of such a morphism need not form an exact sequence; they must be linked up with the cokernels as in the Snake Lemma. In fact it is straightforward to check, using the obvious flasque resolution

$$
\begin{array}{cccccc}
0 & \longrightarrow & \mathcal{F}_1 & \longrightarrow & \mathcal{F}_1 \oplus \mathcal{F}_2 & \longrightarrow & \mathcal{F}_2 & \longrightarrow & 0 \\
& & \downarrow & & \downarrow \pi_2 & & \downarrow & & \\
0 & \longrightarrow & \mathcal{F}_2 & \longrightarrow & \mathcal{F}_2 & \longrightarrow & 0 & \longrightarrow & 0,
\end{array}
$$

that

$$H^p_1(S, \mathcal{F}) = \begin{cases} 
\text{Ker}(\mathcal{F}_1 \to \mathcal{F}_2), & p = 0 \\
\text{Cok}(\mathcal{F}_1 \to \mathcal{F}_2), & p = 1 \\
0, & p > 1
\end{cases}$$

Notice that, furthermore, the forgetful functor $\mathbf{Ab}(S) \to \mathbf{PAb}(S)$ is exact, so $\Gamma_1$ is not even exact as a functor $\Gamma_1 : \mathbf{PAb}(X) \to \mathbf{Ab}$. Even worse, there is no hope of “making it exact” (computing its derived functors) by using Čech cohomology. Indeed, the trivial cover $\{S\}$ refines every other cover, and, since we may compute with alternating Čech cohomology, the Čech cohomology groups in this case are simply

$$\check{H}^p_1(S, \mathcal{F}) = \begin{cases} 
\Gamma_1(S, \mathcal{F}), & p = 0 \\
0, & p > 0
\end{cases}$$

This shows the necessity of the “neighborhoods” assumption in (7.3).

The specialization ordering $(Y, \leq)$ and the category $\mathbf{Ouv}(Y)^{\text{op}}$ look like
respectively. The sheafification functor

\[ \mathbf{PAb}(Y) \rightarrow \mathbf{Ab}(Y) \]

is given by:

\[
\begin{array}{ccc}
\mathcal{F}(Y) & \rightarrow & \mathcal{F}(U_0B) \\
\mathcal{F}(U_0A) & \rightarrow & \mathcal{F}(U_1A) \\
\mathcal{F}(X) & \rightarrow & \mathcal{F}(U_1B) \\
\mathcal{F}(U_1A) & \rightarrow & \mathcal{F}(U_2) \\
\mathcal{F}(\emptyset) & \rightarrow & \mathcal{F}(0)
\end{array}
\]

Note the “double fiber product” formula for global sections \( \mathcal{F}^+(Y) \). The fact that sheafification preserves stalks is manifest in the fact that \( \mathcal{F}(U) \rightarrow \mathcal{F}^+(U) \) is always an isomorphism for

\[ U \in \{U_{0A}, U_{0B}, U_{1A}, U_{1B}, U_2\} \]

as these are the smallest neighborhoods of the five points of \( Y \).
The open cover \( \mathcal{U} = \{ U_{0A}, U_{0B} \} \) of \( Y \) refines any other open cover, hence we can use ordered Čech cohomology with \( \mathcal{U} \) to compute the absolute Čech cohomology of any \( \mathcal{F} \in \text{PAb}(Y) \). The ordered Čech complex \( \check{\mathcal{C}}(\mathcal{U}, <, \mathcal{F}) \) is simply

\[
\check{\mathcal{F}}(U_{0A}) \oplus \check{\mathcal{F}}(U_{0B}) \to \check{\mathcal{F}}(X),
\]

where the map is the difference of the restriction maps. For example, if \( \mathcal{F} \) is the presheaf

\[
\begin{array}{c}
Z \\
\downarrow 1 \\
\downarrow (1,0) \\
Z \oplus Z \\
\downarrow (1,0) \\
Z \\
\downarrow 1 \\
Z
\end{array}
\]

this Čech complex is

\[
\begin{pmatrix}
1 & -1 \\
0 & -1
\end{pmatrix} : Z^2 \to Z^2,
\]

which has no homology at all. On the other hand, \( \mathcal{F}^+ \) (or even \( \mathcal{F}^\text{sep} \)) is just the constant sheaf \( \mathbb{Z}_Y \) and the sheafification map \( \mathcal{F} \to \mathcal{F}^+ \) is an isomorphism on every open set except that \( \mathcal{F}(X) \to \mathcal{F}^+(X) \) is given by \( (1,0) : Z \oplus Z \). The monomorphism

\[
\check{H}^0(Y, \mathcal{F}) \to \Gamma(Y, \mathcal{F}^+)
\]

of Lemma 8.4 is simply \( 0 \to \mathbb{Z} \). The “reason” this is not surjective is because the two sections \( 1 \in \check{\mathcal{F}}(U_{0A}) \) and \( 1 \in \check{\mathcal{F}}(U_{0B}) \) “should” glue to form a global section since their difference in \( \check{\mathcal{F}}(X) \) is zero when restricted to \( \check{\mathcal{F}}(U_{1A}) \) and \( \check{\mathcal{F}}(U_{1B}) \) and \( U_{1A} \) and \( U_{1B} \) cover \( X \).

Higher Čech cohomology of sheaves on \( Y \) is also poorly behaved. We will now construct a sheaf \( \mathcal{F}' \in \text{Ab}(Y) \) where the natural monomorphism

\[
\check{H}^2(Y, \mathcal{F}') \to H^2(Y, \mathcal{F}')
\]

of Theorem 5.1 is given by \( 0 \to \mathbb{Z} \). Of course, we already know \( \check{H}^2(Y, \mathcal{F}') \) vanishes for any \( \mathcal{F}' \in \text{PAb}(Y') \) as the Čech complex is supported in degrees 0, 1, so we simply seek an \( \mathcal{F}' \in \text{Ab}(Y) \) with nonvanishing \( H^2 \). Given that \( Y \) has noetherian dimension two, it is not terribly surprising that we can find such a sheaf.
Our sheaf $\mathcal{F}'$ is part of the short exact sequence

\[(8.1)\quad 0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0\]

in $\text{Ab}(Y)$ given below.

Note that exactness in $\text{Ab}(Y)$ can be checked on stalks, which is to say it can be checked on $U_{0A}, U_{0B}, U_{1A}, U_{1B},$ and $U_2$. To compute the cohomology groups of $\mathcal{F}''$, we consider the SES of sheaves

\[(8.2)\quad 0 \to \mathcal{F}'' \to \mathcal{G} \to \mathcal{G}'' \to 0\]

given by the diagram below.

Here the maps on global sections

$\mathcal{F}''(Y) = \mathcal{Z} \to \mathcal{G}(Y) = \mathcal{Z} \oplus \mathcal{Z} \to \mathcal{G}''(Y) = \mathcal{Z} \oplus \mathcal{Z}$
are given by $\Delta$ and $\Delta\pi_2$, respectively. The sheaves $\mathcal{G}$ and $\mathcal{G}''$ are flasque, hence have no higher cohomology, so by examining the LES in cohomology, we compute:

$$H^p(Y, \mathcal{F}'') = \begin{cases} \mathbb{Z} & p = 0, 1 \\ 0 & p > 1 \end{cases}$$

The sheaf $\mathcal{F}$ in the appearing in the sequence (8.1) is also flasque, so we can now take the LES in cohomology there to show that

$$H^p(X', \mathcal{F}') = \begin{cases} \mathbb{Z} & p = 2 \\ 0 & \text{otherwise} \end{cases}$$

as claimed.

Note that the exact sequence

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{G} \to \mathcal{G}'' \to 0$$

obtained by concatenating the exact sequences (8.1) and (8.2) (which we have used to work out the cohomology of $\mathcal{F}'$) is nothing but the coniveau resolution of $\mathcal{F}'$ constructed in (6.2).

9. Exercises

Exercise 9.1. Let $Y \subseteq X$ be a closed subspace of a finite space $X$. Write down a formula for $\Gamma_Y(X, \mathcal{F})$ in the language of (4) (i.e. making reference only to the stalks of $\mathcal{F}$ and the maps between them arising from specializations). Specialize your formula to the case where $\mathcal{F}$ is a skyscraper.

Exercise 9.2. Extend the above Poincaré Duality to cohomology with support in a closed subspace $Y \subseteq X$ (c.f. (1)).

Exercise 9.3. A ringed space $(X, \mathcal{O}_X)$ is specialization flat if the natural ring map $\mathcal{O}_{X,x} \to \mathcal{O}_{X,y}$ is flat for any specialization $x \leq y$. Show that every scheme is specialization flat, but show that every ring map arises in this manner for some appropriate $(X, \mathcal{O}_X)$. 
CHAPTER VIII

Čech Theory

Čech cohomology is named for the Čech mathematician Eduard Čech (1893-1960), after whom the Stone-Cech compactification is also named.

1. Čech resolution

Let $X$ be a topological space, $\mathcal{F} \in \text{PAb}(X)$. It will be important to develop a theory of Čech cohomology for presheaves as well as sheaves. For an open set $i : U \hookrightarrow X$, we set $\mathcal{F}_U = i_* i^{-1}_\text{pre} \mathcal{F}$. Note that $\mathcal{F}_U$ is given by $V \mapsto \mathcal{F}(U \cap V)$, and that this is a sheaf if $\mathcal{F}$ is a sheaf ($i^{-1}_\text{pre} = i^{-1}$ in this situation). Furthermore, $\mathcal{F}_U$ is clearly a flasque sheaf if $\mathcal{F}$ is a flasque sheaf.

If $V \subset U$ then there is a natural morphism $\mathcal{F}_U \to \mathcal{F}_V$ of abelian presheaves on $X$ induced by the restriction maps of $\mathcal{F}$. Note that the stalk of this map is an isomorphism at any $x \in V$ (both stalks are just $\mathcal{F}_x$). Let $\mathcal{U} = \{U_i : i \in I\}$ be a family of open sets of $X$. For a finite subset $\{i_0, \ldots, i_p\} \subseteq I$, or any vector $(i_0, \ldots, i_p) \in I^{p+1}$, we use the following notation:

$$U(i_0, \ldots, i_p) := U_{i_0} \cap \cdots \cap U_{i_p}$$
$$\mathcal{F}(i_0, \ldots, i_p) := \mathcal{F}_{U(i_0, \ldots, i_p)}.$$

The convention

$$\mathcal{F}(\cdot) := \mathcal{F}$$

is convenient. For $j \in \{0, \ldots, p\}$ the natural inclusion $U(i_0, \ldots, i_p) \to U(i_0, \ldots, \widehat{i}_j, \ldots, i_p)$ of open sets induces a $\text{PAb}(X)$ morphism

$$\mathcal{F}(i_0, \ldots, \widehat{i}_j, \ldots, i_p) \to \mathcal{F}(i_0, \ldots, i_p)$$
$$s \mapsto s|_{U(i_0, \ldots, i_p)}.$$

For each nonnegative integer $p$, define

$$\check{C}^p(U, \mathcal{F}) := \prod_{(i_0, \ldots, i_p) \in I^{p+1}} \mathcal{F}(i_0, i_1, \ldots, i_p).$$
VIII. ČECH THEORY

The presheaves \( \check{C}^p(U, \mathcal{F}) \in P\text{Ab}(X) \) can be assembled into a complex \( \check{C}^\bullet(U, \mathcal{F}) \) whose boundary maps

\[
d^p : \check{C}^p(U, \mathcal{F}) \to \check{C}^{p+1}(U, \mathcal{F})
\]

are given by

\[
(d^ps)(i_0, \ldots, i_{p+1}) := \sum_{j=0}^{p+1} (-1)^j s(i_0, \ldots, \hat{i}_j, \ldots, i_{p+1})|_{U(i_0, \ldots, i_{p+1})}.
\]

There is a natural augmentation morphism

\[
d^{-1} : \mathcal{F} \to \check{C}^0(U, \mathcal{F})
\]

given by setting \( s(i_0) := s|_{U(i_0)} \) (the usual formula using convention (1.1)). The abelian presheaves \( \check{C}^p(U, \mathcal{F}) \) are called the Čech presheaves associated to the presheaf \( \mathcal{F} \) and the covering sieve \( U \).

**Theorem 1.1.** If \( U \) is a cover of \( X \), then \( d^{-1} : \mathcal{F} \to \check{C}^\bullet(U, \mathcal{F}) \) is a resolution for any \( \mathcal{F} \in \text{Ab}(X) \).

**Proof.** We have already noted that, since \( \mathcal{F} \) is a sheaf, each \( \mathcal{F}(i_0, \ldots, i_p) \) is a sheaf, and hence \( \check{C}^p(U, \mathcal{F}) \) (being a product of such sheaves) is also a sheaf. It therefore suffices to check exactness of the augmented complex \( \mathcal{F} \to \check{C}^\bullet(U, \mathcal{F}) \) when restricted to each of the open sets \( U_i \), since the \( U_i \) form a cover.\(^1\) In fact, we will show that

\[
(\mathcal{F} \to \check{C}^\bullet(U, \mathcal{F}))|_{U_i}
\]

is null-homotopic. Define a contraction map

\[
c_p : \prod_{i_0, \ldots, i_p} \mathcal{F}(i_0, \ldots, i_p)|_{U_i} \to \prod_{i_0, \ldots, i_{p-1}} \mathcal{F}(i_0, \ldots, i_{p-1})|_{U_i}
\]

by

\[
(c_ps)(i_0, \ldots, i_{p-1}) := s(i, i_0, \ldots, i_{p-1}).
\]

In making this definition we are using the obvious identification

\[
\mathcal{F}(i, i_0, \ldots, i_p)|_{U_i} = \mathcal{F}(i_0, \ldots, i_p)|_{U_i}.
\]

By our convention (1.1), \( c_0 : \check{C}^0(U, \mathcal{F})|_{U_i} \to \mathcal{F}|_{U_i} \) simply sends \( s \) to \( s(i) \).

Now we just check that the \( c_p \) form a retraction for \( \check{C}^\bullet(U, \mathcal{F})|_{U_i} \). That is,

\[
d^{p-1}c_p + c_{p+1}d^p = \text{Id}.
\]

\(^1\)It would even be good enough to check exactness at all stalks; we are proving a stronger statement.
Indeed, we have
\[
(d^{p-1}c_p s)(i_0, \ldots, i_p) = \sum_{j=0}^{p-1} (c_p s)(i_0, \ldots, \hat{i}_j, \ldots, i_p)
\]
and
\[
(c_{p+1}d^p s)(i_0, \ldots, i_p) = (d^p s)(i, i_0, \ldots, i_p) + \sum_{j=0}^{p-1} (-1)^{j+1} s(i, i_0, \ldots, \hat{i}_j, \ldots, i_p).
\]

The resolution of the theorem is called the Čech resolution of \( \mathcal{F} \in \text{Ab}(X) \).

2. Refinement maps

Suppose that \( U = \{U_i : i \in I\} \) and \( V = \{V_j : j \in J\} \) are two families of open sets in \( X \). Suppose \( r : J \to I \) is a refinement in the sense that \( V_j \subseteq U_{r(j)} \) for all \( j \in J \). Then we have a natural restriction map \( \mathcal{F}_{U(r(j))} \to \mathcal{F}_{V_j} \) for every \( j \), and in fact a map of complexes
\[
r^* : \check{C}^\bullet(U, \mathcal{F}) \to \check{C}^\bullet(V, \mathcal{F})
\]
\[
s \mapsto s(r(j_0), \ldots, r(j_p))|_{V(j_0, \ldots, j_p)}.
\]

**Lemma 2.1.** Suppose \( r_1, r_2 : J \to I \) are two refinements. Then the maps
\[
r_1^*, r_2^* : \check{C}^\bullet(U, \mathcal{F}) \to \check{C}^\bullet(V, \mathcal{F})
\]
are homotopic.

**Proof.** The maps
\[
h_p : \check{C}^p(U, \mathcal{F}) \to \check{C}^{p-1}(V, \mathcal{F})
\]
given by
\[
(h_p s)(j_0, \ldots, j_{p-1}) := \sum_{k=0}^{p-1} (-1)^k s(r_1(j_0), \ldots, r_1(j_k), r_2(j_k), \ldots, r_{p-1}(j_k))
\]
provide a homotopy from \( r_1^* \) to \( r_2^* \). \( \square \)
3. Alternating variant

The alternating (or ordered) Čech complex is a “smaller” version of the Čech complex which is convenient for computations, and has theoretical implications (relations between covering dimension and cohomological dimension). Let \( \{ U_i : i \in I \} \) be a family of open sets in \( X \). Fix a total ordering \( \leq \) of \( I \). The sheaves
\[
\check{C}_p(U, \leq, F) := \prod_{i_0 < \ldots < i_p} F(i_0, \ldots, i_p)
\]
form a complex \( \check{C}^\bullet(U, \leq, F) \) whose boundary map is defined by the same formula (1.2) as the boundary map for \( \check{C}^\bullet(U, F) \). There is an obvious projection map
\[
\check{C}_p(U, F) \to \check{C}_p(U, \leq, F)
\]
which is a map of complexes.

**Lemma 3.1.** The map (3.1) is a homotopy equivalence.

**Proof.** We will obtain this later from the simplicial point of view. \( \square \)

For now, we just note that the alternating Čech complex certainly provides a resolution of any \( F \in \text{Ab}(X) \). This can be proved directly (without the above lemma) just as in the proof of (1.1) using the contraction operator
\[
(c_p)(i_0, \ldots, i_{p-1}) := \begin{cases} s(i, i_0, \ldots, i_{p-1}), & i < i_0 \\ 0, & \text{otherwise.} \end{cases}
\]

4. Čech cohomology for a fixed cover

Finally we define Čech cohomology. Let \( U = \{ U_i : i \in I \} \) be a cover of \( X \). Let \( F \in \text{PAb}(X) \), let \( \check{C}^\bullet(U, F) \) be the complex in \( \text{Ab}(X) \) obtained by applying \( \Gamma \Phi \) to the Čech complex \( \check{C}^\bullet(U, F) \).

**Warning.** The obvious inclusion
\[
\check{C}_p(U, F) \subseteq \prod_{(i_0, \ldots, i_p) \in I^{p+1}} \Gamma_{\Phi \cap U(i_0, \ldots, i_p)}(U(i_0, \ldots, i_p), F)
\]
is not generally an equality because \( \Gamma_{\Phi} \) does not generally commute with products. This formula is, however, valid if \( \Gamma_{\Phi} = \Gamma \), or if \( I \) is finite. In general, \( \check{C}_p(U, F) \) is the subcomplex of \( \check{C}^p(U, F) \) consisting of those
\[
s = (s(i_0, \ldots, i_p)) \in \prod_{(i_0, \ldots, i_p) \in I^{p+1}} F(U(i_0, \ldots, i_p))
\]
such that there is a \( Y \in \Phi \) such that
\[
\text{Supp } s(i_0, \ldots, i_p) \subseteq Y
\]
for all \( (i_0, \ldots, i_p) \in I^{p+1} \).

The abelian groups
\[
\check{H}^n(U, F) := H^n(\check{C}^\bullet(U, F))
\]
are called the Čech cohomology\(^2\) groups of \(\mathcal{F}\) with respect to the cover \(\mathcal{U}\).

5. Basic properties

We now prove some of the basic properties of the Čech cohomology groups \(\check{H}^n_{\Phi}(\mathcal{U}, \mathcal{F})\).

**Theorem 5.1.** For any \(\mathcal{F} \in \text{Ab}(X)\), there is a first quadrant spectral sequence

\[
E_2^{p,q} = H^p(\check{H}^q_{\Phi}(\mathcal{U}, \mathcal{F})) \implies H^{p+q}_\Phi(X, \mathcal{F}).
\]

In particular, there are natural maps \(E_2^{p,0} = \check{H}^p_{\Phi}(\mathcal{U}, \mathcal{F}) \to H^p_\Phi(X, \mathcal{F})\) for every \(p\).

**Proof.** Since the Čech complex is a resolution by (1.1), this is just a special case of the spectral sequence associated to a resolution of an object in an abelian category. \(\square\)

When each of the sheaves \(\check{\mathcal{C}}^p(\mathcal{U}, \mathcal{F})\) is \(\Gamma_\Phi\)-acyclic, we say that \(\mathcal{U}\) is a *good cover* for \(\mathcal{F}\). We say \(\mathcal{U}\) is a *good cover* if it is a good cover for every abelian sheaf \(\mathcal{F}\).

**Corollary 5.2.** If \(\mathcal{U}\) is a good cover for \(\mathcal{F}\), then the natural map

\[
\check{H}^p_{\Phi}(\mathcal{U}, \mathcal{F}) \to H^p_\Phi(X, \mathcal{F})
\]

is an isomorphism for every \(p\).

**Proof.** The spectral sequence of (5.1) degenerates. \(\square\)

6. Čech cohomology

In this section we introduce the (absolute) Čech cohomology groups, which have no dependence on a choice of cover.

Set

\[
\check{H}^p_\Phi(X, \mathcal{F}) := \lim_{\mathcal{U}} \check{H}^p_{\Phi}(\mathcal{U}, \mathcal{F}),
\]

where the direct limit is over the the category of covers \(\mathcal{U} = \{U_i : i \in I\}\) where there is a unique morphism from \(\mathcal{U} \to \mathcal{V} = \{V_j : j \in J\}\) iff there is a refinement map \(r : J \to I\). The restriction maps are defined by taking the maps \(H^p(r^*)\) induced on cohomology and noting that there are independent of the choice of \(r\).

An alternative means of calculating the Čech cohomology groups is by means of covering sieves (c.f. (9)). Let \(\mathcal{U} = \{U_i : i \in I\}\) be the set of all open subsets of \(X\) (the trivial sieve on \(X \in \text{Ouv}(X)\)). By slight abuse of notation, say \(J \subseteq I\) is a covering sieve if \(\{U_j : j \in J\}\) is a covering sieve. Order the set of covering sieves

\(^2\)Our definition agrees with Godement’s [G, 5.2].
by reverse inclusion. The resulting set is filtered and we may compute the Čech cohomology as the cohomology of the direct limit complex:

\[ \check{H}^p(\Phi, \mathcal{F}) := H^n \left( \lim_{\to J} \check{C}^p_\Phi(\{U_j : j \in J\}, \mathcal{F}) \right). \]

This is because filtered direct limits commute with cohomology and the set of covering sieves is cofinal in the set of all covers, ordered by the existence of a refinement.

The abelian groups \( \check{H}^p(\Phi, \mathcal{F}) \) are called the (absolute) Čech cohomology groups of \( \mathcal{F} \). We will revisit these constructions are first proving some “classical” facts about Čech cohomology.

### 7. Čech acyclic sheaves

As with usual sheaf cohomology, we will need a supply of presheaves with vanishing higher Čech cohomology. Unfortunately, our supply of such presheaves will be rather lame.

**Theorem 7.1.** Let \( U \) be an open cover of \( X, \mathcal{F} \in \mathbf{PAb}(X) \). The Čech cohomology groups \( \check{H}^p(U, \mathcal{F}) \) vanish for \( p > 0 \) under either of the following hypotheses:

1. \( \mathcal{F} \) is flasque and is in \( \mathbf{Ab}(X) \)
2. \( \mathcal{F} \) is \( \Phi \) soft, \( \Phi \) is paracompactifying, and \( \mathcal{F} \in \mathbf{Ab}(X) \)

**Proof.** We first prove the statement when \( \mathcal{F} \) is flasque. Then it is clear from the definition that \( \iota_* \iota^* \mathcal{F} \) is flasque for any inclusion \( \iota : U \hookrightarrow X \). Furthermore, a product of flasque sheaves is flasque (since \( (\prod_i \mathcal{F}_i)(U) = \prod_i \mathcal{F}_i(U) \) and a product of surjections is surjective). Hence the Čech resolution \( \varepsilon : \mathcal{F} \to \check{C}^*(U, \mathcal{F}) \) is a flasque resolution of \( \mathcal{F} \), so

\[ \check{H}^p(U, \mathcal{F}) = H^p(\Gamma_\Phi \check{C}^*(U, \mathcal{F})) = H^p(X, \mathcal{F}) \]

by 10.5, but this vanishes for \( p > 0 \) by 10.4 because \( \mathcal{F} \) is flasque. \( \square \)

The above theorem says that Čech cohomology is, in some sense, effaceable. However, it turns out that Čech cohomology does not generally form a \( \delta \)-functor from \( \mathbf{Ab}(X) \) to \( \mathbf{Ab} \): a short exact sequence of sheaves does not generally induce a long exact sequence of Čech cohomology groups. This is the only obstruction to identifying Čech cohomology with sheaf cohomology.

If we replace sheaves with presheaves, Čech cohomology is better behaved:

**Theorem 7.2.** For any cover \( U \) of \( X \), the Čech cohomology groups \( \check{H}^*(U, -) \) form an effaceable \( \delta \)-functor from \( \mathbf{PAb}(X) \) to \( \mathbf{Ab} \), hence \( \check{H}^*(U, \mathcal{F}) = R^n \check{H}^0(U, \mathcal{F}) \) for every \( \mathcal{F} \in \mathbf{PAb}(X) \).
Proof. Suppose
\[ \mathcal{F} = 0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0 \]
is a short exact sequence in \( \mathbf{PAb}(X) \). Then
\[ 0 \to \mathcal{F}'(U) \to \mathcal{F}(U) \to \mathcal{F}''(U) \to 0 \]
is exact for every \( U \in \text{Ouv}(X) \) (this is why it is important to use presheaves). Consequently,
\[ 0 \to \check{\mathcal{C}}^\bullet(U, \mathcal{F}') \to \check{\mathcal{C}}^\bullet(U, \mathcal{F}) \to \check{\mathcal{C}}^\bullet(U, \mathcal{F}'') \to 0 \]
is a short exact sequence of complexes of abelian groups, since, in each degree it is just a product of short exact sequences of the above form. The associated long exact cohomology sequence is natural in \( \mathcal{F} \) hence the first claim is proved. To see that \( \check{H}^\bullet(U, -) \) is effaceable it is enough to check that it is left exact and preserves finite direct sums (clear) because then it can be erased by injectives, which \( \mathbf{PAb}(X) \) has enough of by (4.6). \( \square \)

With supports, we will need to be able to refine covers. The best we can do is:

**Theorem 7.3.** Let \( X \) be a topological space, \( \Phi \) a family of supports with neighborhoods. The absolute Čech cohomology groups \( \check{H}^\bullet_{\Phi}(X, -) \) form an effaceable \( \delta \)-functor from \( \mathbf{PAb}(X) \) to \( \mathbf{Ab} \), hence \( \check{H}^n_{\Phi}(U, \mathcal{F}) = R^n \check{H}^0_{\Phi}(U, \mathcal{F}) \) for every \( \mathcal{F} \in \mathbf{PAb}(X) \).

**Proof.** Let \( \mathcal{U} = \{ U_i : i \in I \} \) be the set of all open subsets of \( X \). By the same argument as in the previous proof, we reduce to proving that
\[ 0 \to \lim_{\longrightarrow \mathcal{I}' \setminus \mathcal{I}} \check{C}^p_{\Phi}(I', \mathcal{F}') \to \lim_{\longrightarrow \mathcal{I}' \setminus \mathcal{I}} \check{C}^p_{\Phi}(I', \mathcal{F}) \to \lim_{\longrightarrow \mathcal{I}' \setminus \mathcal{I}} \check{C}^p_{\Phi}(U, \mathcal{F}'') \to 0 \]
(the filtered direct limit is over all \( I' \subseteq I \) such that \( \{ U_i : i \in I' \} \) is a covering sieve, ordered by reverse inclusion) is exact for any exact sequence of presheaves
\[ 0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0. \]

Exactness on the left and in the middle follows from exactness of filtered direct limits and left exactness of \( \Gamma_{\Phi} \), so the issue is to prove surjectivity on the right. As mentioned in (??), an element \([ (s, I') \] on the right is represented by a pair \(( s, I' \) where \( I' \) indexes a covering sieve and \( s \) is a set of sections
\[ s(i_0, \ldots, i_p) \in \mathcal{F}''(U(i_0, \ldots, i_p)) \]
such that there is a \( Y \in \Phi \) with
\[ \text{Supp } s(i_0, \ldots, i_p) \subseteq Y \]
for all \(( i_0, \ldots, i_p ) \in (I')^{p+1} \).

Since \( \Phi \) has neighborhoods, we can find an open set \( V \in \text{Ouv}(X) \) with \( Y \subseteq V \) and \( V^- \in \Phi \). By choosing a common refinement of the cover indexed by \( I' \) and the cover \( \{ X \setminus Y, V \} \) we can an \( I'' \subseteq I' \) indexing a covering sieve, such that, for
every \( i \in I' \), either \( U_i \subseteq V \) or \( U_i \cap Y = \emptyset \). In particular, for \((i_0, \ldots, i_p) \in (I')^{p+1}, s(i_0, \ldots, i_p) = 0 \) unless \( U(i_0, \ldots, i_p) \subseteq V \). The presheaf exactness implies that

\[
\mathcal{F}(U(i_0, \ldots, i_p)) \rightarrow \mathcal{F}'(U(i_0, \ldots, i_p))
\]

is always surjective, so we can find, for each \((i_0, \ldots, i_p) \in (I')^{p+1} \), \( \tau(i_0, \ldots, i_p) \in \mathcal{F}(U(i_0, \ldots, i_p)) \) mapping to \( s(i_0, \ldots, i_p) \). By setting \( \tau(i_0, \ldots, i_p) = 0 \) for those \((i_0, \ldots, i_p) \in (I')^{p+1} \) where \( U(i_0, \ldots, i_p) \) is not contained in \( V \), we obtain a preimage \([(t, I')]\) for \([(s, I')] = [(s|_{I'}, I')]\). Note that \( \text{Supp} \; \tau(i_0, \ldots, i_p) \subseteq V^{-} \in \Phi \) for all \((i_0, \ldots, i_p) \in (I')^{p+1} \).

The assumption that \( \Phi \) has neighborhoods cannot be removed. See (??).

8. Čech presheaf

Let \( X \) be a topological space, \( \Phi \) a family of supports on \( X \), \( \mathcal{F} \in \text{Ab}(X) \). For \( n \geq 0 \), let \( \mathcal{H}^n_{\Phi}(\mathcal{F}) \in \text{PAb}(X) \) be the presheaf

\[
\mathcal{H}^n_{\Phi}(\mathcal{F}) : \text{Ouv}(X)^{op} \rightarrow \text{Ab}
\]

\[
U \mapsto H^n_{\Phi \cap U}(U, \mathcal{F}|_U).
\]

As usual, when \( \Phi \) is the family of all closed sets, we drop it from the notation. Note that \( \mathcal{H}^0 : \text{Ab}(X) \rightarrow \text{PAb}(X) \) is just the usual forgetful functor.

**Lemma 8.1.** The functor

\[
\mathcal{H}^n_{\Phi} : \text{Ab}(X) \rightarrow \text{PAb}(X)
\]

\[
\mathcal{F} \mapsto \mathcal{H}^n_{\Phi}(\mathcal{F})
\]

coincides with the \( n \)th right derived functor of \( \mathcal{H}^0 : \text{Ab}(X) \rightarrow \text{PAb}(X) \).

**Proof.** The functors \( \mathcal{F} \mapsto \mathcal{H}^n_{\Phi}(\mathcal{F}) \) form an effaceable (by flasque sheaves, say) \( \delta \)-functor which manifestly has the right degree zero term.

**Remark 8.2.** The functor \( \mathcal{H}^0_{\Phi} \) does not necessarily take a flasque sheaf to a sheaf. If \( X \) is an infinite discrete space, \( \Phi \) is the family of compact (i.e. finite) subsets of \( X \), and \( \mathcal{F} \) is the flasque sheaf of continuous (i.e. arbitrary) \( \mathbb{Z} \) valued functions, then \( \mathcal{H}^0_{\Phi} \) is not even a sheaf.

The restriction maps for the presheaf \( \mathcal{H}^n_{\Phi}(\mathcal{F}) \) can be described concretely as follows. Fix an injective resolution \( \varepsilon : \mathcal{F} \rightarrow \mathcal{I}^\bullet \) of \( \mathcal{F} \). Then an inclusion of open sets \( V \hookrightarrow U \) induces a morphism of complexes

\[
\Gamma_{\Phi \cap U}(U, \mathcal{I}^\bullet) \rightarrow \Gamma_{\Phi \cap V}(V, \mathcal{I}^\bullet)
\]

and thus a morphism on the cohomology of these complexes. On the other hand, we have

\[
\mathcal{H}^n_{\Phi}(\mathcal{F})(U) = H^n(\Gamma_{\Phi \cap U}(U, \mathcal{I}^\bullet))
\]
and similarly with $U$ replaced by $V$. The abelian presheaves $\mathcal{H}_\Phi^0(\mathcal{F})$ are called the local cohomology presheaves of $\mathcal{F}$ with support in $\Phi$.

**Lemma 8.3.** Assume $\Phi$ has neighborhoods. Then the sheaf (or even the separated presheaf) associated to the abelian presheaf $\mathcal{H}_\Phi^q(\mathcal{F})$ is zero for all $q > 0$.

**Proof.** Fix a $q > 0$, an injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$, an open set $U$, and an element $s \in \mathcal{H}_\Phi^q(\mathcal{F})(U)$. Then $s$ is represented by some $q$-cocycle $\tilde{s}$ in the complex $\Gamma_{\Phi \cap U}(U, \mathcal{I}^\bullet)$, in other words by an element $\tilde{s} \in \mathcal{I}^q(U)$ where the closure $S$ of $\text{Supp} \tilde{s}$ in $X$ is in $\Phi$. Since $\Phi$ has neighborhoods, there is a $B \in \Phi$ such that $S$ is contained in the interior $V$ of $B$. Since the complex $\mathcal{I}^\bullet$ is exact in positive degrees, the $q$-cocycle $\tilde{s}|_{U \cap V}$ is a “coboundary on a cover.” That is, there is an open cover $\{U_i\}$ of $U \cap V$ and sections $s_i \in \mathcal{I}^{q-1}(U_i)$ such that $d^{q-1}(U_i)(s_i) = s|_{U \cap V}$ for all $i$. Since $U_i$ is contained in $V$, the closure in $X$ of $\text{Supp} s_i$ is contained in $B$, hence is in $\Phi$, so $s_i \in \Gamma_{\Phi \cap U_i}(U_i, \mathcal{I}^{q-1})$. The $U_i$ together with the set $U \setminus \text{Supp} \tilde{s}$ form an open cover of $U$, and taking the zero section of $\mathcal{I}^{q-1}$ on the latter open set we find that $\tilde{s}$ is a coboundary on each set in this open cover, hence the restriction of $s$ to every set in this open cover is zero in cohomology. \qed

**Lemma 8.4.** Let $X$ be a topological space, $\Phi$ a family of supports. The natural map

$$\Pi_\Phi^0(X, \mathcal{F}) \to \Gamma_\Phi(X, \mathcal{F}^+)$$

is injective for any $\mathcal{F} \in \text{PAb}(X)$ and an isomorphism if $\mathcal{F}$ is separated.

**Proof.** Let $\{U_i : i \in I\}$ be the open sets of $X$. The natural map is induced by the natural map of presheaves $\mathcal{F} \to \mathcal{F}^+$, naturality of the Čech complex associated to a presheaf, and the fact that, for any $\mathcal{G} \in \text{Ab}(X)$, we have a natural isomorphism

$$\Pi_\Phi^0(X, \mathcal{G}) = \Gamma_\Phi$$

given by sending $s \in \Gamma_\Phi(X, \mathcal{G})$ to the class of $(s|_{U_i}, I)$ in $s \in \Gamma_\Phi(X, \mathcal{G})$ (this is injective by separation for $\mathcal{G}$ and surjective by gluing for $\mathcal{G}$, taking care to keep track of supports by noting that the support of the section obtained by gluing is the unions of the supports of the sections being glued). Now let us prove the lemma. Making the natural identification just established, this map send a class $[(s, I')] \in \Pi_\Phi^0(X, \mathcal{F})$ to the corresponding class $[(s^+, I')] \in \Pi_\Phi^0(X, \mathcal{F}^+)$. Viewing $\mathcal{F}^+$ as the sheaf of maps to the stalks with the local coherence condition, it is clear that $[(s^+, I')]$ is zero iff we can find a finer cover (hence a finer covering seive) $I'' \subseteq I'$ on which $(s, I')$ is zero.

Now let us try to prove surjectivity. We will ignore the supports for simplicity. A section $s \Gamma(X, \mathcal{F}^+)$ is an element $s \in \prod_{x \in X} \mathcal{F}_x$ satisfying the local coherence condition (c.f. (??)), so we can find an $I' \subseteq I$ indexing a cover and sections $s(i) \in \mathcal{F}(U_i)$ such that $s(i)_x = s(x)$ for all $x \in U_i$. Of course, by adding in every set which is a subset of some $U_i$ and restricting sections, we may assume $I'$ indexes a covering seive, so $(I', s) \in \mathcal{O}^0(I', \mathcal{F})$ is a class mapping to $s \in \Gamma(X, \mathcal{F}^+)$ under the natural
map... but the difficulty is in showing that \((I', s)\) is in the kernel of \(d^0\) of the Čech complex, and this is where we need separation of \(\mathcal{F}\). When \(\mathcal{F}\) is separated, we simply note that the stalk of \(s(i)|_{U(i,j)} - s(j)|_{U(i,j)}\) is zero at every point \(x \in U(i, j)\) (because it is just \(s(x) - s(x)\) by local coherence), hence \(s(i)|_{U(i,j)} - s(j)|_{U(i,j)} = 0\) by separation. \(\square\)

We encountered an example where \(\check{H}^0(X, \mathcal{F}) \to \Gamma(X, \mathcal{F}^+)\) is not surjective in (8).

**Theorem 8.5.** Let \(X\) be a topological space, \(\mathcal{F} \in \text{Ab}(X)\), \(U\) a cover of \(X\). There is a first quadrant spectral sequence

\[
E_2^{p,q} = \check{H}^p(U, \mathcal{H}^q(\mathcal{F})) \implies H^{p+q}(X, \mathcal{F}).
\]

If \(\Phi\) is a family of supports with neighborhoods, then there is a similar spectral sequence

\[
E_2^{p,q} = \check{H}_\Phi^p(U, \mathcal{H}^q(\mathcal{F})) \implies H^{p+q}_\Phi(X, \mathcal{F})
\]

using the absolute Čech cohomology.

**Proof.** The functor \(\Gamma : \text{Ab}(X) \to \text{Ab}\) is the composition of the forgetful functor \(\mathcal{H}^0 : \text{Ab}(X) \to \text{PAb}(X)\) and the functor \(\check{H}^0(U, -) : \text{PAb}(X) \to \text{Ab}\), hence we have the Grothendieck spectral sequence

\[
E_2^{p,q} = R^p\check{H}^0(U, R^q\mathcal{H}^0\mathcal{F}) \implies H^{p+q}(X, \mathcal{F}).
\]

This spectral sequence is identified with the desired one by (8.1) and (7.2). The second statement is proved similarly using (7.3).

Note that, for the existence of the Grothendieck spectral sequence we need to know that \(\mathcal{H}^0\) takes injectives in \(\text{Ab}(X)\) to \(\check{H}^0(U, -)\) acyclics (\(\check{H}^0_\Phi(X, -)\) acyclics for the second statement); this follows from (7.1). \(\square\)

**Theorem 8.6.** For any topological space \(X\) and any abelian sheaf \(\mathcal{F}\) on \(X\), the natural maps

\[
\check{H}^p(X, \mathcal{F}) \to H^p(X, \mathcal{F})
\]

are isomorphisms for \(p = 0, 1\) and injective for \(p = 2\).

**Proof.** This follows from the exact sequence of low order terms

\[
0 \to \check{H}^1(X, \mathcal{F}) \to H^1(X, \mathcal{F}) \to H^0(X, \mathcal{H}^1\mathcal{F}) \to \check{H}^2(X, \mathcal{F}) \to H^2(X, \mathcal{F})
\]

in the spectral sequence of (8.5), together with (8.4) and (8.3), which together imply \(H^0(X, \mathcal{H}^1\mathcal{F}) = 0\). \(\square\)
9. Comparison theorems

We conclude our treatment of classical Čech theory with two theorems about the relationship between Čech cohomology and the usual sheaf cohomology.

**Theorem 9.1. (Cartan’s Criterion)** Let $X$, be a topological space, $\Phi$ a family of supports, $\mathcal{F} \in \text{Ab}(X)$. Suppose $\mathcal{U}$ is an open cover of $X$ satisfying:

1. If $U, V \in \mathcal{U}$, then $U \cap V \in \mathcal{U}$.
2. $\mathcal{U}$ contains arbitrarily small neighborhoods of any $x \in X$.
3. $\check{H}_p^\Phi(U, \mathcal{F}|_U) = 0$ for any $p > 0$ and any $U \in \mathcal{U}$.

Then the natural map

$$\hat{H}_p^\Phi(X, \mathcal{F}) \to H_p^\Phi(X, \mathcal{F})$$

is an isomorphism for every $p$.

**Proof.** We first prove, by induction on $p$, that the natural map $\hat{H}_p^\Phi(U, \mathcal{F}|_U) \to H_p^\Phi(U, \mathcal{F}|_U)$ is an isomorphism for all $p$, and all $U \in \mathcal{U}$. The case $p = 0$ is trivial. Suppose we know this for all $q$ with $0 \leq q < p$. Because of our assumptions (1),(2) about $\mathcal{U}$, we note that, for any $U \in \mathcal{U}$, the covers of $U$ by open sets from $\mathcal{U}$ are cofinal in the family of all open covers of $U$, so we may compute the Čech cohomology groups $\check{H}_p^\Phi(U, \mathcal{F}|_U)$ using only such covers. It then follows from assumption (3) and the induction hypothesis that $H_p^\Phi(U, \mathcal{F}|_U) = 0$ for every $U \in \mathcal{U}$ and every $q$ with $1 \leq q < p$, so that $\check{H}_q^\Phi(U, \mathcal{F}|_U) = 0$ for every such $q$ because we can compute Čech cohomology only using covers by open sets from $\mathcal{U}$. Furthermore, $\check{H}_p^\Phi(U, \mathcal{F}|_U) = 0$ by (8.3) and (8.4). Consider the spectral sequence of 8.5 on $U$. The natural map

$$E_2^{ij} = \hat{H}_p^\Phi(U, \mathcal{F}|_U) \to H_p^\Phi(U, \mathcal{F}|_U)$$

in this spectral sequence must be an isomorphism because we have shown that $E_2^{ij} = 0$ in this spectral sequence for every $i, j$ with $i + j = p$ and $j > 0$.

Finally, we conclude the desired result by again considering the spectral sequence of 8.5, using the first part of the proof and the assumption (3) to know that $\check{H}_p^\Phi(U, \mathcal{F}|_U) = 0$ for every $q > 0$, so that the SES degenerates. □

**Theorem 9.2.** Suppose $\Phi$ is a paracompactifying family of supports (§7) on a topological space $X$. Then for every $p \geq 0$ and every $\mathcal{F} \in \text{Ab}(X)$, the natural map

$$\hat{H}_p^\Phi(X, \mathcal{F}) \to H_p^\Phi(X, \mathcal{F})$$

is an isomorphism.

**Proof.** □

**Corollary 9.3.** For a paracompact topological space $X$, the natural map

$$\check{H}_p^\Phi(X, \mathcal{F}) \to H_p^\Phi(X, \mathcal{F})$$

is an isomorphism for every $p$. □
10. The unit interval

It is possible to compute the Čech cohomology groups $\check{H}^p(X, \mathcal{F})$ directly from the definitions for reasonably simple topological spaces $X$ and sheaves $\mathcal{F}$. For example, the interval $I = [0, 1]$ can be covered by connected open subintervals $U_1, \ldots, U_n$ of roughly equal length where $U_i$ intersects only $U_{i-1}$ and $U_{i+1}$ (and this intersection is connected). (Note $U_1$ only meets $U_2$ and $U_n$ only meets $U_{n-1}$ and we assume $n \geq 2$.) These sort of covers are cofinal in all covers of the interval. Global sections of the Čech complex for this cover and the constant sheaf $\mathbb{Z}$ give the complex $d: \mathbb{Z}^n \to \mathbb{Z}^{n-1}$ (in degrees 0, 1) where the boundary map is given by the matrix:

$$
\begin{pmatrix}
1 & -1 & 0 & \cdots & 0 & 0 \\
0 & 1 & -1 & \cdots & 0 & 0 \\
0 & 0 & 1 & -1 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & -1 & 0 \\
0 & 0 & 0 & \cdots & 1 & -1
\end{pmatrix}
$$

This map has cohomology only in degree zero, given by the diagonal $\Delta: \mathbb{Z} \to \mathbb{Z}^n$. Since the interval is certainly paracompact, Čech cohomology and the usual sheaf cohomology coincide by Corollary 9.3, so we find

$$
H^p(I, \mathbb{Z}) = \begin{cases} 
\mathbb{Z}, & p = 0 \\
0, & p > 0.
\end{cases}
$$

11. Simplicial objects

For a nonnegative integer $n$, let $[n] := \{0, \ldots, n\}$. Let $\Delta$ denote the category whose objects are the finite sets $[0], [1], \ldots$ and where a morphism $[m] \to [n]$ is a nondecreasing function. For each $i \in [n]$, let $s^n_i : [n-1] \to [n]$ be the unique $\Delta$ morphism whose image is $[n] \setminus \{i\}$ (the pneumonic naming $s^n_i$ is for “skip $i$”). For $i \in [n]$, let $e^n_i : [n+1] \to [n]$ be the unique nondecreasing map which is surjective and hits $i \in [n]$ twice.

Let $\mathbf{C}$ be a category. A simplicial object in $\mathbf{C}$ is a functor $\Delta^{\text{op}} \to \mathbf{C}$. For example, any object $C \in \mathbf{C}$ determines a constant simplicial object (also denoted $C \in s\mathbf{C}$) via the constant functor $[n] \mapsto C$ (which takes every $\Delta^{\text{op}}$ morphism to $\text{Id}_C$). Simplicial objects in $\mathbf{C}$ form a category $s\mathbf{C} := \text{Hom}_{\text{Cat}}(\Delta^{\text{op}}, \mathbf{C})$, which evidently contains $\mathbf{C}$ as a full subcategory by identifying $\mathbf{C}$ with the constant simplicial objects in $s\mathbf{C}$. For $C \in s\mathbf{C}$, it is customary to write $C_n$ for $C([n])$ and $d^n_i : C_n \to C_{n-1}$ for $C(s^n_i)$. The maps $d^n_i$ are called the boundary maps of $C$. Similarly, we will write $e^n_i : C_n \to C_{n+1}$ as abuse of notation for $C(e^n_i)$. The maps $e^n_i$ are called degeneracy maps.

More generally, given any surjective $\Delta$ morphism $\sigma : [m] \to [n]$, the map $C(\sigma) : C_n \to C_m$ is often called a degeneracy map. Notice that every surjective $\Delta$ morphism
σ has a section (any set-theoretic section of a Δ morphism is again a Δ morphism), so every degeneracy map C(σ) of a simplicial object has a retract.

12. The standard simplex

For a non-negative integer n, let \( \Delta_n^{\text{top}} \subseteq \mathbb{R}^{n+1} \) denote the convex hull of the standard basis vectors \( e_0, \ldots, e_n \):

\[
\Delta_n^{\text{std}} := \{ t_0e_0 + \cdots + t_ne_n \in \mathbb{R}^{n+1} : t_i \in [0,1], \sum t_i = 1 \}.
\]

A morphism \( \sigma : [m] \to [n] \) in the simplicial category \( \Delta \) induces a morphism of topological spaces

\[
\Delta^{\text{top}}(\sigma) : \Delta^{\text{top}}_m \to \Delta^{\text{top}}_n
\]

\[
(t_0e_0 + \cdots + t_me_m) \mapsto \left( \sum_{i \in \sigma^{-1}(0)} t_i e_0 + \cdots + \sum_{i \in \sigma^{-1}(n)} t_i e_n \right),
\]

so we can regard \( \Delta^{\text{top}} \) as a cosimplicial topological space called the standard (topological) simplex.

The algebraic analog of the standard simplex is the following: For a non-negative integer n, let \( A_n \) denote the ring

\[
A_n := \mathbb{Z}[t_0, \ldots, t_n]/(1 - t_0 - t_1 - \cdots - t_n).
\]

A \( \Delta \)-morphism \( \sigma : [m] \to [n] \) induces a ring homomorphism

\[
A(\sigma) : A_n \to A_m
\]

\[
t_i \mapsto \sum_{j \in \sigma^{-1}(i)} t_j,
\]

so that the \( A_n \) form a simplicial ring \( A \), and hence the \( \Delta^{\text{alg}}_n := \text{Spec} A_n \) form a cosimplicial scheme \( \Delta^{\text{alg}} \) called the standard (algebraic) simplex.

13. Truncations and (co)skeleta

For a nonnegative integer n, let \( \Delta_n \) denote the full subcategory of \( \Delta \) whose objects are \([0], \ldots, [n] \). An n-truncated simplicial object in \( C \) is a functor \( \Delta_n^{\text{op}} \to C \). Let

\[
s_nC := \text{Hom}_{\text{Cat}}(\Delta_n^{\text{op}}, C)
\]

be the category of n-truncated simplicial objects in \( C \). The inclusion \( \Delta_n^{\text{op}} \hookrightarrow \Delta^{\text{op}} \) induces a restriction functor

\[
\text{tr}_n : sC \to s_nC
\]

\[
C \mapsto C|_{\Delta_n^{\text{op}}},
\]

which we call the n truncation.
Let $\Delta_{\text{mon}}$ (resp. $\Delta_{\text{mon}}^n$) denote the subcategory of $\Delta$ (resp. $\Delta_n$) with the same objects, but where a morphism $[m] \to [k]$ is also required to be monic. Define $\Delta_{\text{epi}}$ and $\Delta_{\text{epi}}^n$ similarly.

Assume $C$ has finite inverse limits. By the Kan Extension Theorem, the truncation $\text{tr}_n : sC \to s_n C$ admits a right adjoint
\[ \text{Cosk}_n : s_n C \to sC \]
called the $n$ \textit{coskeleton} functor. Specifically,
\[
\text{(Cosk}_n C)_m := \lim_{\leftarrow} \{ C_k : (m \to [k]) \in [m] \downarrow \Delta_{\text{op}}^m \} \\
= \lim_{\leftarrow} \{ C_k : ([k] \to [m]) \in \Delta_n \downarrow [m] \} \\
= \lim_{\leftarrow} \{ C_k : ([k] \to [m]) \in \Delta_{\text{mon}}^n \downarrow [m] \}.
\]
The first equality is the usual “formula” for Kan extension; the second is a translation of it. To be clear, we emphasize that the inverse limit is the inverse limit of the functor
\[
(\Delta_n \downarrow [m])^{\text{op}} \to C \\
([k] \to [m]) \mapsto C_k.
\]
The third equality is by cofinality of $\Delta_{\text{mon}}^n \downarrow [m]$ in $\Delta_n \downarrow [m]$: every object $([k] \to [m])$ in $\Delta_n \downarrow [m]$ admits a map to an object $([l] \to [m])$ where $[l] \to [m]$ is monic.

In particular, notice that $C_m = (\text{Cosk}_n C)_m$ for $m \leq n$. Indeed, when $m \leq n$, $\text{Id} : [m] \to [m]$ is the terminal object of $\Delta_n \downarrow m$.

The adjunction isomorphism
\[ \text{Hom}_{s_n C}(\text{tr}_n C, D) \to \text{Hom}_{sC}(C, \text{Cosk}_n D) \]
takes a map $f : \text{tr}_n C \to D$ to the map $C \to \text{Cosk}_n D$ given in degree $m$ by the map $C_m \to (\text{Cosk}_n D)_m$ obtained as the inverse limit of the maps
\[
f_k X(\sigma) : X_m \to Y_k
\]
over all $\sigma : [k] \to [m]$ in $\Delta_n \downarrow [m]$.

For $C \in sC$, the abuse of notation
\[ \text{Cosk}_n C := \text{Cosk}_n(\text{tr}_n C) \]
is standard.

Dually, if $C$ has finite direct limits, then the Kan Extension Theorem implies that truncation $\text{tr}_n$ admits a left adjoint
\[ \text{Sk}_n : s_n C \to sC \]
called the $n$ \textit{skeleton} functor, given on $C \in s_n C$ by the formula
\[
(\text{Sk}_n C)_m = \lim_{\rightarrow} \{ C_k : ([m] \to [k]) \in [m] \downarrow \Delta_n \} \\
= \lim_{\rightarrow} \{ C_k : ([m] \to [k]) \in [m] \downarrow \Delta_{\text{epi}}^n \}.
\]
14. Simplicial homotopy

Let $f, g : X \rightarrow Y$ be $sC$ morphisms. A homotopy $h$ from $f$ to $g$ consists of a $C$ morphism $h(\phi) : X_n \rightarrow Y_n$, defined for each morphism $\phi : [n] \rightarrow [1]$ of $\Delta$, satisfying the conditions:

1. For any commutative triangle

\[
\begin{array}{ccc}
[m] & \xrightarrow{\psi} & [n] \\
\downarrow{\phi} & & \downarrow{\theta} \\
\end{array}
\]

in $\Delta$, the square

\[
\begin{array}{ccc}
X_n & \xrightarrow{h(\phi)} & Y_n \\
\downarrow{X(\psi)} & & \downarrow{Y(\psi)} \\
X_m & \xrightarrow{h(\theta)} & Y_m \\
\end{array}
\]

commutes in $C$.

2. For any $n$, if $\phi : [n] \rightarrow [1]$ is the constant map to $0 \in [1]$, then $h(\phi) = f_n$ and if $\phi$ is the constant map to $1 \in [1]$, then $h(\phi) = g_n$.

The relation consisting of pairs

\[(f, g) \in \text{Hom}_{sC}(C, D) \times \text{Hom}_{sC}(C, D)\]

defined by “there is a homotopy from $f$ to $g$” is not generally an equivalence relation. It is clearly reflexive, but not generally symmetric or transitive. However, if there is a homotopy $h$ from $f$ to $g$, and $k, l : D \rightarrow E$ are $sC$ morphisms with a homotopy $h'$
from $k$ to $l$, then there is a homotopy from $kf$ to $lg$, obtained by composing $h$ and $h'$:

$$(\phi : [n] \to [1]) \mapsto (h'(\phi)h(\phi) : C_n \to E_n).$$

We let $\sim$ be the equivalence relation generated by this relation and we say $f$ is homotopic to $g$ if $f \sim g$. In other words, $f \sim g$ means there is a finite string of morphisms $f = f_0, f_1, \ldots, f_n = g$ such that, for every $i$, either there is a homotopy from $f_i$ to $f_{i+1}$ or there is a homotopy from $f_{i+1}$ to $f_i$.

An $s\mathbf{C}$ morphism $f : C \to D$ is a homotopy equivalence if there is an $s\mathbf{C}$ morphism $g : D \to C$ such that $fg \sim \text{Id}_D$ and $gf \sim \text{Id}_C$.

Let $\text{Hots}\mathbf{C}$ denote the category with the same objects as $s\mathbf{C}$ where

$$\text{Hom}_{\text{Hots}\mathbf{C}}(C, D) := \text{Hom}_{s\mathbf{C}}(C, D)/\sim.$$ 

Composition is defined by choosing representatives of equivalence classes and composing them. This is well defined because of the remarks about composition of homotopies made above. There is an obvious forgetful functor $s\mathbf{C} \to \text{Hots}\mathbf{C}$. Evidently $f : C \to D$ is a homotopy equivalence iff its image in $\text{Hots}\mathbf{C}$ is an isomorphism.

**Lemma 14.1.** Suppose $\mathbf{C}$ has products and $C \in \mathbf{C}$ is such that the unique map $f : C \to 1$ to the terminal object (empty product) admits section $s$ (i.e. there is some map $1 \to C$). Then

$$\text{Cosk}_0 f : \text{Cosk}_0 C \to 1$$

is a homotopy equivalence in $s\mathbf{C}$ with homotopy inverse

$$\text{Cosk}_0 s : 1 \to \text{Cosk}_0 C.$$

**Proof.** Certainly $(\text{Cosk}_0 f)(\text{Cosk}_0 s) = \text{Cosk}_0(fs)$ is the identity of $\text{Cosk}_0 1 = 1$ because $\text{Cosk}_0$ is a functor and $fs = \text{Id}_1$ by definition of “section”. We need to show that

$$(\text{Cosk}_0 s)(\text{Cosk}_0 f) = \text{Cosk}_0(sf)$$

is homotopic to the identity of $\text{Cosk}_0 C$. The map

$$(\text{Cosk}_0 (sf))_n : C^{n+1} \to C^{n+1}$$

is given by

$$(\text{Cosk}_0 (sf))_n(c)(i) = sc(i)$$

$(i = 0, \ldots, n)$. We define a homotopy to the identity by associating to $\phi : [n] \to [1]$ the “straight line homotopy” map

$$h(\phi) : C^n \to C^{n+1}$$

given by

$$h(\phi)(c)(i) := \begin{cases} sc(i), & \phi(i) = 0 \\ c(i), & \phi(i) = 1. \end{cases}$$
When $\phi$ is the constant map to $0 \in [1]$, obviously $h(\phi) = (\text{Cosk}_0(sf))_n$ and when $\phi$ is the constant map to $1 \in [1]$ obviously $h(\phi)$ is the identity. To finish the proof that $h$ is the desired homotopy, we must show that, for any commutative square

$$
\begin{array}{ccc}
[m] & \xrightarrow{\psi} & [n] \\
\downarrow{\phi} & & \downarrow{\theta} \\
[1] & \xrightarrow{} & [1]
\end{array}
$$

in $\Delta$, the square

$$
\begin{array}{ccc}
C_{n+1} & \xrightarrow{h(\theta)} & C_{n+1} \\
\downarrow{\psi^*} & & \downarrow{\psi^*} \\
C_{m+1} & \xrightarrow{h(\phi)} & C_{m+1}
\end{array}
$$

commutes in $C$.

Here $\psi^*$ is short for $(\text{Cosk}_0 C)(\psi)$. It is given by $(\psi^*c)(i) = c(\psi(i))$. This is a simple computation:

$$(\psi^* \circ h(\theta))(c)(i) = \begin{cases} 
  sfc(\psi(i)), & \theta(\psi(i)) = 0 \\
  c(\psi(i)), & \theta(\psi(i)) = 1 
\end{cases}
$$

$$(\psi^* \circ h(\theta))(c)(i) = \begin{cases} 
  sfc(\psi(i)), & \phi(i) = 0 \\
  c(\psi(i)), & \phi(i) = 1 
\end{cases}
$$

$$(\psi^* \circ h(\theta))(c)(i) = \begin{cases} 
  sf(\psi^*c)(i), & \phi(i) = 0 \\
  (\psi^*c)(i), & \phi(i) = 1 
\end{cases}
$$

$$(\psi^* \circ h(\theta))(c)(i) = (h(\phi) \circ \psi^*)(c)(i).
$$

15. Simplicial sets

In this section we study the category $\textbf{sEns}$ of simplicial sets. For a simplicial set $X$, we refer to the elements of $X_n$ as simplicies of $X$ of dimension $n$. Fix a simplex $x \in X_n$. For any injective $\Delta$ morphism $\sigma : [k] \to [n]$, the simplex $X(\sigma)(x) \in X_k$ is called a face of $x$. For any surjective $\Delta$ morphism $\sigma : [k] \to [n]$, the simplex $X(\sigma)(x) \in X_k$ is called a degeneracy of $x$. Recall that the degeneracy map $X(\sigma)$ corresponding to a $\Delta$ surjection $\sigma$ always has a retract, so in particular it is injective. The simplex $x$ is called non-degenerate if it is a degeneracy only of itself. It is easy to see that any simplex $x \in X_m$ is a degeneracy of a unique non-degenerate simplex $y$: just choose $l$ minimal such that there is a $\Delta$ surjection $\sigma : [m] \to [l]$ and a $y \in X_l$ with $X(\sigma)(y) = x$.

The $n$ skeleton functor of §13 behaves like the one the reader may be familiar with from topology:

**Lemma 15.1.** For a simplicial set $X$ and a non-negative integer $n$, the image of the adjunction morphism $\text{Sk}_n X := \text{Sk}_n(\text{tr}_n X) \to X$ consists of the simplices of $X$.
which are degeneracies of some $k$ simplex with $k \leq n$. This adjunction morphism is an isomorphism iff all non-degenerate simplices of $X$ have dimension $\leq n$.

**Proof.** According to the general formulas of §13, an element $[\sigma, x]$ of $(\text{Sk}_n X)_m$ is represented by a pair $(\sigma, x)$ consisting of a surjective $\Delta$ morphism $\sigma : [m] \rightarrow [k]$, with $k \leq n$, and an element $x \in X_k$. Given any map $\tau : [k] \rightarrow [k']$ in $[m] \downarrow \Delta_n$ and any $x' \in X_{k'}$, we introduce the relation $(\sigma', x') \sim (\sigma, X(\tau)(x'))$. Two such pairs $(x, \sigma)$ and $(x', \sigma')$ represent the same element of $(\text{Sk}_n X)_m$ iff they are equivalent under the smallest equivalence relation containing the relations $\sim$. The adjunction morphism $(\text{Sk}_n X)_m \rightarrow X_m$ takes $[\sigma, x]$ to $X(\sigma)(x) \in X_m$. The description of the image of the adjunction morphism is now clear.

It remains only to prove that the adjunction morphism is injective when all non-degenerate simplices of $X$ have dimension $\leq n$. Suppose $[x, \sigma], [x', \sigma'] \in (\text{Sk}_n X)_m$ have $X(\sigma)(x) = X(\sigma')(x') = y \in X_m$ and we wish to show that $[x, \sigma] = [x', \sigma]$. By hypothesis, we can write $y = X(\rho)(z)$ for some surjective $\Delta$ morphism $\rho : [m] \rightarrow [l]$ with $l \leq n$ and some $z \in X_l$. We can complete the diagram

$$
\begin{array}{ccc}
\sigma & \rightarrow & [m] \\
\downarrow & & \downarrow \\
[k] & \rightarrow & [k'] \\
\downarrow \sigma' & & \downarrow \rho \\
[\tau] & \rightarrow & [l] \\
\tau' & \rightarrow & \\
\end{array}
$$

in $\Delta$ as indicated (for example, by choosing sections of the surjections $\sigma, \sigma'$). We have $X(\tau)(z) = x$ because this is true after applying the monic map of sets $X(\sigma)$ (both sides are then $y$), so $(\sigma, x) \sim (\rho, z)$. Similarly we have $X(\tau')(z) = x'$, so $(\sigma', x') \sim (\rho, z)$. From transitivity of the equivalence relation induced by $\sim$, we conclude $[x, \sigma] = [x', \sigma] \in (\text{Sk}_n X)_m$. \qed

16. Standard simplicial sets

There are several special simplicial sets that will be useful later. We collect them here as a series of examples.

**Example 16.1.** For each $n \in \mathbb{N}$ we have the $n$-simplex $\Delta[n] \in \text{sEns}$ defined by setting

$$
\Delta[n]_m := \{(i_0, \ldots, i_m) : 0 \leq i_0 \leq \cdots \leq i_m \leq n\}.
$$

The boundary maps for $\Delta[n]$ are given by

$$
d^i_m : \Delta[n]_m \rightarrow \Delta[n]_{m-1} \quad (i_0, \ldots, i_m) \mapsto (i_0, \ldots, \hat{i}_i, \ldots, i_m)
$$
and the basic degeneracy maps for $\Delta[n]$ are given by
$$s^j_m : \Delta[n]_m \to \Delta[n]_{m+1} \quad (i_0, \ldots, i_m) \mapsto (i_0, \ldots, i_{j-1}, i_j, i_{j+1}, \ldots, i_m).$$
It is easy to check that for any $X \in \mathbf{sEns}$, the map
$$\text{Hom}_{\mathbf{sEns}}(\Delta[n], X) \ni f \mapsto f_n((0, 1, \ldots, n))$$
is bijective.

**Example 16.2.** For each $n \in \mathbb{N}$ we also have the boundary $\partial \Delta[n]$ of $\Delta[n]$, which is the sub-simplicial set of $\Delta[n]$ generated by
$$(1, 2, \ldots, n), (0, 2, \ldots, n), \ldots, (0, 1, \ldots, n-1) \in \Delta[n]_{n-1}.$$ The map
$$\text{Hom}_{\mathbf{sEns}}(\partial \Delta[n], X) \to \text{Hom}_{\mathbf{Ens}}([n], X_{n-1})$$
is a bijection onto the set of $f \in \text{Hom}_{\mathbf{Ens}}([n], X_{n-1})$ satisfying
$$d^i_{n-1}f(j) = d^{i-1}_{n-1}f(i) \quad \text{for all } 0 \leq i < j \leq n.$$

**Example 16.3.** For example, the simplicial $n$-sphere $S^n \in \mathbf{sEns}$ is the simplicial set with exactly two non-degenerate simplices: one in dimension zero and one in dimension $n$. Using Lemma 15.1 it is not hard to show that such a simplicial set exists and is characterized up to isomorphism (unique isomorphism unless $n = 0$) by this property.

### 17. Geometric realization

Given a simplicial set $X$, we define a topological space $|X|$ called the geometric realization of $X$ as follows. We start with the space
$$\overline{X} := \bigsqcup \bigsqcup \bigsqcup \Delta^\text{top}_n,$$
where $\Delta^\text{top}_n$ is the standard topological $n$ simplex of $\Sect$. For clarity, we write $\Delta^\text{top}_n(x)$ for the component of the coproduct indexed by an $n$ simplex $x \in X_n$. Given any $\Delta$ morphism $\sigma : [m] \to [n]$, we have the map of sets $X(\sigma) : X_n \to X_m$ and the map of topological spaces $\Delta^\text{top}(\sigma) : \Delta^\text{top}_m \to \Delta^\text{top}_n$ reflecting the cosimplicial structure of $\Delta^\text{top}$. For any $x \in X_n$ we glue the component $\Delta^\text{top}_m(X(\sigma)(x))$ of $\overline{X}$ to the component $\Delta^\text{top}_n(x)$ of $\overline{X}$ via the map
$$\Delta^\text{top}(\sigma) : \Delta^\text{top}_m(X(\sigma)(x)) \to \Delta^\text{top}_n(x).$$
That is, we introduce the relation $\sim$ on $\overline{X}$ where $P \sim \Delta^\text{top}(\sigma)(P)$ for all $P \in \Delta^\text{top}_m(X(\sigma)(x))$. We let $|X|$ be the space obtained from $\overline{X}$ by making all such gluings. That is, we let $|X|$ be the quotient of $\overline{X}$ (with the quotient topology) by the equivalence relation generated by the relations $\sim$ introduced above.
The construction is clearly functorial in $X$, hence defines a functor $|| : s\text{Ens} \to \text{Top}$. In fact, the standard $n$ simplex $\Delta^n_{\text{top}}$ is compact, so $\overline{X}$ is a compactly generated space (a set is open iff its intersection with each compact set is open), hence so is $|X|$ since it is a quotient of $\overline{X}$. We can therefore view $||$ as a functor $|| : s\text{Ens} \to \text{K}$, where $\text{K}$ is the category of compactly generated topological spaces. As such, the functor $||$ preserves finite inverse limits [Hov, 3.2.4].

For more on geometric realization, see [Mil].

18. The abelian setting

Here we briefly review the theory of simplicial objects in an abelian category $A$. The basic theorems of Dold-Kan-Puppe assert that simplicial objects in $A$ are the same thing is chain complexes in $A$ supported in non-negative degrees.

Given an $A \in sA$ and an $n \in \mathbb{N}$, we set

$$\mathcal{N}_n(A) := \cap_{i=0}^{n-1} \ker(d_n^i : A_n \to A_{n-1}).$$

(Note that the intersection is over the kernels of all but the last boundary map $d_n^i : A_n \to A_{n-1}$. By convention, $\mathcal{N}_0(A) := A_0$. It is easy to check that the last boundary map $d_n^n$ takes $\mathcal{N}_n(A)$ into $\mathcal{N}_{n-1}(A)$ and that $d_n^{n-1}d_n^n = 0$, so that we have a chain complex

$$\mathcal{N}(A) := (\mathcal{N}_n(A), d_n^\bullet)$$

in $A$ supported in non-negative degrees called the normalized chain complex of $A$. This is functorial in $A$ and defines a functor

$$sA \to \text{Ch}_{\geq 0}A,$$

which is in fact an equivalence of categories. Furthermore, this functor takes homotopies in the simplicial sense of §14 to homotopies in the usual sense of chain complexes. When $A$ is the category of modules over a ring $R$, then the simplicial set underlying any simplicial $R$ module $M$ is fibrant and the homotopy groups of this simplicial set (with any choice of basepoint; the natural one being $0 \in M$) are identified with the homology of the normalized chain complex $\mathcal{N}_n(A)$.

19. Simplicial rings

Let $A\text{n}$ denote the category of commutative rings with unit. A simplicial object in $A\text{n}$ is called a simplicial ring. Given an $A \in sA\text{n}$, a module over $A$ is a simplicial abelian group $M$ such that each $M_n$ is equipped with the structure of an $A_n$ module compatibly with the simplicial structures: that is, for each $\Delta$-morphism $\sigma : [m] \to [n]$, the map of abelian groups $M(\sigma) : M_n \to M_m$ should be $A(\sigma) : A_n \to A_m$ linear:

$$M(\sigma)(a \cdot m) = A(\sigma) \cdot M(\sigma)(m)$$

for all $a \in A_n$, $m \in M_n$. For example, if $A \to B$ is a morphism of simplicial rings, then the modules of Kähler differentials $\Omega_{B/A_n} \in \text{Mod}(B_n)$ fit together to form a $B$ module $\Omega_{B/A}$.
For a simplicial ring $A$, we set

\[
N_n(A) := \cap_{i=0}^{n-1} \ker(d_i^n : A_n \to A_{n-1})
\]

\[
Z_n(A) := \cap_{i=0}^{n-1} \ker(d_i^n : A_n \to A_{n-1})
\]

\[
H_n(A) := N_n(A) / d_{n+1}^{n+1}Z_{n+1}(A).
\]

Note that $N_n(A), Z_n(A) \subseteq A_n$ are ideals of $A_n$. The subset $d_{n+1}^{n+1}Z_{n+1}(A) \subseteq A_n$ is also an ideal because $d_{n+1}^{n+1} : A_{n+1} \to A_n$ has a section $s_n^n : A_n \to A_{n+1}$ so for $a \in A_n$ and $b \in Z_{n+1}(A)$ we have

\[
ad_{n+1}^{n+1}(b) = d_{n+1}^{n+1}(s_n^n(a))d_{n+1}^{n+1}(b) = d_{n+1}^{n+1}(s_n^n(a)b)
\]

and $s_n^n(a)b \in Z_{n+1}(A)$ because $Z_{n+1}(A) \subseteq A_{n+1}$ is an ideal. Consequently, $H_n(A)$, being a quotient of two ideals of $A_n$, becomes an $A_n$ module.

If $A_n$ is a noetherian ring, then the $A_n$ modules $Z_n(A), d_{n+1}^{n+1}Z_{n+1}(A), H_n(A)$ are all finitely generated.

The homology groups $H_n(A)$ fit together into a graded-commutative ring

\[
H_* (A) := \bigoplus_{n=0}^\infty H_n(A).
\]

The fastest way to define the multiplication maps

\[
H_m(A) \otimes H_n(A) \to H_{m+n}(A)
\]

is via geometric realization: we have

\[
H_m(A) = \pi_m(A)
\]

\[
= \pi_m(|A|),
\]

where $|A|$ is the compactly generated topological space obtained as the geometric realization §17 of the simplicial set underlying the simplicial ring $A$. The space $|A|$ is of course pointed by $0 \in |A|$. Since $A$ is a ring object in simplicial sets and geometric realization (viewed as a functor to $K$) preserves finite inverse limits, $|A|$ is a ring object in $K$. An element $[f]$ of $\pi_m(|A|)$ is represented by a map $f : I^m \to A$ taking the boundary of the cube $I^m$ to 0 in $|A|$ and an element $[g]$ of $\pi_n(A)$ is represented by a map $g : I^n \to A$ taking the boundary of $I^n$ to 0. Using the ring structure on $A$, we can define a map

\[
f \circ g : I^{m+n} \to |A|
\]

\[
(x, y) \mapsto f(x)g(y)
\]

which also clearly takes the boundary of $I^{m+n}$ to 0 in $|A|$, hence represents a class $[fg] \in \pi_{m+n}(|A|)$. The multiplication $[f][g] := [fg]$ is clearly well-defined; it is graded commutative because the automorphism $(x, y) \mapsto (y, x)$ of $I^{m+n}$ acts by $(-1)^{mn}$ on the orientation of $I^{m+n}$.

Similarly, the normalized chain complex $N(A)$ of a simplicial ring $A$ carries a natural differential graded ring structure. The multiplication maps

\[
N_m(A) \otimes N_n(A) \to N_{m+n}(A)
\]
are defined using the “shuffle product” map

\[ A_m \otimes A_n \to A_{m+n} \]

\[ a \otimes b \mapsto \sum_{(\sigma, \tau)} \text{Sign}(\sigma, \tau)A(\sigma)(a)A(\tau)(b). \]

The sum here....

20. Simplicial sheaves

Let \( X \) be a topological space, \( \mathbf{Sh}(X) \) the category of sheaves on \( X \). The category \( \mathbf{sSh}(X) \) is called the category of simplicial sheaves on \( X \).

For \( \mathcal{F} \in \mathbf{Sh}(X) \), we will write \( s \in \mathcal{F} \) to mean \( s \in \text{Hom}(\mathcal{G}, \mathcal{F}) \) for some \( \mathcal{G} \) and “locally” means “after precomposing with an epimorphism \( \mathcal{H} \to \mathcal{G} \).”

A simplicial sheaf \( \mathcal{F} \in \mathbf{sSh}(X) \) is called fibrant in degree \( n \) if, for every \( k \in [n+1] \) and every

\[ x_0, \ldots, \hat{x}_k, \ldots, x_{n+1} \in \mathcal{F}_n \]

such that \( d^n_i x_j = d^n_{j-1} x_i \) for all \( i < j \) in \( [n] \) both distinct from \( k \), there is (locally) a \( y \in X_{n+1} \) with \( d^n_{i+1} y = x_i \) for all \( i \neq k \) in \( [n] \). A simplicial sheaf is fibrant if it is fibrant in every degree.

The definition is motivated as follows: Label the \( i \)th codimension one face \( \{ t_i = 0 \} \) of the standard topological \( n + 1 \) simplex

\[ \Delta_{n+1}^{\text{top}} := \{ (t_0, \ldots, t_{n+1}) \in \mathbb{R}^{n+1} : t_0 + \cdots + t_{n+1} = 1 \} \]

with \( x_i \), so the \( k \)th face remains unlabelled; the labelled faces form the so called \( k \)-horn \( \Delta_{n+1}^{\text{top}} [k] \) of the standard \( n + 1 \) simplex (I would call it the closed star of the vertex \( e_k \) opposite the \( k \)th codimension one face). The condition \( d^n_i x_j = d^n_{j-1} x_i \) says that the labellings “agree” on the codimension two faces and the fibrancy condition asserts the existence of a “map on the whole \( n + 1 \) simplex” whose restriction to each of the faces in the \( k \)-horn is the map indicated by the labelling. Note that we make no demands on the restriction of this “filling map” to the \( k \)th face. Such a filling map always exists in topology because the inclusion of the \( k \)-horn \( \Delta_{n+1}^{\text{top}} [k] \to \Delta_{n+1}^{\text{top}} \) admits a retract \( r \), hence any map \( f \) from the \( k \)-horn to a topological space \( X \) can be lifted to a map \( fr : \Delta_{n+1}^{\text{top}} \to X \).
We can abstractly define the $k$-horn of a simplicial sheaf $\mathcal{F}$ to be

$$\mathcal{F}_{n+1}[k] := \lim_{\leftarrow} \{ \mathcal{F}_m : \phi : [m] \to [n+1], \ m \leq n, \ k \in \text{Im} \phi \}.$$

Then a section of $\mathcal{F}_{n+1}[k]$ is nothing but a tuple $x_0, \ldots, \hat{x}_k, \ldots, x_n \in \mathcal{F}_n$ as above.

There is an obvious restriction map $\mathcal{F}_{n+1} \to \mathcal{F}_{n+1}[k]$; the fibrancy condition says that this should be surjective.

A map from the terminal object $1$ (the terminal object of $\text{sSh}(X)$ regarded as a constant simplicial sheaf) to a simplicial sheaf $\mathcal{F}$ will be called a basepoint of $\mathcal{F}$. To give such a map is the same thing as giving a map $1 \to \mathcal{F}_0$, for then the map $1 \to \mathcal{F}_n$ must be the composition of $1 \to \mathcal{F}_0$ and any degeneracy map $\mathcal{F}_0 \to \mathcal{F}_n$, since this degeneracy map is an isomorphism in the constant simplicial object $1$. We will systematically use "1" as abuse of notation for the map $1 \to \mathcal{F}_0$. The existence of a basepoint implies, in particular, that each map $\mathcal{F}_n \to 1$ is an epimorphism (since it has a section). The category of based or pointed simplicial sheaves is the category of simplicial sheaves under $1$.

Let $1 \to \mathcal{F}$ be a based fibrant simplicial sheaf, $n$ a nonnegative integer. Let $Z_n^1\mathcal{F}$ be the subsheaf of $\mathcal{F}_n$ consisting of those $x \in \mathcal{F}_n$ with $d^n_{i+1}z = 1$ for all $i \in [n]$. The sheaf $Z_n^1\mathcal{F}$ is called the sheaf of $n$ cycles in $\mathcal{F}$. Define a relation $\sim$ on $Z_n^1\mathcal{F}$ by declaring $x \sim y$ iff, locally, there is some $z \in \mathcal{F}_{n+1}$ such that

1. $d^n_{i+1}z = 1$ for $i \in [n+1] \setminus \{n, n+1\}$ and
2. $d^n_{n+1} = x$, $d^{n+1}_{n+1}z = y$.

I claim $\sim$ is an equivalence relation.

**Reflexive:** (This is the most difficult property to establish.) Given $x \in Z_n^1\mathcal{F}$, we seek, locally, a $z \in \mathcal{F}_{n+1}$ such that

1. $d^n_{i+1}z = 1$ for $i \in [n-1]$ and
2. $d^n_{n+1}z = x$ for $i \in \{n, n+1\}$.

First check that

$$(1, \ldots, 1, x) \in \mathcal{F}_{n+1}[n],$$

so by fibrancy in degree $n$ we can find, locally, a $y \in \mathcal{F}_{n+1}$ such that $d^n_{i+1}y = 1$ for $i \in [n-1]$ and $d^n_{n+1}y = x$ (we know nothing about $d^{n+1}_{n+1}y$). Next check that

$$(1, \ldots, 1, y, y) \in \mathcal{F}_{n+2}[n+2],$$

so by fibrancy in degree $n+1$ there is (further locally) a $w \in \mathcal{F}_{n+2}$ such that

1. $d^n_{i+2}w = 1$ for $i \in [n-1]$ and
2. $d^n_{n+2}w = d^n_{n+2}w = y$ for $i \in \{n, n+1\}$.
I claim $z := d_{n+2}^{m+2} w \in \mathcal{F}_{n+1}$ is as desired. Indeed, for $i \in [n-1]$, we compute
\begin{align*}
d_i^{m+1} z &= d_i^{m+1} d_{n+2}^{m+2} w \\
          &= d_{n+1}^{m+1} d_{i}^{n+2} w \\
          &= d_{n+1}^{m+1} 1 \\
          &= 1,
\end{align*}
and for $i \in \{n, n+1\}$, we compute
\begin{align*}
d_i^{n+1} z &= d_i^{n+1} d_{n+2}^{n+2} w \\
          &= d_{n+1}^{n+1} d_{i}^{n+2} w \\
          &= d_{n+1}^{n+1} y \\
          &= y.
\end{align*}

**Symmetric:** Suppose $x, y \in Z_n \mathcal{F}$ are such that (locally), there is some $z \in \mathcal{F}_{n+1}$ with
\begin{enumerate}
  \item $d_i^{n+1} z = 1$ for $i \in [n-1]$ and 
  \item $d_n^{n+1} z = x, d_{n+1}^{n+1} z = y$.
\end{enumerate}
By reflexivity, further locally, there is $r \in \mathcal{F}_{n+1}$ such that
\begin{enumerate}
  \item $d_i^{n+1} r = 1$ for $i \in [n-1]$ and 
  \item $d_n^{n+1} r = d_{n+1}^{n+1} r = y$.
\end{enumerate}
Since $d_{n+1}^{n+1} r = d_{n+1}^{n+1} z$ it follows that 
\begin{align*}
(1, \ldots, 1, z, r) &\in \mathcal{F}_{n+2}[n+2],
\end{align*}
so there is, further locally, some $u \in \mathcal{F}_{n+2}$ with
\begin{enumerate}
  \item $d_i^{n+2} u = 1$ for $i \in [n-1]$ and 
  \item $d_n^{n+2} z = z, d_{n+1}^{n+2} u = r$.
\end{enumerate}
Now we check, just as above, that $w := d_{n+2}^{n+2} u$ satisfies
\begin{enumerate}
  \item $d_i^{n+1} w = 1$ for $i \in [n-1]$ and 
  \item $d_n^{n+1} w = y, d_{n+1}^{n+1} w = x$.
\end{enumerate}

**Transitive:** Suppose $x, y, z \in Z_n \mathcal{F}$ have $x \sim y$ and $y \sim z$. We know $\sim$ is symmetric so, since $y \sim x$, locally there is some $u \in \mathcal{F}_{n+1}$ with
\begin{enumerate}
  \item $d_i^{n+1} u = 1$ for $i \in [n-1]$ and 
  \item $d_n^{n+1} u = y, d_{n+1}^{n+1} u = x$.
\end{enumerate}
and since $y \sim z$ there is locally some $v \in \mathcal{F}_{n+1}$ with
\begin{enumerate}
  \item $d_i^{n+1} v = 1$ for $i \in [n-1]$ and 
  \item $d_n^{n+1} v = y, d_{n+1}^{n+1} v = z$.
From the equality $d^{n+1}_n u = d^{n+1}_n v$ it follows that

$$(1, \ldots, 1, u, v) \in \mathcal{F}_{n+2}[n+2].$$

By fibrancy, there is locally some $w \in \mathcal{F}_{n+2}$ with

1. $d^{n+2}_n i w = 1$ for $i \in [n-1]$ and
2. $d^{n+2}_n n w = u, d^{n+2}_{n+1} w = v$

and one checks easily that $d^{n+2}_{n+2} w \in \mathcal{F}_{n+1}$ witnesses $x \sim z$.

Because of the judicious insertions of the word “locally,” the quotient

$$\pi_n(\mathcal{F}) := (Z_n \mathcal{F})/\sim$$

is easily seen to be a sheaf, which we will call the $n^{th}$ homotopy sheaf of $\mathcal{F}$.

The sheaf $\pi_0(\mathcal{F})$ does not depend on the choice of basepoint. In fact, one can see easily from the definition that

$$\pi_0(\mathcal{F}) = \lim_{\rightarrow} \left( \begin{array}{c} \mathcal{F}_1 \\ d_i^0 \\ d_i^1 \end{array} \mathcal{F}_0 \right).$$

**Theorem 20.2.** Let $f, g : \mathcal{F} \to \mathcal{G}$ be two maps of based fibrant simplicial sheaves which are simplicially homotopic. Then

$$\pi_n(f) = \pi_n(g) : \pi_n(\mathcal{F}) \to \pi_n(\mathcal{G})$$

for every $n$.

**Proof.** Exercise. $\square$

## 21. Weak equivalences

A morphism $f : \mathcal{F} \to \mathcal{G}$ of based fibrant simplicial sheaves induces a morphism $Z_n \mathcal{F} \to Z_n \mathcal{G}$ for all $n$ and a morphism

$$\pi_n(\mathcal{F}) : \pi_n(f) \to \pi_n(\mathcal{G})$$

for all $n$. Indeed, $\pi_n$ is a functor from based fibrant simplicial sheaves to sheaves. Such a morphism $f$ is called a weak equivalence if $\pi_n(f)$ is an isomorphism for every $n$.

A morphism of (unbased) fibrant simplicial sheaves $f : \mathcal{F} \to \mathcal{G}$ is called a weak equivalence if, for every $U \in \text{Ouv}(X)$, every basepoint $1 \to \mathcal{F}|_U$, and every $n$, the map $\pi_n(f|_U)$ is an isomorphism. It is necessary to look at all locally defined basepoints in order to have a notion of weak equivalence that is local (even the existence of a global basepoint is a serious restriction, so otherwise too many maps would vacuously be weak equivalences).

---

3Note here that $f|_U : \mathcal{F}|_U \to \mathcal{G}|_U$ can be viewed as a map of based fibrant simplicial sheaves on $U$ for a unique choice of basepoint on $\mathcal{G}|_U$, namely the one given by the composition $1 \to \mathcal{F}|_U \to \mathcal{G}|_U$. 

---
Lemma 21.1. A morphism of fibrant simplicial sheaves $f : \mathcal{F} \to \mathcal{G}$ is a weak equivalence iff $f_x : \mathcal{F}_x \to \mathcal{G}_x$ is a weak equivalence of fibrant simplicial sets (bijection on homotopy sets for every basepoint) for every point $x \in X$.

Proof. $f$ is a weak equivalence iff $\pi_n(f|_U) : \mathcal{F}|_U \to \mathcal{G}|_U$ is an isomorphism for every basepoint $1 \to \mathcal{F}|_U$. This can be checked on stalks by (6.1). Formation of homotopy sheaves commutes with stalks since the construction of $\pi_n(\mathcal{F}|_U)$ can be phrased in terms of finite inverse limits (to construct $Z_n(\mathcal{F}|_U)$ and finite direct limits (to take the quotient $(Z_n(\mathcal{F}|_U))/\sim$) and these limits commute with stalks. □

A fibrant simplicial sheaf $\mathcal{F} \in s\text{Sh}(X)$ is called acyclic (or a hypercover) if the map to the terminal object $\mathcal{F} \to 1$ is a weak equivalence. It is clear that $\pi_n(1) = 1$ for all $n$, so $\mathcal{F}$ is a hypercover iff $\pi_n(\mathcal{F}|_U) = 1$ for all $n$ for all $U \in \text{Ouv}(X)$ and for all basepoints $1 \to \mathcal{F}|_U$.

The category $\text{HCov}(X)$ of hypercovers of $X$ is the full subcategory of $\text{HotsSh}(X)$ consisting of hypercovers. Note, by (20.2), that the property of being a hypercover is invariant under homotopy equivalences.

22. Properties of coskeleta

The coskeleta functors play well with fibrancy and homotopy sheaves. In order to streamline the proofs in this section, we begin with some notation for coskeleta. Like any inverse limit, $(\text{Cosk}_n C)_m$ sits inside the corresponding product, so we may view a section of (“map to”) $(\text{Cosk}_n C)_m$ as a section

$$s = (s(\phi)) \in \prod_{\phi : [k] \to [m], k \leq n} C_k$$

satisfying the condition: whenever $\phi : [k] \to [m]$ can be factored in $\Delta$ as

$$[k] \xrightarrow{\psi} [l] \xrightarrow{\theta} [m]$$

with $l \leq n$, we have $s(\phi) = C(\psi)(s(\theta))$.

Of course, if $m \leq n$, then such a section $s$ is uniquely determined by $s(\text{Id} : [m] \leftrightarrow [m]) \in C_m$ and conversely any such $s(\text{Id} : [m] \leftrightarrow [m])$ determines a unique section of $(\text{Cosk}_n C)_m$ by setting

$$s(\phi : [k] \leftrightarrow [m]) := C(\phi)(s(\text{Id} : [m] \leftrightarrow [m]))$$

and we see that $C_m = (\text{Cosk}_n C)_m$ as we already noted in (11).

Evidently then, we should concentrate on the case $m \geq n$. Here, $s$ is determined by the $s(\phi : [n] \leftrightarrow [m])$ since any map $[k] \to [m]$ with $k \leq n$ factors through such a
$\phi$. The condition that

$$s = (s(\phi : [n] \hookrightarrow [m])) \in \prod_{\phi : [n] \hookrightarrow [m]} C_n$$

actually determines a section of $(\text{Cosk}_n C)_m$ is the following: For any two monic $\Delta$ morphisms $\phi, \psi : [n] \hookrightarrow [m]$, define an integer $k$ by

$$k + 1 := |\text{Im } \phi \cap \text{Im } \psi|.$$ 

If $k = -1$, then no condition is imposed. Otherwise, there is a (unique) commutative diagram

$$\begin{array}{ccc}
[n] & \xrightarrow{\theta} & [k] \\
\downarrow{\phi} & & \downarrow{\psi} \\
[m] & \xleftarrow{\eta} & [n]
\end{array}$$

of $\Delta$ monomorphisms and the requirement is:

$$C(\theta)(s(\phi)) = C(\eta)(s(\psi)).$$

**Lemma 22.1.** For any $C \in sC$, and any $0 \leq m \leq n$, the adjunction morphism

$$\text{Cosk}_m C \to \text{Cosk}_n(\text{Cosk}_m C)$$

is an isomorphism.

**Proof.** Follows directly from the definitions and the fact that inverse limits commute amongst themselves. $\square$

**Lemma 22.2.** For any $\mathcal{F} \in s\mathsf{Sh}(X)$, $\text{Cosk}_n \mathcal{F}$ is fibrant in degrees $> n$. If $\mathcal{F}$ is fibrant, then $\text{Cosk}_n \mathcal{F}$ is fibrant.

**Proof.** Suppose $m > n$ and consider

$$x = (x_1, \ldots, \hat{x}_k, \ldots, x_{m+1}) \in (\text{Cosk}_n \mathcal{F})_{m+1}[k].$$

Note each $x_i$ is in $(\text{Cosk}_n \mathcal{F})_m$ and we view the latter as being contained in

$$\prod_{[l] \hookrightarrow [m], l \leq n} \mathcal{F}_n$$

as in the above discussion. We want to find a $y \in (\text{Cosk}_n \mathcal{F})_{m+1}$ with $d_i^{m+1}y = x_i$ for $i \in [m+1] \setminus \{k\}$. Using the above description of sections $y \in (\text{Cosk}_n \mathcal{F})_{m+1}$, we define $y(\phi : [n] \hookrightarrow [m + 1])$ as follows: Since $m > n$, the image of $\phi$ must miss at
least two elements of \([m + 1]\) (this is the key point). In particular, it misses some \(i \in [m + 1]\) with \(i \neq k\), which is to say: \(\phi\) can be factored as

\[
\begin{array}{c}
[\n] \xrightarrow{\psi} [m] \xrightarrow{s_i} [m + 1].
\end{array}
\]

Declare \(y(\phi) := x_i(\psi)\). I claim this is independent of the choice of such a factorization of \(\phi\). Indeed, suppose \(\phi\) can be factored through both \(s_i, s_j : [m] \to [m + 1]\) (with \(i < j\), say). Then we have a commutative diagram

![Diagram](image)

of monomorphisms in \(\Delta\). Using the definition of the boundary maps in the coskeleton and the fact that \(x\) is in the \(k\)-horn \(F_{n+1}[k]\), we then compute

\[
\begin{align*}
x_i(\psi) &= x_i(s_{j-1}\theta) \\
&= (d_{j-1}^m)(\theta) \\
&= (d_i^m x_j)(\theta) \\
&= x_j(s_i\theta) \\
&= x_j(\eta).
\end{align*}
\]

It easily follows that \(y\) is actually a section of \((\text{Cosk}_n \mathcal{F})_{m+1}\). We now easily compute

\[
(d_i^{m+1}y)(\phi : [n] \to [m]) = y(s_i\phi) = x_i(\phi).
\]

To prove the second statement, first note that \(\text{Cosk}_n \mathcal{F}\) is certainly fibrant in degrees \(m < n\) since fibrancy in degree \(m\) only depends on the degree \(m - 1, m, m + 1\) parts, and \(\mathcal{F} \to \text{Cosk}_n \mathcal{F}\) is an isomorphism in these degrees. By what we just proved above, it remains only to prove fibrancy of \(\text{Cosk}_n \mathcal{F}\) in degree \(n\). Since \(\mathcal{F}_n = (\text{Cosk}_n \mathcal{F})_n\), and \(\mathcal{F}_{n-1} = (\text{Cosk}_n \mathcal{F})_{n-1}\) we have

\[
\mathcal{F}_{n+1}[k] = (\text{Cosk}_n \mathcal{F})_{m+1}[k].
\]

Given

\[
x = (x_1, \ldots, \hat{x}_k, \ldots, x_{n+1}) \in (\text{Cosk}_n \mathcal{F})_{m+1}[k] = \mathcal{F}_{n+1}[k],
\]

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\]

Given

\[
x = (x_1, \ldots, \hat{x}_k, \ldots, x_{n+1}) \in (\text{Cosk}_n \mathcal{F})_{m+1}[k] = \mathcal{F}_{n+1}[k],
\]
we use the fact that $\mathcal{F}$ is fibrant to find $y \in \mathcal{F}_{n+1}$ such that $d_i^{n+1} = x_i$ for $i \in [n+1] \setminus \{k\}$. Define

$$z \in \prod_{[n] \to [n+1]} \mathcal{F}_n$$

by

$$z(s_i : [n] \hookrightarrow [n+1]) := d_i^{n+1} y \in \mathcal{F}_n.$$ 

One easily checks that in fact $z \in (\text{Cosk}_n \mathcal{F})_{n+1}$ and that $d_i^{n+1} z = x_i$. □

**Lemma 22.3.** For any based fibrant simplicial sheaf $\mathcal{F}$, $\pi_m(\text{Cosk}_n \mathcal{F}) = 1$ for $m \geq n$.

**Proof.** Suppose $x \in Z_m(\text{Cosk}_n \mathcal{F})$ for $m \geq n$. Define $y \in (\text{Cosk}_n \mathcal{F})_{m+1}$ as follows: Given $\phi : [n] \to [m+1]$, choose a factorization

$$[n] \xrightarrow{\psi} [m] \xrightarrow{s_i} [m+1]$$

and declare

$$y(\phi) := \begin{cases} x(\psi), & i = m + 1 \\ 1, & \text{otherwise}. \end{cases}$$

I claim that this is independent of the choice of factorization. The only issue is when $\phi$ can be factored as $s_i \psi$ for $i < m + 1$ and as $s_{m+1} \eta$, in which case we have a commutative diagram

$$\begin{array}{ccc}
[n] & \xrightarrow{\psi} & [m] \\
\downarrow{\theta} & \nearrow{s_{j-1}} & \searrow{s_i} \\
[m-1] & \xrightarrow{s_i} & [m+1] \\
\downarrow{s_i} & \nearrow{s_{m+1}} & \searrow{[m]} \\
[m] & \xrightarrow{\eta} & [m] \\
\end{array}$$

of monomorphisms in $\Delta$. But then we just compute

$$x(\psi) = x(s_m \theta) = (d_m^n x)(\theta) = 1$$

since $x \in Z_m(\text{Cosk}_n \mathcal{F})$. This independence of choice of factorization implies that

$$y \in \prod_{[n] \to [m+1]} \mathcal{F}_n.$$
is in fact in \((\text{Cosk}_n F)_{m+1}\). Now we compute, for \(i \in [m]\) that
\[
(d^{m+1}_{i}(\phi : [n] \hookrightarrow [m]) = y(s_i \phi) = 1
\]
and
\[
(d^{m+1}_{m+1}y)(\phi : [n] \hookrightarrow [m]) = y(s_{m+1} \phi) = x(\phi),
\]
\(y\) witnesses \(x \sim 1\) in \(\pi_m(F)\).

**Lemma 22.4.** Let \(F\) be a based fibrant simplicial sheaf. Suppose that the degree \(n+1\) part \(\pi_{n+1} : F_{n+1} \to (\text{Cosk}_n F)_{n+1}\) of the adjunction morphism \(F \to \text{Cosk}_n F\) is an epimorphism for every \(n\). Then \(F \to 1\) is a weak equivalence (i.e. \(\pi_n(F) = 1\) for every \(n\)).

**Proof.** Given \(x \in Z_n(F) = Z_n(\text{Cosk}_n F)\), we know from the previous lemma that \(\pi_n(\text{Cosk}_n F) = 1\). So, since \(x \sim 1\) in \(Z_n(\text{Cosk}_n F)\), there is locally (in fact, from the proof of the previous lemma, we don’t need “locally here”) some \(z \in (\text{Cosk}_n F)_{n+1}\) such that
\[
\begin{align*}
(1) \quad d^{n+1}_{i} &= 1 \text{ for } i \in [n] \text{ and} \\
(2) \quad d^{n+1}_{n+1}y &= x.
\end{align*}
\]
Note that we always give \(\text{Cosk}_n F\) the basepoint determined by the composition
\[
1 \to F \to \text{Cosk}_n F.
\]
Since \(F_{n+1} \to (\text{Cosk}_{n-1} F)_n\) is epic, we also know that, further locally, there is some \(z \in F_{n+1}\) with \(\pi_{n+1}z = y\). But \(F \to \text{Cosk}_n F\) is a map of simplicial objects, so the diagrams
\[
\begin{array}{ccc}
F_{n+1} & \xrightarrow{\pi_{n+1}} & (\text{Cosk}_n F)_{n+1} \\
d^{n+1}_{i} \downarrow & & \downarrow d^{n+1} \\
F_n = (\text{Cosk}_n F)_n & \xrightarrow{d^{n+1}_{n+1}} & (\text{Cosk}_n F)_n \\
\end{array}
\]
commute for every \(i \in [n+1]\), hence \(z\) witnesses \(x \sim 1\), so \([x] = 1\) in \(\pi_n(F)\).

**23. Čech theory revisited**

Given a sheaf \(F \in \text{Ab}(X)\) we used the Čech complex (1) of a cover \(\{U_i : i \in I\}\) of \(X\) to produce a resolution of \(F\), and hence a spectral sequence (5.1) converging to the cohomology \(H^n(X, F)\). This is putting the cart before the horse. Actually,

\[\text{And we know from the lemma before that that Cosk}_n F \text{ is fibrant, so it makes sense to speak of } \pi_n(\text{Cosk}_n F).\]
we will see in a moment that the Čech complexes are obtained by taking resolutions “to the left” of $\mathbb{Z}_X$, then applying $\text{Hom}(\mathbb{Z}, F)$, so we are really trying to compute
\[ H^n(X, F) = \text{Ext}^n(\mathbb{Z}_X, F) \]
(c.f. (6.1)) by resolving $\mathbb{Z}_X$. We cannot generally find a projective resolution of $\mathbb{Z}_X$ as there are not generally enough projectives in $\text{Ab}(X)$, but to correctly compute this Ext group, we only need a resolution $G^\bullet \to \mathbb{Z}_X$ where the $G^i$ are $\text{Hom}(\mathbb{Z}, F)$ acyclic for this particular $F$. Even better, if we are willing to use a sequence
\[ \cdots G^2 \to G^1 \to G^0 \]
of such resolutions (really a filtered category of resolutions), then we could get away with the hypothesis:
\[ \lim_{n \to \infty} \text{Ext}^0(G^m, F) = 0 \]

The next thing to notice is that the failure of Čech cohomology to compute sheaf cohomology in general has to do with the fact that, no matter how fine a cover $U = \{U_i : i \in I\}$ of $X$ we choose, we will have little control over the pairwise intersections $U_i \cap U_j$ (and the triple intersections $U_i \cap U_j \cap U_k$, etc.). We saw an extreme example of this in (8) where there was a finest possible cover $U = \{U_{0A}, U_{0B}\}$ of a space $Y$, but where many sheaves on $Y$ had higher cohomology on $X = U_{0A} \cap U_{0B}$.

“The solution” is to use hypercovers (21) keep track of not only a cover $U = \{U_i\}$ of $X$ (i.e. a refinement of the trivial cover), but we also allow the possibility of taking a nontrivial cover $V_{i,j}^k$ of each $U_i \cap U_j$. Of course, once we do this, we will also want to allow nontrivial covers of all new pairwise intersections $V_{i,j}^k \cap U_k$, etc.

Given any $\mathcal{G} \in \text{Sh}(X)$, we can form its 0-coskeleton
\[ \text{Cosk}_0 \mathcal{G} \in \text{sSh}(X), \]
which is just the simplicial sheaf with
\[ (\text{Cosk}_0 \mathcal{G})_n = \mathcal{G}^{n+1} \]
(for reasons of notation, it is convenient to view $\mathcal{G}^{n+1}$ as $\mathcal{G}^{[n]} = \prod_{[n]} \mathcal{G}$). For a Δ morphism $\phi : [m] \to [n]$, 
\[ (\text{Cosk}_0 \mathcal{G})(\phi) : \mathcal{G}^{n+1} \to \mathcal{G}^{m+1} \]
is given by $(\text{Cosk}_0 \mathcal{G})(\phi)(s)(i) = s(\phi(i))$. The boundary map $d^i_n : \mathcal{G}^{n+1} \to \mathcal{G}^n$ is given by omitting the $i^{th}$ entry of an $n$-tuple and the degeneracy map $e^i_n : \mathcal{G}^n \to \mathcal{G}^{n+1}$ repeats the $i^{th}$ entry.

If $\mathcal{G} = \coprod_i h_{U_i}$ for a cover $\{U_i\}$, then since finite products commute with coproducts in $\text{Sh}(X)$ and the Yoneda functor $h$ preserves inverse limits (intersections in
(Cosk\textsubscript{0} \mathcal{F})_n = \coprod_{(i_0, \ldots, i_n)} h_{U_{i_0} \cap \cdots \cap U_{i_n}} \]

To obtain the Čech complex constructed in (1) for an \( \mathcal{F} \in \text{Ab}(X) \), we simply apply the sheaf Hom functor

\[ \mathcal{H}om(\_ , \mathcal{F}) \]

to \( \text{Cosk}_0 \mathcal{G} \) and note that the result is a cosimplicial object in \( \text{Ab}(X) \). In this case we have

\[ \mathcal{H}om(\text{Cosk}_0 \mathcal{G}, \mathcal{F})_n = \mathcal{H}om((\text{Cosk}_0 \mathcal{G})_n, \mathcal{F}) \]

\[ = \mathcal{H}om(\coprod_{(i_0, \ldots, i_n)} h_{U_{i_0} \cap \cdots \cap U_{i_n}}, \mathcal{F}) \]

\[ = \prod_{(i_0, \ldots, i_n)} \mathcal{H}om(h_{U_{i_0} \cap \cdots \cap U_{i_n}}, \mathcal{F}) \]

\[ = \prod_{(i_0, \ldots, i_n)} \mathcal{F}_{U_{i_0} \cap \cdots \cap U_{i_n}}. \]

To this cosimplicial object, one can associate a chain complex supported in nonnegative degrees, which is just the Čech resolution of \( \mathcal{F} \). Of course, there are varying ways to make this complex (the unnormalized chain complex, normalized chain complex, complement of the degeneracies in the normalized complex, etc.—all are homotopy equivalent) and these correspond to the variations on the Čech complex (e.g. the alternating Čech complex).

The fact that \( \mathcal{F} \to \mathcal{H}om(\text{Cosk}_0 \mathcal{G}, \mathcal{F}) \) is a resolution of \( \mathcal{F} \) (i.e. a quasi-isomorphism of complexes) boils down to the following simple observations:

1. It suffices to check this on each \( U_i \), since the \( U_i \) form a cover.
2. The coskeleton is constructed via finite inverse limits, hence it certainly commutes with restriction to \( U_i \), as does \( \mathcal{H}om \).
3. The restriction \( \mathcal{G}|_{U_i} \to h\chi|_{U_i} \) has an obvious section obtained from the structure map of \( h\chi|_{U_i} = h\chi|_{U_i} \) into the coproduct \( \mathcal{G} \).
4. For any object \( C \) in any category (with finite inverse limits, say) if the map to the terminal object \( 1 \) has a section, then \( \text{Cosk}_0 C \to 1 \) is a homotopy equivalence (14.1). Hence \( \mathcal{G}|_{U_i} \to h\chi|_{U_i} \) is a homotopy equivalence in \( \mathcal{S} \mathcal{S} \mathcal{H} \mathcal{A} \mathcal{B}(U_i) \).
5. Functors preserve homotopy equivalences, so \( \mathcal{F}|_{U_i} \to \mathcal{H}om(\text{Cosk}_0 \mathcal{G}|_{U_i}, \mathcal{F}|_{U_i}) \) is also a homotopy equivalence in \( \mathbf{c} \mathbf{s} \mathbf{A} \mathbf{b}(X) \).
6. The Dold-Kan theorem identifies \( \mathbf{c} \mathbf{s} \mathbf{A} \mathbf{b}(X) \) with the category of nonnegatively graded cochain complexes in \( \text{Ab}(X) \) and this identification takes homotopies in the sense of simplicial objects to homotopies in the sense of chain complexes, hence \( \mathcal{F}|_{U_i} \to \mathcal{H}om(\text{Cosk}_0 \mathcal{G}|_{U_i}, \mathcal{F}|_{U_i}) \) is a homotopy equivalence.
24. Exercises

Many of the constructions of this section involve factoring $\Gamma_\Phi$ as $\mathcal{H}^0$ followed by $\hat{H}^{\Phi}(X, -)$, under the assumption that $\Phi$ has neighborhoods (then applying the Grothendieck spectral sequence). On the other hand, we can always factor $\Gamma_\Phi$ as $\mathcal{H}^0_\Phi$ followed by the usual global section functor $\Gamma : \mathcal{PAb}(X) \to \mathcal{Ab}$, even without the neighborhoods assumption. However, it is not so clear whether we can apply the Grothendieck spectral sequence.

Exercise 24.1. The class of flasque presheaves satisfies the following properties:

1. For any short exact sequence
   
   \[ 0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0 \]
   
   in $\mathcal{PAb}(X)$ with $\mathcal{F}'$ and $\mathcal{F}$ flasque, $\mathcal{F}''$ is flasque.

2. A direct summand of a flasque presheaf is flasque.

3. For any $\mathcal{F} \in \mathcal{PAb}(X)$, there is a flasque $\mathcal{G} \in \mathcal{PAb}(X)$ and a monomorphism $\mathcal{F} \hookrightarrow \mathcal{G}$.

Exercise 24.2. For any family of supports $\Phi$ on a topological space $X$, show that the functor $\mathcal{H}^0_\Phi : \mathcal{Ab}(X) \to \mathcal{PAb}(X)$ takes flasque sheaves to separated flasque presheaves. If $X$ is notherian, show that $\mathcal{H}^0_\Phi$ takes flasque sheaves to flasque sheaves.

Exercise 24.3. Is the $\delta$-functor $\hat{H}^*(\mathcal{U}, -)$ from $\mathcal{PAb}(X) \to \mathcal{Ab}$ (c.f.) acyclic for the class of flasque presheaves? Assume $\Phi$ has neighborhoods, so that $\hat{H}^*_\Phi(X, -)$ is also a $\delta$-functor. Is it acyclic for flasque presheaves?
Bibliography

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