Small Gaps Between Almost Primes, the Parity Problem, and Some Conjectures of Erdős on Consecutive Integers

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In a previous paper, the authors proved that in any system of three linear forms satisfying obvious necessary local conditions, there are at least two forms that infinitely often assume \( E_2 \)-values; that is, values that are products of exactly two primes. We use this result to prove that there are infinitely many positive integers \( x \) such that both \( x \) and \( x+1 \) have prime factorizations of the form \( p_1^2 p_2 p_3 p_4 \). Consequently, there are infinitely many integers \( x \) that simultaneously satisfy \( \omega(x) = \omega(x + 1) = 4 \), \( \Omega(x) = \Omega(x + 1) = 5 \), and \( d(x) = d(x + 1) = 24 \). We prove several other similar theorems. Our results sharpen earlier works by Heath-Brown and Schlage-Puchta.

1 Introduction

Erdős had many favorite problems on consecutive integers (see the work of Hildebrand [13]). We will discuss among others the celebrated Erdős–Mirsky conjecture [5] on consecutive values of the divisor function:

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Conjecture C1. \[ d(x) = d(x + 1) \] infinitely often.

We will also deal with the analogous conjectures for \( \Omega(x) \) and \( \omega(x) \), which denote the number of prime factors of a positive integer \( x \) counted with and without multiplicity, respectively.

Conjecture C2. \[ \Omega(x) = \Omega(x + 1) \] infinitely often.

Conjecture C3. \[ \omega(x) = \omega(x + 1) \] infinitely often.

Conjectures C1–C3 follow from some analogues of the twin prime conjecture. For example, it is conjectured that \( 2p + 1 \) is prime for infinitely many primes \( p \), and Chen’s method (see Chapter 11 of [10]) proves that there are infinitely many \( p \in \mathcal{P} \) (\( \mathcal{P} \) denotes the set of prime numbers) such that either

\[ 2p + 1 \in \mathcal{P} \] (1)

or

\[ 2p + 1 = p_1 p_2 \quad p_1, p_2 \in \mathcal{P}, \quad p_1 \neq p_2. \] (2)

If, as we believe, (2) holds infinitely often, then C1–C3 hold; more precisely

\[ d(2p) = d(2p + 1) = 4, \quad \omega(2p) = \Omega(2p) = \omega(2p + 1) = \Omega(2p + 1) = 2. \] (3)

Due to the connection to the problem in (2) (“which is believed to be of the same depth as the twin prime conjecture” [13, p. 309]), Conjectures C1–C3 were also considered extremely difficult, if not hopeless.

It was therefore a great surprise for Erdős [4] when C. Spiro [16] proved in 1981 that

\[ d(x) = d(x + 5040) \] infinitely often.

At about the same time, Heath-Brown [11] found a conditional proof of C2 under a hypothesis slightly weaker than the Elliott–Halberstam conjecture [3]. In 1984, he succeeded [12] in proving the original Erdős–Mirsky Conjecture C1 by using Spiro’s
approach in combination with other new ideas. His method also yielded C2, but not C3. About two decades later, J.-C. Schlage-Puchta [15] gave the first proof of C3.

A common feature of all these proofs was their intimate connection with almost primes. More precisely (as suggested by (1–2)), all numbers produced that satisfied the relations C1–C3 had a bounded number of prime factors, but this number (even its parity) was left unspecified by the nature of the methods applied.

The phenomenon which “prevents us from showing that (2) has infinitely many solutions” is called the “parity obstacle” or “parity problem.” Selberg gave the two sets [9, Chapter 4]

$$A'(X) = \{a: X < a \leq 2X, \; \Omega(a) \equiv r \pmod 2\} \; (r = 0, 1),$$

which show that the upper and lower bounds obtained under general conditions of the linear sieve of Rosser are optimal. The parity problem is expressed informally as saying that sieve methods cannot differentiate between integers with an even and an odd number of prime factors. Therefore, the parity obstacle prevents the sieve method from revealing the existence of primes in a suitable set as formulated by Greaves [9, p. 171]. For example, while Chen’s method yields that $2p + 1$ has infinitely often at most two prime factors, the method is unable to specify the parity of the number of prime factors of $2p + 1$. As a result, the seemingly much easier assertion that for infinitely many primes $p$, $2p + 1$ has an odd (or even) number of prime factors, is still open [13, p. 310]. The same comments apply when $2p + 1$ is replaced by $p + 2$.

The general view about the parity problem and Conjectures C1–C3 can be described by again citing the survey paper of Hildebrand [13, p. 310]: “However, there is one crucial difference which makes this conjecture [the Erdős–Mirsky conjecture] more accessible than ‘twin-prime type’ conjectures. Namely, in contrast to the above-mentioned problem on the parity of the number of prime factors of $2p + 1$ or $p + 2$ when $p$ is prime (and thus, in particular, has an odd number of prime factors), when trying to prove that $d(n) = d(n + 1)$ holds infinitely often one does not need to specify the parity of the number of prime factors of $n$. It is this fact that allowed the solution of the Erdős–Mirsky conjecture, while bypassing the deeper problems related to the twin prime conjecture.”

Heath-Brown describes the situation in exactly the same way. In reference to the proof of (4) by Spiro, (and his remark also applies to the solution he gives for C1 and C2): “Thus one does not know the value of $\Omega(n)$ for the particular $n$ which satisfies $d(n) = d(n + 5040)$. In this way, one sidesteps the ‘parity problem’.”
In the present work, we shall show that the method which yielded the existence of short gaps between primes [8] and $E_2$ numbers with bounded differences [6, 7] is able to prove a stronger version of Conjectures C1–C3. (An $E_2$ number is a product of two distinct primes. Such numbers are also known as semiprimes.) In this variant, the parity problem is not bypassed but overcome. We can show the above conjectures in the stronger form where the value of the relevant arithmetic function $d, \Omega$ or $\omega$ is specified. Moreover, we prove the existence of infinitely many $x$ that satisfy Conjectures C1–C3 simultaneously.

Before stating our first result, we need a definition. For a positive integer $n$ with prime factorization $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$, we define the exponent pattern of $n$ to be the multiset $\{a_1, a_2, \ldots, a_k\}$. We follow the usual conventions for multisets; in particular, an element may appear more than once, and the order of elements does not matter. The relevance of this definition comes from the observation that if $x$ and $x + 1$ have the same exponent pattern, then $\omega(x) = \omega(x + 1), \Omega(x) = \Omega(x + 1), \text{ and } d(x) = d(x + 1)$. We prove

**Theorem 1.1.** Let $\mathcal{A}$ be any multiset of positive integers that contains $\{2, 1, 1, 1\}$ as a subset. There exist infinitely many integers $x$ such that $x$ and $x + 1$ both have exponent pattern $\mathcal{A}$. Consequently, for any integer $B \geq 0$, there exist infinitely many integers $x$ such that

$$\omega(x) = \omega(x + 1) = 4 + B, \quad \Omega(x) = \Omega(x + 1) = 5 + B, \quad \text{and } d(x) = d(x + 1) = 24 \cdot 2^B. \quad (5)$$

Furthermore, for any integer $B \geq 1$, there exist infinitely many integers $x$ such that

$$\omega(x) = \omega(x + 1) = 5, \quad \Omega(x) = \Omega(x + 1) = 5 + B, \quad \text{and } d(x) = d(x + 1) = 24(B + 1). \quad (6)$$

Note that in fact, we have $f(x) = f(x + 1)$ infinitely often for all number-theoretic functions $f$ with the property that $f(n)$ depends only on the exponent pattern of $n$.

We can also prove that there are infinitely many $x$ with $\omega(x) = \omega(x + 1) = 3$, and a similar result for $\Omega(x) = \Omega(x + 1) = 4$. However, these sharpenings come at the price of simultaneity; that is, the values of $x$ with $\omega(x) = \omega(x + 1) = 3$ are not necessarily the same as the values of $x$ with $\Omega(x) = \Omega(x + 1) = 4$.

**Theorem 1.2.** There are infinitely many integers $x$ with

$$\omega(x) = \omega(x + 1) = 3.$$

□
Theorem 1.3. There are infinitely many integers $x$ with

$$\Omega(x) = \Omega(x + 1) = 4.$$

\[\square\]

Eggleton and MacDougall [2, Theorem 4] proved, conditionally upon a natural generalization of the twin prime conjecture, that for any $n \geq 2$, there are infinitely many integers $x$ such that $\omega(x) = \omega(x + 1) = n$. From Theorems 1.1 and 1.3, we see that this is true unconditionally for $n \geq 3$.

For a given multiset $\mathcal{A}$, let $\mathcal{R}_\mathcal{A}$ be the set of positive integers with exponent pattern $\mathcal{A}$. Let $r_1 < r_2 < \ldots$ be the ordering of the elements of $\mathcal{R}_\mathcal{A}$. Theorem 1.1 states that when $\mathcal{A} = \{2, 1, 1, 1\},$

$$\liminf_{n \to \infty} (r_{n+1} - r_n) = 1.$$

In [7], we proved that if $\mathcal{A} = \{1, 1\}$, then

$$\liminf_{n \to \infty} (r_{n+1} - r_n) \leq 6.$$

Intermediate to these two results, we have

Theorem 1.4. Let $e$ be any integer with $e \geq 1$, and let $\mathcal{A}$ be any multiset containing $\{1, 1, e\}$. In the notation given above,

$$\liminf_{n \to \infty} (r_{n+1} - r_n) \leq 2.$$

\[\square\]

All of the theorems here are done in a straightforward and uniform way. In fact, all of our theorems are simple consequences of the following result proved in [7, Theorem 2].

Basic Theorem. We say that

$$L_i(x) = a_ix + b_i \quad (1 \leq i \leq 3) \quad a_i, b_i \in \mathbb{Z}, \ a_i > 0,$$

is an admissible triplet of linear forms if for every prime $p$ there exists $x_p \in \mathbb{Z}$ such that

$$p \mid L_1(x_p)L_2(x_p)L_3(x_p).$$
Let \( C \) be any constant. If \( \{L_1, L_2, L_3\} \) is an admissible triplet, then there are two forms 
\( L_i(x), L_j(x) \) \((i \neq j)\) in the triplet that simultaneously take \( E_2 \)-values with both prime factors exceeding \( C \) for infinitely many integer values \( x \). In other words, there are two forms \( L_i, L_j \) such that

\[
\omega(L_i(x)) = \Omega(L_i(x)) = \omega(L_j(x)) = \Omega(L_j(x)) = 2,
\]

and

\[
\left( \prod_{p \leq C} p, L_i(x)L_j(x) \right) = 1
\]

for infinitely many integers \( x \). □

In a forthcoming paper, we will show how the methods here can be extended to prove corresponding results for an arbitrary shift \( b \). In particular, we can prove that for any positive integer \( b \), there are infinitely many integers \( x \) such that \( x \) and \( x + b \) have the same exponent pattern. This generalizes the works of Pinner [14] and Buttkewitz [1]. The former proved that \( d(x) = d(x + b) \) infinitely often for any \( b \); the latter proved that \( \omega(x) = \omega(x + b) \) infinitely often for an infinite set of shifts \( b \).

2 Proofs of the Theorems

We begin with the proof of Theorem 1.2; this is a straightforward application of the Basic Theorem. Consider the system

\[
L_1(m) = 6m + 1, \quad L_2(m) = 8m + 1, \quad L_3(m) = 9m + 1. \tag{7}
\]

This system is admissible because we may take \( x_p = 0 \) for all primes \( p \). We note the relations

\[
4L_1(m) = 3L_2(m) + 1, \quad 3L_1(m) = 2L_3(m) + 1, \quad 9L_2(m) = 8L_3(m) + 1. \tag{8}
\]

By the Basic Theorem, at least two of the forms will be simultaneously \( E_2 \) numbers for infinitely many values of \( x \). We take \( C = 3 \), so that resulting \( E_2 \) numbers have both prime factors exceeding 3.
Now suppose the two forms giving infinitely many $E_2$ numbers are $L_1$ and $L_2$. Then we let

$$ x = 3L_2(m), \quad x + 1 = 4L_1(m). $$

If the two relevant forms are $L_1$ and $L_3$, we let

$$ x = 2L_3(m), \quad x + 1 = 3L_1(m). $$

If the two relevant forms are $L_2$ and $L_3$, we let

$$ x = 8L_3(m), \quad x + 1 = 9L_2(m). $$

In all cases, we obtain infinitely many positive integers $x$ with $\omega(x) = \omega(x + 1) = 3$.

As this proof illustrates, our approach is to combine the Basic Theorem with an appropriate choice of linear forms. Another element of our approach is the idea of “adjoining” extra prime factors. We illustrate this with the proof of Theorem 1.3. Take the system

$$ L_1(m) = 4m + 1, \quad L_2(m) = 5m + 1, \quad L_3(m) = 6m + 1. \quad (9) $$

This is admissible; let $x_p = 0$ for all $p$. Note the relations

$$ 5L_1 = 4L_2 + 1, \quad 3L_1 = 2L_3 + 1, \quad 6L_2 = 5L_3 + 1. \quad (10) $$

If we apply the Basic Theorem directly to this system, we fail to get the desired result. For example, if $L_1(m)$ and $L_2(m)$ are both $E_2$ numbers, and if $x = 4L_2(m)$, then $\Omega(4L_2(m)) = 4$ and $\Omega(5L_1(m)) = 3$. To remedy this, we modify our system so that $4m + 1$ and $6m + 1$ each have one additional prime factor. We use the Chinese Remainder Theorem to solve the system of congruences

$$ 4m + 1 \equiv 0 \pmod{11}, \quad 6m + 1 \equiv 0 \pmod{7}. $$

The solution is $m \equiv 8 \pmod{77}$. Accordingly, we set $K(\ell) = 77\ell + 8$. We then define three new forms $K_1, K_2, K_3$ by setting $r_1 = 11, r_2 = 1, r_3 = 7$, and

$$ K_i(\ell) = \frac{L_i(K(\ell))}{r_i} \quad (i = 1, 2, 3). $$
In other words,

\[ K_1(\ell) = 28\ell + 3, \quad K_2(\ell) = 385\ell + 41, \quad K_3(\ell) = 66\ell + 7. \tag{11} \]

We claim that this is an admissible system. For if \( p \neq 7 \) and \( p \neq 11 \), then we may take \( x_p \) to be the solution of \( 77x_p + 8 \equiv 0 \pmod{p} \). Otherwise, we take \( x_7 = 1 \) and \( x_{11} = 0 \). By the Basic Theorem, at least two forms in (11) are \( E_2 \) numbers with all prime factors exceeding 11. Accordingly, at least two of the statements

\[ \Omega(4m+1) = 3, \quad \Omega(5m+1) = 2, \quad \Omega(6m+1) = 3, \]

are simultaneously true for infinitely many integer values \( m \). In view of the relations (10), we obtain infinitely many integers \( x \) with

\[ \Omega(x) = \Omega(x + 1) = 4. \tag{12} \]

Note that the multiplications in (10) adjoin one new prime factor to \( 4m+1 \) and \( 6m+1 \) and two prime factors to \( 5m+1 \).

Now we prove Theorem 1.1. We begin with the case \( \mathcal{A} = \{2, 1, 1, 1\} \). Let

\[ L_1(m) = 3m + 2, \quad L_2(m) = 4m + 3, \quad L_3(m) = 10m + 7. \]

This is admissible; take \( x_p = 1 \) if \( p = 2 \) or \( 3 \), \( x_p = 2 \) if \( p = 7 \), and \( x_p = 0 \) otherwise. Note the relations

\[ 3L_2 = 4L_1 + 1, \quad 3L_3 = 10L_1 + 1, \quad 5L_2 = 2L_3 + 1. \tag{13} \]

Now let \( r_1 = 5, r_2 = 7^2, r_3 = 11^2 \), and find \( m \) such that

\[ L_i(m) \equiv 0 \pmod{r_i} \quad \text{and} \quad (L_i(m)/r_i, r_1r_2r_3) = 1 \quad \text{for} \ i = 1, 2, 3. \tag{14} \]

A calculation shows that one may take \( m = 3956 \). Consider the three reduced forms

\[ K_i(\ell) = \frac{L_i(m + r_1r_2r_3\ell)}{r_i}. \]

By calculation,

\[ K_1(\ell) = 3 \cdot 7^2 \cdot 11^2 \ell + 2374, \quad K_2(\ell) = 2^2 \cdot 5 \cdot 11^2 \ell + 323, \quad K_3(\ell) = 2 \cdot 5^2 \cdot 7^2 \ell + 327. \]
From (13), we have the relations

\[ 3r_2K_2 = 4r_1K_1 + 1, \quad 3r_3K_3 = 10r_1K_1 + 1, \quad 5r_2K_2 = 2r_3K_3 + 1. \tag{15} \]

We can show that \( \{K_1, K_2, K_3\} \) is admissible by using the admissibility of \( \{L_1, L_2, L_3\} \). Suppose first that \( p \) is a prime with \( p \nmid r_1r_2r_3 \). Take \( y_p \) to be the solution of \( m + r_1r_2r_3y_p \equiv x_p \pmod{p} \). Then

\[ r_1r_2r_3K_1K_2K_3(y_p) = L_1L_2L_3(x_p) \not\equiv 0 \pmod{p}. \]

Next, suppose that \( p| r_1r_2r_3 \); that is, \( p = 5, 7 \) or 11. Here, we may take \( y_p = 0 \) because \( p \) cannot appear as a factor of \( K_1(0) \) by the second condition in (14).

We apply the Basic Theorem and deduce that at least two of the \( K_i \) are infinitely often \( E_2 \) numbers with all prime factors exceeding 11. If, for example, \( K_1(m) = p_1p_2 \) and \( K_2(m) = p_3p_4 \), then we may take

\[ x = 4r_1 \cdot K_1(m) = 2^2 \cdot 5 \cdot p_1p_2, \quad x + 1 = 3r_2 \cdot K_2(m) = 3 \cdot 7^2 \cdot p_3p_4. \]

The other cases may be treated in a similar fashion. We conclude that there are infinitely many \( x \) such that both \( x \) and \( x + 1 \) have exponent pattern \( \{2, 1, 1, 1\} \).

To complete the proof of Theorem 1.1, we adapt the previous proof to a more general situation. One of the referees suggested the following corollary and its proof.

**Corollary 2.1.** Suppose \( L_1, L_2, L_3 \) is an admissible triplet, and that \( r_1, r_2, r_3 \) are coprime integers with \( (r_i, a_i) = 1 \) for each \( i \) and \( (r_i, a_ib_j - a_jb_i) = 1 \) for each \( i \neq j \). Then there exists \( 1 \leq i < j \leq 3 \) such that there are infinitely many integers \( n \) for which \( L_k(n) \) equals \( r_k \) times an \( E_2 \) number that is coprime to all primes \( \leq C \), for \( k = i, j \). \( \square \)

**Proof.** By the Chinese Remainder Theorem, there exists an integer \( m \) such that \( a_im + b_i \equiv r_i \pmod{r_i^2} \) because \( (a_i, r_i) = 1 \) for \( i = 1, 2, 3 \). Define

\[ K_i(\ell) = L_i(m + \ell(r_1r_2r_3)^2)/r_i \]

for \( i = 1, 2, 3 \). We will show that \( K_1, K_2, K_3 \) is an admissible triplet.

We begin by proving that \( (K_i(\ell), r_1r_2r_3) = 1 \) for each \( i = 1, 2, 3 \) and each integer \( \ell \). We first note that \( K_i(\ell) \equiv L_i(m)/r_i \equiv 1 \pmod{r_i} \). If \( i \neq j \), then \( K_i(\ell) \equiv L_i(m)/r_i \pmod{r_j} \).
If $p|(r_j, K_i(ℓ))$, then $p|(L_i(m), L_j(m))$, and therefore $p|\alpha_j L_i(m) - \alpha_i L_j(m) = \alpha_b j - \alpha_i b_j$. However, this contradicts the hypothesis that $(r_j, \alpha_b j - \alpha_i b_j) = 1$.

If the prime $p$ does not divide $r_1 r_2 r_3$, then we use the fact that $L_1, L_2, L_3$ is admissible. We select $t_p$ to be an integer for which $m + t_p (r_1 r_2 r_3)^2 \equiv x_p \pmod{p}$; hence $K_i(t_p) \equiv L_i(x_p)/r_i \not\equiv 0 \pmod{p}$.

The result now follows from the Basic Theorem.

In applying this corollary, we use triples for which

$$\frac{a_j b_i - a_i b_j}{(a_i, a_j)} = 1 \text{ for } 1 \leq i < j \leq 3,$$

so that

$$\frac{a_j}{(a_i, a_j)}(a_i x + b_i) = \frac{a_i}{(a_i, a_j)}(a_j x + b_j) + 1.$$

In the proof of Theorem 1.2, we used the forms with $6m + 1, 8m + 1, 9m + 1$ with $r_i = 1$. The ratios $a_i/(a_i, a_j)$ are all powers of a single prime, so we obtain infinitely many pairs of consecutive integers that have exactly three distinct prime factors.

In the proof of Theorem 1.3, we used the forms $4m + 1, 5m + 1, 6m + 1$ with $(r_1, r_2, r_3) = (11, 1, 7)$. Each $r_j a_i/(a_i, a_j)$ has exactly two prime factors counting multiplicity.

To prove Theorem 1.1 in the general case, we take the forms $4m + 3, 10m + 7, 3m + 2$ and $(r_1, r_2, r_3) = (5q_1, 7^2 q_2, 11^2 q_3)$, where each $q_i$ has exponent pattern $\{e_1, e_2, \ldots, e_k\}$, with the prime factors all distinct and all $\geq 13$. Then each $r_j a_i/(a_i, a_j)$ has exponent pattern $\{2, 1, e_1, \ldots, e_k\}$, and we obtain infinitely many $x$ such that $x$ and $x + 1$ both have exponent pattern $\{2, 1, 1, 1, e_1, \ldots, e_k\}$.

For a simple proof of Theorem 1.4 in the case $A = \{1, 1, 1\}$, one may apply the Basic Theorem with the forms $L_1 = 6m - 1, L_2 = 15m - 2, L_3 = 10m - 1$. Note that

$$5L_1 = 30m - 3, \quad 2L_2 = 30m - 4, \quad 3L_3 = 30m - 3.$$

Therefore, we obtain infinitely many pairs of integers with exponent pattern $\{1, 1, 1\}$ and difference at most 2.

In the general case $A = \{1, 1, e, e_1, \ldots, e_k\}$, we start with a value of $c$ that satisfies

$$c - 1 \equiv 3^e \pmod{3^{e+1}}, \quad c \equiv 2^e \pmod{2^{e+1}}, \quad c + 1 \equiv 5^e \pmod{5^{e+1}}.$$
We then consider the forms
\[
L_1(m) = \frac{30e+1m + c - 1}{3^e}, \quad L_2(m) = \frac{30e+1m + c}{2^e}, \quad L_3(m) = \frac{30e+1m + c + 1}{5^e}.
\]
We claim that \(L_1, L_2, L_3\) is admissible. If \(p = 2, 3, 5\), then \(p\) divides exactly one of the three numbers, and the highest power of \(p\) dividing that numerator is \(p^e\). Consequently, none of the \(L_i(m)\) is divisible by \(p\) for \(p \leq 5\). For \(p > 5\), each of the congruences \(L_i(m) \equiv 0 \pmod{p}\) has exactly one solution, so there are at least \(p - 3\) residue classes for which \(L_1L_2L_3(m)\) is not divisible by \(p\). We claim that all prime factors of \(a_ib_j - a_jb_i\) are below 7. This follows by noting that
\[
a_1b_2 - a_2b_1 = \frac{30e+1}{3^e}, \quad a_2b_3 - a_3b_2 = \frac{30e+1}{2^e}, \quad a_1b_3 - a_3b_1 = \frac{30e+1}{5^e}.
\]
We now apply Corollary 2.1 with \((r_1, r_2, r_3) = (q_1, q_2, q_3)\), where each \(q_i\) has exponent pattern \(\{e_1, \ldots, e_k\}\) with the prime factors all distinct and all \(\geq 7\). Consequently, there is a pair \(i, j\) with \(1 \leq i < j \leq 3\) such that \(L_i(m)\) and \(L_j(m)\) both have exponent pattern \(\{1, 1, e_1, \ldots, e_k\}\). The result now follows because \(3^eL_1 + 2 = 2^eL_2 + 1 = 5^eL_3\).

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