Small gaps between primes

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Abstract. This paper describes the authors' joint research on small gaps between primes in the last decade and how their methods were developed further independently by Zhang, Maynard, and Tao to prove stunning new results on primes. We now know that there are infinitely many primes differing by at most 246, and that one can find k primes a bounded distance (depending on k) apart infinitely often. These results confirm important approximations to the Hardy–Littlewood Prime Tuples Conjecture.

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1. History

The twin prime conjecture that n and $n+2$ are both primes for infinitely many positive integers n, may have been conceived around the time of Euclid, more than two thousand years ago. Among as yet unsolved problems in mathematics it is one of the oldest. The purpose of the present article is to give an overview of the progress in the last nine years in this subject, in particular, of the results of the authors.

As a young boy Gauss observed in 1792 or 1793 that the primes around x have an average distance $\log x$ which led him to conjecture that

$$\pi(x) := \sum_{\substack{p \le x \\ p:\text{ prime}}} 1 \sim \mathrm{li}\ x := \int_2^x \frac{dt}{\log t} \sim \frac{x}{\log x} \qquad (x \to \infty). \tag{1.1}$$

This conjecture was proved in 1896 (independently) by Hadamard and de la Vallée Poussin, and is now called the Prime Number Theorem.

A relevant quantity in the study of small gaps between primes is

$$\Delta := \liminf_{n\to\infty} \frac{d_n}{\log n} = \liminf_{n\to\infty} \frac{p_{n+1}-p_n}{\log n}, \tag{1.2}$$

where $\{p_i\}_{i=1}^{\infty} =: \mathcal{P}$ is the set of primes sequenced in increasing order and $d_n := p_{n+1} - p_n$. The Prime Number Theorem, (1.1), immediately implies $\Delta \leqslant 1$, so the first task concerning an upper estimation of Δ was to show an estimate of the type $\Delta < 1$. During the twentieth century there were many papers on upper estimates for Δ. First, in 1926, Hardy and

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Littlewood (unpublished, see [32]) succeeded in showing, assuming the Generalized Riemann Hypothesis (GRH), that

$$\Delta \leqslant 2/3. \tag{1.3}$$

The first unconditional bound

$$\Delta \leqslant 1 - c_1, \tag{1.4}$$

with an unspecified but explicitly calculable $c_1 > 0$, was shown by Erdős in 1940 [5] who used Brun's sieve. The next big step was made by Bombieri and Davenport [2] who removed the assumption of GRH in Hardy and Littlewood's method by using Bombieri's work [1] on the large sieve and showed that

$$\Delta \leqslant \left(2 + \sqrt{3}\right)/8 = 0.466\ldots . \tag{1.5}$$

Their method gave $\Delta \leqslant 1/2$ but they were also able to combine this with an explicit version of Erdős's [5] proof which led them to (1.5). After several smaller improvements (Huxley and others), Maier [23] succeeded in combining the matrix method he developed with the ideas of Bombieri–Davenport, Erdős and Huxley, making it possible to multiply the best known bound by $e^{-\gamma}$ (γ is Euler's constant) and reach

$$\Delta \leqslant 0.248\ldots . \tag{1.6}$$

In 2005 the authors proved (see [14]; or for a brief account §2, §3 below)

$$\Delta = 0. \tag{1.7}$$

2. Ideas behind the proofs of some results concerning small gaps between consecutive primes

We begin by recounting a number of conjectures related to the twin prime conjecture and more generally to small gaps between consecutive primes. Some of them have been known for a long time, some of them were introduced by us.

Conjecture 2.1 (Twin Prime Conjecture). $d_n = 2$ *infinitely often.*

A generalization of this was formulated in 1849 by de Polignac.

Conjecture 2.2 (De Polignac's Conjecture [29]). *For every given positive even integer* h, $d_n = h$ *infinitely often.*

For a further generalization we need the notion of *admissible* k*-tuples*.

Definition 2.3. $\mathcal{H} = \{h_i\}_{i=1}^{k}$ $(0 \leqslant h_1 < h_2 \cdots < h_k,\, h_k \in \mathbb{Z})$ is *admissible* if the h_i's do not cover all residue classes mod p for any prime p.

This is clearly a necessary condition that $n + h_i \in \mathcal{P}$ for all integers $1 \leq i \leq k$ holds for infinitely many numbers n.

Dickson formulated in 1904 the conjecture that this condition was also sufficient. Although his conjecture included linear forms of type $a_i n + b_i$ $(a_i, b_i \in \mathbb{Z})$ we will consider the special case $a_i = 1$ for all $i \in [1, k]$.

Conjecture 2.4 (Dickson's Conjecture [3]). *If $\mathcal{H}$ is admissible, then $n + h_i \in \mathcal{P}$ for all $i \in [1, k]$ holds for infinitely many values of n.*

About twenty years later, in 1923, Hardy and Littlewood formulated this in a quantitative form as

Conjecture 2.5 (Hardy–Littlewood Prime-Tuples Conjecture [20]). *If $\mathcal{H}$ is an admissible k-tuple, then*

$$\sum_{\substack{n\leqslant x\\ \{n+h_i\}_{i=1}^k\in\mathcal{P}^k}} 1 \sim \mathfrak{S}(\mathcal{H})\frac{x}{\log^k x}, \tag{2.1}$$

where

$$\mathfrak{S}(\mathcal{H}) := \prod_p \left(1-\frac{\nu_{\mathcal{H}}(p)}{p}\right)\left(1-\frac{1}{p}\right)^{-k} > 0, \tag{2.2}$$

and $\nu_{\mathcal{H}}(p)$ denotes the number of distinct residue classes modulo p occupied by the elements of $\mathcal{H}$.

Note that the relation $\mathfrak{S}(\mathcal{H}) > 0$ is equivalent to $\mathcal{H}$ being admissible.

Until now the conjectures were listed in increasing strength. We introduced a weaker form of Dickson's Conjecture:

Conjecture 2.6 (Conjecture DHL$(k, 2)$). *If $\mathcal{H}$ is an admissible k-tuple, then $n + \mathcal{H}$ contains at least two primes infinitely often.*

If the above conjecture is true for at least one admissible k-tuple, then it implies another conjecture which is a good approximation to the Twin Prime Conjecture. This we called the

Conjecture 2.7 (Bounded Gaps Conjecture). *There exists an absolute constant C such that $d_n = p_{n+1} - p_n \leqslant C$ for infinitely many n.*

A still weaker form of the Bounded Gap Conjecture is

Conjecture 2.8 (Small Gaps Conjecture). $\Delta = \liminf_{n\to\infty} \dfrac{p_{n+1}-p_n}{\log p_n} = 0.$

Within the scope of our work the existence of small or bounded gaps between consecutive primes is intimately connected with the distribution of primes in arithmetic progressions. The following definition of an admissible level ϑ of primes was already known and used in sieve theory.

Definition 2.9. ϑ is called an *admissible level of distribution of primes* if for any $\varepsilon > 0$, $A > 0$ we have for any $X > 2$

$$\sum_{q\leqslant X^{\vartheta-\varepsilon}} \max_{\substack{a\\ (a,q)=1}} \left| \sum_{\substack{p\equiv a(q)\\ p\leqslant X}} \log p - \frac{X}{\varphi(q)} \right| \leqslant \frac{C(A,\varepsilon)X}{(\log X)^A}, \tag{2.3}$$

where $C(A, \varepsilon)$ is an ineffective constant depending on A and ε.

The largest known level $\vartheta = 1/2$ is the celebrated Bombieri–Vinogradov [1, 38] Theorem. The strongest possibility, $\vartheta = 1$, is the Elliott–Halberstam [4] Conjecture, and more generally one can introduce

Conjecture 2.10 (Conjecture EH(ϑ)). (2.3) *is true for a fixed* $\vartheta \in (\frac{1}{2}, 1]$.

We succeeded in showing in 2005 the following result.

Theorem 2.11 ([14]). *If* EH(ϑ) *is true for some fixed* $\vartheta > 1/2$, *then* DHL$(k, 2)$ *is true for* $k > k_0(\vartheta)$ *and consequently the Bounded Gaps Conjecture is true, i.e.* $\liminf_{n\to\infty} d_n < \infty$.

Theorem 2.12 ([14]). *The Small Gaps Conjecture is true, i.e.* $\Delta = 0$.

We improved this somewhat later to

Theorem 2.13 ([15]). $\liminf_{n\to\infty} \dfrac{d_n}{(\log n)^{1/2}(\log\log n)^2} < \infty$.

Concerning the frequency of small gaps we showed

Theorem 2.14 ([17, 18]). *Given any fixed* $\eta > 0$ *the relation*

$$d_n = p_{n+1} - p_n < \eta \log n \tag{2.4}$$

holds for a positive proportion of all gaps.

One of the important ideas which yielded a proof of the Small Gaps Conjecture in [14] and which – along with the work of Y. Motohashi and J. Pintz [25] – represented an important step in the first proof of the Bounded Gaps Conjecture by Y. Zhang [39] was to attack, among the listed seven conjectures, particularly DHL$(k, 2)$. The idea was to find suitable non-negative weights a_n for $n \in [N, 2N)$ to be abbreviated later as $n \sim N$, such that a_n should be relatively large compared with $S = \sum_{n\sim N} a_n > 0$ if the set

$$n + \mathcal{H}_k = \{n + h_i\}_{i=1}^{k} \tag{2.5}$$

contains some (possibly several) primes. A good quantitative formulation is to consider (and try to maximize) the ratio

$$E_j = \frac{S_j}{S^*} := \frac{\sum\limits_{n\sim N} a_n \chi_{\mathcal{P}}(n + h_j)\log(n + h_j)}{\sum\limits_{n\sim N} a_n \log 3N}, \tag{2.6}$$

where $\chi_{\mathcal{P}}(m)$ denotes the characteristic function of primes, that is, $\chi_{\mathcal{P}}(m) = 1$ if m is prime and 0 otherwise.

The quantity

$$\alpha(\mathcal{H}_k) = \sum_{j=1}^{k} E_j \tag{2.7}$$

describes the (weighted) average number of primes in $n + \mathcal{H}_k$ if n runs between N and $2N$, i.e. $n \sim N$. If we succeed in obtaining for a k-tuple $\mathcal{H} = \mathcal{H}_k$ a lower bound greater than 1 for the quantity in (2.7), then DHL$(k, 2)$ is proved (at least for a single $\mathcal{H} = \mathcal{H}_k$), and from this the Bounded Gaps Conjecture follows immediately.

(i) If we start with the simple uniform choice $a_n \equiv 1$ we obtain

$$\alpha(\mathcal{H}_k) \sim \frac{k}{\log N} \quad \text{as } N \to \infty, \tag{2.8}$$

which clearly tends to 0.

(ii) Choosing $a_n = 1$ if $\{n + h_i\}_{i=1}^k \in \mathcal{P}^k$ and 0 otherwise, we can seemingly reach the optimal value

$$\alpha(\mathcal{H}_k) = k \ \text{ unless } \ S = \sum_{n \sim N} a_n = 0. \tag{2.9}$$

Unfortunately, to exclude the possibility $S = S(N) = 0$ for $N > N_0$ is equivalent to the proof of Dickson's Conjecture, so we arrive at a tautology.

In the following $\mathcal{H} = \mathcal{H}_k$ will always be an admissible k-tuple, but to simplify notation we often write simply $\mathcal{H}$ instead of $\mathcal{H}_k$.

(iii) An essentially equivalent formulation of the above is to use the generalized von Mangoldt function

$$a_n = \Lambda_k\big(P_{\mathcal{H}}(n)\big) := \sum_{d \mid P_{\mathcal{H}}(n)} \mu(d) \left(\log \frac{P_{\mathcal{H}}(n)}{d}\right)^k, \ \ P_{\mathcal{H}}(n) = \prod_{i=1}^k (n + h_i) \tag{2.10}$$

which vanishes if $P_{\mathcal{H}}(n)$ has more than k distinct prime factors. However, in this case a direct evaluation of S seems to be hopeless, since d can be as large as N^k.

(iv) It was an idea of Selberg to approximate (2.10) with the divisors cut at $R = N^c$ and accordingly use

$$\sum_{\substack{d \mid P_{\mathcal{H}}(n) \\ d \leqslant R}} \mu(d) \log^k \frac{R}{d}. \tag{2.11}$$

However, this might be negative.

(v) So the next idea is the weight used in the so-called k-dimensional Selberg sieve, i.e., simply the square of (2.11), namely,

$$a_{n,k} = \Bigg(\sum_{d \leqslant R, d \mid P_{\mathcal{H}}(n)} \mu(d) \log^k \frac{R}{d}\Bigg)^2. \tag{2.12}$$

In this case choosing $R \leqslant N^{1/2} L^{-A}$, $L = \log N$, $A > A_0(k)$, S can be readily evaluated. Assuming EH(ϑ), the unconditional case being EH$(1/2)$ (the Bombieri–Vinogradov Theorem), the more difficult sum S_j can also be evaluated, but only under the stronger constraint

$$R \leqslant N^{(\vartheta - \varepsilon)/2}. \tag{2.13}$$

This yields for the crucial quantity $\alpha(\mathcal{H}_k)$ in (2.7)

$$\alpha(\mathcal{H}_k) = \vartheta - \varepsilon + O\left(\frac{1}{k}\right) \tag{2.14}$$

primes on average, which is unfortunately still less than 1 even under the strongest hypothesis $\vartheta = 1$, the original Elliott–Halberstam Conjecture.

(vi) The winning choice is if we are more modest and instead of Dickson's Conjecture approximate the situation when $\prod\limits_{i=1}^{k} (n + h_i)$ has at most $k + \ell$ different prime factors where $\ell \geqslant 0$ is a free parameter. (The choice $\ell = 1$ was used earlier by Heath-Brown

[22], however, not to localize primes in $n+\mathcal{H}$ but to find n values where all components $n+h_i$ are almost primes). This means that we use (2.12) with $k+\ell$ instead of k, i.e. our choice in [14] was

$$a_{n,k+\ell} = \Bigg(\sum_{d \leqslant R, d \mid P_{\mathcal{H}}(n)} \mu(d) \log^{k+\ell} \frac{R}{d} \Bigg)^2. \tag{2.15}$$

This yielded under the condition (2.13) a gain of a factor 2, rather surprisingly. More precisely we got

$$\alpha(\mathcal{H}_k) = 2(\vartheta - \varepsilon) + O\left(\frac{\ell}{k}\right) + O\left(\frac{1}{\ell}\right). \tag{2.16}$$

Under the optimal choice $\ell = \left[\sqrt{k}/2\right]$ this meant

$$\alpha(\mathcal{H}_k) = 2(\vartheta - \varepsilon) + O\left(\frac{1}{\sqrt{k}}\right). \tag{2.17}$$

Consequently if EH(ϑ) is true for some $\vartheta > 1/2$ we obtain $\alpha(\mathcal{H}_k) > 1$ primes on average if $k > C/(\vartheta - 1/2)^2$.

In the unconditional case $\vartheta = 1/2$, this yielded Theorem 2.12 but missed the goal DHL$(k, 2)$ by a hairbreadth.

The way to see how this argument could lead to a proof of the Small Gaps Conjecture begins by observing that on average only $\left(2\varepsilon + \dfrac{c_1}{\sqrt{k}}\right)$ primes were “missing” to obtain more than one prime on average. Using all numbers of the form

$$n + h, \;\; h \in [1, H], \;\; H = \eta \log N \tag{2.18}$$

with an arbitrarily small but fixed $\eta > 0$ instead of only

$$n + h_i, \;\; h_i \in \mathcal{H}_k \tag{2.19}$$

we could pick up more primes so as to fill the missing part.

If in case of $h \in [1, H] \setminus \mathcal{H}_k$ we expect heuristically $n+h$ to be prime with a probability $1/\log N$, we can hope to collect

$$\eta > 2\varepsilon + \frac{c_1}{\sqrt{k}} + O\left(\frac{k}{\log N}\right) \tag{2.20}$$

primes among $n+h$ on average if $n \sim N$, $h \in [1, H] \setminus \mathcal{H}_k$.

The condition (2.20) is clearly satisfied if

$$\varepsilon < \frac{\eta}{3}, \;\; k > C_2 \eta^{-2}, \;\; N > N_0(k, \varepsilon, \eta). \tag{2.21}$$

In the original work [14] we used a result of Gallagher [11] and an averaging procedure over all $\mathcal{H}_k \subset [1, H]$ to show that the above sketched heuristic works in practice. In the next section we use a simpler way, which avoids Gallagher's Theorem and uses a single, suitably chosen k-tuple $\mathcal{H}_k$ for all k.

We will not sketch the rather complicated procedure to show Theorem 2.13. We just mention here that it needs the investigation of k-tuples with

$$k \asymp \frac{(\log N)^{1/2}}{(\log\log N)^2}, \quad \ell \asymp \sqrt{k}. \tag{2.22}$$

In the work [27] it was shown that using a suitable polynomial $P(x)$ instead of the simple $x^{k+\ell}$ in (2.15) $(x = \log(R/d))$ one can improve Theorem 2.13 further to

Theorem 2.15 ([27]). $\liminf\limits_{n\to\infty} \dfrac{d_n}{(\log N)^{3/7}(\log\log N)^{4/7}} < \infty.$

One can raise the more general question of finding the optimal polynomial, or more generally the optimal function $P(x)$. B. J. Conrey calculated the optimal weight function, actually a Bessel-type function. Later in the work [10] an exact analysis confirmed the optimality of the Bessel-type function and the fact that it yielded instead of (2.17) the sharper estimate

$$\alpha(\mathcal{H}_k) = 2(\vartheta - \varepsilon) + O(k^{-2/3}). \tag{2.23}$$

This was, however, the same strength as the polynomial in [27] and [10] apart from the implicit constant in the above O symbol. Therefore the result in Theorem 2.15 can be considered as the limit of the original GPY method.

Concerning Theorem 2.14 the crucial idea is the fact, discovered by the second named author ([26]), and independently by Friedlander and Iwaniec [9] that the weights a_n are strongly concentrated on numbers n where all components $n + h_i$ are almost prime, more precisely for numbers n with

$$P^-\left(\prod_{i=1}^{k}(n + h_i)\right) > N^\delta, \ \ n \sim N, \tag{2.24}$$

where δ is an arbitrarily small fixed positive constant and $P^-(m)$ denotes the smallest prime factor of n. In fact, it was proved in [26] that

$$\sum_{\substack{n\sim N \\ P^-(P_{\mathcal{H}}(n))\leqslant N^\delta}} a_n \leqslant C\delta \sum_{n\sim N} a_n \tag{2.25}$$

with a constant $C = C(k)$. (The factor $C(k)\delta$ was improved to $C'k^3\delta^2$ with an absolute constant C' in [17]).

3. Sketch of the proof of Theorems 2.11 and 2.12

In the following we consider a general sieve situation when the number of residues sieved out mod p satisfies

$$\Omega_{\mathcal{H}}(p) = \Omega(p) = k \ \text{ for } \ p \nmid \Delta(\mathcal{H}) := \prod_{i>j}(h_i - h_j), \ k \text{ fixed} \tag{3.1}$$

and let $\Omega(n)$ be extended multiplicatively for all squarefree values of n. Actually we have $\Omega(p) = \Omega_{\mathcal{H}}(p) = \nu_{\mathcal{H}}(p)$. There are three possibilities:

(i) to work analytically with two complex variables (cf. [14]);

(ii) to work elementarily (cf. [13] using pure sieve methods beyond (2.3));

(iii) to work partially elementarily and partially analytically with one complex variable.

Here we will pursue the third possibility, worked out in an unpublished note of K. Soundararajan [37].

We use a somewhat more general weight function: a polynomial $P(y)$ but note that the argument would work the same for a function $P(y)$ analytic on $[0,1]$, if $P(y)$ has at least a kth order zero at 0.

First we evaluate the sum of the weights a_n, where in the following we will define

$$a_n = \left(\sum_{d \leqslant R} \mu(d) P\left(\frac{\log(R/d)}{\log R} \right) \right)^2, \tag{3.2}$$

$$S = \sum_{n \sim N} a_n \sim N \sideset{}{'}\sum_{d,e \leqslant R} \mu(d)\mu(e) \frac{\Omega([d,e])}{[d,e]} P\left(\frac{\log(R/d)}{\log R} \right) P\left(\frac{\log(R/e)}{\log R} \right) \tag{3.3}$$

(we ignored a negligible error of size $O(R^{2+\varepsilon})$) and $\sum'$ will always denote summation over squarefree variables).

Introducing the notation $(d,e) = u$, $d = um$, $e = un$, $(m,n) = 1$ we obtain

$$S \sim N \sideset{}{'}\sum_{u \leqslant R} \sideset{}{'}\sum_{\substack{m,n \leqslant R/u \\ (m,n)=1 \\ (m,u)=(n,u)=1}} \frac{\mu(m)\mu(n)\Omega(u)\Omega(m)\Omega(n)}{umn} P\left(\frac{\log(R/um)}{\log R} \right) P\left(\frac{\log R/un}{\log R} \right). \tag{3.4}$$

We can rewrite the condition $(m,n) = 1$ using the relation

$$\sum_{\beta | m, \beta | n} \mu(\beta) = \begin{cases} 1 & \text{if } (m,n) = 1, \\ 0 & \text{otherwise} \end{cases} \tag{3.5}$$

as

$$S \sim N \sideset{}{'}\sum_{u \leqslant R} \sideset{}{'}\sum_{\beta \leqslant R/u} \mu(\beta) \frac{\Omega(u)\Omega^2(\beta)}{u\beta^2} \left(\sideset{}{'}\sum_{\substack{m' \leqslant R/u\beta \\ (m',u)=1}} \frac{\mu(\beta m')\Omega(m')}{m'} P\left(\frac{\log(R/u\beta m')}{\log R} \right) \right)^2. \tag{3.6}$$

Grouping terms with the same value of $u\beta =: \gamma$ with notation $m = m'$ we have

$$S \sim N \sideset{}{'}\sum_{\gamma \leqslant R} \frac{\Omega(\gamma)}{\gamma} \left(\sideset{}{'}\sum_{\beta | \gamma} \frac{\mu(\beta)\Omega(\beta)}{\beta} \right) \left(\sideset{}{'}\sum_{\substack{m \leqslant R/\gamma \\ (m,\gamma)=1}} \frac{\mu(m)\Omega(m)}{m} P\left(\frac{\log(R/\gamma m)}{\log R} \right) \right)^2. \tag{3.7}$$

Let us denote the inner sum by $J\left(\gamma, \frac{R}{\gamma}\right)$ where the first variable refers to the condition $(m,\gamma)=1$, the second to $m \leqslant R/\gamma$. Further let for a squarefree γ

$$G(s+1,\gamma) := \sideset{}{'}\sum_{\substack{m\\(m,\gamma)=1}} \frac{\mu(m)\Omega(m)}{m^{s+1}} =: \zeta(s+1)^{-k}F(s+1,\gamma). \tag{3.8}$$

Here we have for Re $s>0$

$$F(s+1,\gamma) = \prod_p \left(1-\frac{\Omega(p)}{p^{s+1}}\right)\left(1-\frac{1}{p^{s+1}}\right)^{-k} \prod_{p\mid\gamma}\left(1-\frac{\Omega(p)}{p^{s+1}}\right)^{-1}. \tag{3.9}$$

Using the Taylor expansion

$$P(x) = \sum_{j=k}^{\infty} \frac{P^{(j)}(0)x^j}{j!} \tag{3.10}$$

and Perron's formula ($c>0$, arbitrary)

$$\frac{1}{2\pi i}\int_{(c)} \frac{x^s}{s^{j+1}}ds = \begin{cases} \frac{(\log x)^j}{j!} & \text{if } x \geqslant 1,\\ 0 & \text{if } 0 \leqslant x \leqslant 1 \end{cases} \qquad (j \in \mathbb{Z}^+) \tag{3.11}$$

we can rewrite $J\left(\gamma, \frac{R}{\gamma}\right)$ as

$$\begin{aligned} J\left(\gamma, \frac{R}{\gamma}\right) &= \sum_{j=k}^{\infty} \frac{P^{(j)}(0)}{(\log R)^j} \sideset{}{'}\sum_{\substack{m\leqslant R/\gamma\\(m,\gamma)=1}} \frac{\mu(m)\Omega(m)}{m}\frac{1}{j!}\left(\log\frac{R/\gamma}{m}\right)^j \\ &= \sum_{j=k}^{\infty} \frac{P^{(j)}(0)}{(\log R)^j}\cdot\frac{1}{2\pi i}\int_{(c)} \sum_{\substack{m=1\\(m,\gamma)=1}}^{\infty} \frac{\mu(m)\Omega(m)}{m^{s+1}}\left(\frac{R}{\gamma}\right)^s \frac{ds}{s^{j+1}} \\ &= \sum_{j=k}^{\infty} \frac{P^{(j)}(0)}{(\log R)^j}\cdot\frac{1}{2\pi i}\int_{(c)} F(s+1,\gamma)\zeta(s+1)^{-k}\left(\frac{R}{\gamma}\right)^s \frac{ds}{s^{j+1}}. \end{aligned} \tag{3.12}$$

Since $F(s+1,\gamma)$ is regular for $\sigma > -\frac{1}{2}$ we can transform the line inside the zero-free region of $\zeta(s+1)$, that is, to $\sigma > 1 - c/(\log(|t|+2))$, $|t| \leqslant \exp\left(\sqrt{\log R}\right)$. The integral is negligible on the new contour and so we obtain by the residue at $s=0$

$$\begin{aligned} J\left(\gamma, \frac{R}{\gamma}\right) &\sim \sum_{j=k}^{\infty} \frac{P^{(j)}(0)}{(\log R)^j} F(1,\gamma)\frac{(\log R/\gamma)^{j-k}}{(j-k)!} \\ &= \frac{F(1,\gamma)}{(\log R)^k}\sum_{\nu=0}^{\infty}\frac{P^{(\nu+k)}(0)}{\nu!}\left(\frac{\log(R/\gamma)}{\log R}\right)^{\nu} \\ &= \frac{F(1,\gamma)}{(\log R)^k}P^{(k)}\left(\frac{\log R/\gamma}{\log R}\right). \end{aligned} \tag{3.13}$$

We remark that although this argument does not work if R/γ is not large enough, that part can be shown to be negligible directly from (3.7). So we obtain

$$\begin{aligned} S &\sim \frac{N}{(\log R)^{2k}} {\sum_{\gamma \leqslant R}}' \frac{\Omega(\gamma)}{\gamma} \prod_{p \mid \gamma} \left(1 - \frac{\Omega(p)}{p}\right) \cdot F(1,\gamma)^2 \left(P^{(k)}\left(\frac{\log R/\gamma}{\log R}\right)\right)^2 \\ &\sim \frac{N}{(\log R)^{2k}} \mathfrak{S}^2(\mathcal{H}) {\sum_{\gamma \leqslant R}}' \frac{\Omega(\gamma)}{\gamma} \prod_{p \mid \gamma} \left(1 - \frac{\Omega(p)}{p}\right)^{-1} \left(P^{(k)}\left(\frac{\log R/\gamma}{\log R}\right)\right)^2 . \end{aligned} \tag{3.14}$$

Since apart from finitely many primes, for which

$$p \mid \Delta(\mathcal{H}) := \prod_{i>j} (h_i - h_j) \tag{3.15}$$

we have $\Omega(p) = k$, the behaviour of $\Omega(n)$ is similar to that of the generalized divisor function

$$\tau_k(n) = \sum_{n_1 n_2 \ldots n_k = n} 1. \tag{3.16}$$

This implies (for the details see Lemma 11 of [13])

$${\sum_{\gamma \leqslant x}}' \frac{\Omega(\gamma)}{\gamma} \prod_{p \mid \gamma} \left(1 - \frac{\Omega(p)}{p}\right)^{-1} \sim \mathfrak{S}(\mathcal{H})^{-1} \frac{(\log x)^k}{k!}. \tag{3.17}$$

The sum in (3.14) can be evaluated from (3.17) by partial summation, and we obtain

$$S \sim \frac{\mathfrak{S}(\mathcal{H}) N}{(\log R)^k (k-1)!} \int_0^1 y^{k-1} \left(P^{(k)}(1-y)\right)^2 dy. \tag{3.18}$$

Let us consider now the quantity

$$S_j = {\sum_{n \sim N}}' a_n \chi_{\mathcal{P}}(n + h_j) \log n, \quad h_j \in \mathcal{H}. \tag{3.19}$$

In this case (if $R < N$) the two conditions

$$n + h_j \in \mathcal{P}, \quad d \mid \prod_{i=1}^{k} (n + h_i), \quad d \leqslant R \tag{3.20}$$

and

$$n + h_j \in \mathcal{P}, \quad d \mid \prod_{\substack{i=1 \\ i \neq j}}^{k} (n + h_i), \quad d \leqslant R \tag{3.21}$$

are equivalent. So the situation is similar to (3.3) if

$$R \leqslant N^{(\vartheta - \varepsilon)/2} \tag{3.22}$$

since it is easy to see that by the condition (2.3) (which is unconditionally true with $\vartheta = 1/2$ by the Bombieri–Vinogradov Theorem) we can substitute $\chi_{\mathcal{P}}(n + h_j) \log n$ by 1. Thus we have

$$S_j \sim \sum_{d,e \leqslant R}{}' \mu(d)\mu(e) \frac{\Omega_j([d,e])}{[d,e]} P\left(\frac{\log(R/d)}{\log R}\right) P\left(\frac{\log(R/e)}{\log R}\right) \tag{3.23}$$

with the only difference that we have now $\Omega_j(p) = \Omega(p) - 1 = k - 1$ if $p \nmid \Delta$. The singular series $\mathfrak{S}_j(\mathcal{H})$ is accordingly

$$\begin{aligned} \mathfrak{S}_j(\mathcal{H}) &= \prod_p \left(1 - \frac{\nu_{\mathcal{H}}(p) - 1}{p - 1}\right)\left(1 - \frac{1}{p}\right)^{-(k-1)} \\ &= \prod_p \left(1 - \frac{\nu_p(\mathcal{H})}{p}\right)\left(1 - \frac{1}{p}\right)^{-k} = \mathfrak{S}(\mathcal{H}). \end{aligned} \tag{3.24}$$

So we obtain for all $j \in [1, k]$ under the stronger condition (3.22) now analogously to (3.18)

$$S_j \sim \frac{\mathfrak{S}(\mathcal{H})N}{(\log R)^{k-1}(k-2)!} \int_0^1 y^{k-2} \left(P^{(k-1)}(1-y)\right)^2 dy \tag{3.25}$$

and this gives in total for $R = N^{(\vartheta - \varepsilon)/2}$, $P^{(k-1)}(x) = Q(x)$

$$\frac{\sum_{j=1}^{k} S_j}{S \log 3N} \sim \frac{\log R}{\log N} k(k-1)M(Q) \sim \frac{k(k-1)(\vartheta - \varepsilon)}{2} M(Q) \tag{3.26}$$

primes on average in $\{n + h_i\}_{i=1}^k$ if n runs between N and $2N$ and the numbers n are weighted by $a_n \log n$, where

$$M(Q) = \frac{\int_0^1 y^{k-2}\big(Q(1-y)\big)^2 dy}{\int_0^1 y^{k-1}\big(Q'(1-y)\big)^2 dy}. \tag{3.27}$$

In case of the simple choice

$$P(x) = x^{k+\ell}, \ \ \ell = \left[\sqrt{k}/2\right] \Leftrightarrow Q(x) = C(k,\ell)x^{\ell+1} \tag{3.28}$$

we obtain

$$\begin{aligned} M(Q) &= \frac{\int_0^1 y^{k-2}(1-y)^{2\ell+2} dy}{(\ell+1)^2 \int_0^1 y^{k-1}(1-y)^{2\ell} dy} = \frac{(k-2)!(2\ell+2)!/(k+2\ell+1)!}{(\ell+1)^2(k-1)!(2\ell)!/(k+2\ell)!} \\ &= \frac{4\left(1 - \frac{1}{2(\ell+1)}\right)}{(k+2\ell+1)(k-1)} \sim \frac{4\left(1 - O\left(\frac{1}{\sqrt{k}}\right)\right)}{k^2}. \end{aligned} \tag{3.29}$$

By (3.26) this yields on the weighted average

$$2(\vartheta-\varepsilon)\left(1-O\left(\frac{1}{\sqrt{k}}\right)\right) \tag{3.30}$$

primes in $\{n+\mathcal{H}\}$ if $n\sim N$.

The quantity above is clearly greater than 1 if

$$\vartheta>1/2, \quad k>k_0(\vartheta), \tag{3.31}$$

which proves Theorem 2.11.

Suppose now $h_0\notin\mathcal{H}$, let $\mathcal{H}_0=\mathcal{H}\cup\{h_0\}$, and $\Omega_0(p)=\Omega_{\mathcal{H}_0}(p)$ is defined as in (3.1) with $k+1$ in place of k,

$$S_0=\sum_{n\sim N}a_n\chi_{\mathcal{P}}(n+h_0)\log n. \tag{3.32}$$

In case of $\nu_{\mathcal{H}_0}(p)=\nu_{\mathcal{H}}(p)$ we have $\Omega_0(p)=\nu_{\mathcal{H}}(p)-1$ residue classes in the sieve mod p (Ω_0 is defined as in (3.1)); if $\nu_{\mathcal{H}_0}(p)=\nu_{\mathcal{H}}(p)+1$, then $\Omega_0(p)=\nu_{\mathcal{H}}(p)$. So we have in both cases $\Omega_0(p)=\nu_{\mathcal{H}_0}(p)-1$ and $\Omega_0(p)=k$ if $p\nmid\Delta(\mathcal{H}_0)$.

This yields an analogous asymptotic to (3.18) for S_0, with $\mathcal{H}$ replaced by $\mathcal{H}_0$:

$$S_0\sim\frac{\mathfrak{S}(\mathcal{H}_0)N}{(\log R)^k(k-1)!}\int_0^1 y^{k-1}\left(P^{(k)}(1-y)\right)^2dy \tag{3.33}$$

and consequently

$$\frac{S_0}{S}\sim\frac{\mathfrak{S}\left(\mathcal{H}\cup\{h_0\}\right)}{\mathfrak{S}(\mathcal{H})}\quad(\text{as } N\to\infty). \tag{3.34}$$

This relation helps us to obtain Theorem 2.12 unconditionally. Let us consider an interval of length

$$H=\eta\log N, \tag{3.35}$$

where η is an arbitrarily small fixed positive constant. Let us suppose that we can find for any k an admissible k-tuple $\mathcal{H}=\mathcal{H}_k$ such that with a fixed absolute constant $c_0>0$

$$\mathfrak{S}\left(\mathcal{H}_k\cup h_0\right)>c_0\mathfrak{S}(\mathcal{H})\ \text{ for any even } h_0. \tag{3.36}$$

In this case using only $\vartheta=1/2$, that is, the Bombieri–Vinogradov Theorem, we obtain on average

$$\frac{\sum_{h=1}^{H}\sum_{n\sim N}a_n\chi_{\mathcal{P}}(n+h)\log n}{\sum_{n\sim N}a_n\log 3N}\geqslant(1-2\varepsilon)\left(1-O\left(\frac{1}{\sqrt{k}}\right)\right)+\frac{c_0\eta}{2}-\frac{k}{\log 3N}>1 \tag{3.37}$$

primes between n and $n+H$ if

$$k>k_0(\eta),\ \ \varepsilon<\varepsilon_0(\eta),\ \ N>N_0(\eta,k,\varepsilon). \tag{3.38}$$

In order to show the existence of $\mathcal{H}_k$ with (3.36) we can just choose

$$\mathcal{H} = \mathcal{H}_k = \Big\{ i \prod_{p \leqslant 2k} p \Big\}_{i=1}^{k}. \tag{3.39}$$

Then we have for any even h with $\nu_p = \nu_{\mathcal{H}}(p)$

$$\begin{aligned} \frac{\mathfrak{S}(\mathcal{H} \cup h)}{\mathfrak{S}(\mathcal{H})} &\geqslant 2 \prod_{2<p\leqslant 2k} \frac{1-2/p}{(1-1/p)^2} \prod_{p>2k} \frac{1-(\nu_p+1)/p}{1-(\nu_p+1)/p+\nu_p/p^2} \\ &\geqslant c_1 \prod_{p>2k} \left(1 + O\left(\frac{k}{p^2}\right)\right) \geqslant c_0. \end{aligned} \tag{3.40}$$

In such a way we obtain Theorem 2.12. We remark that the above proof avoids Gallagher's Theorem [11]. Another proof, also avoiding Gallagher's Theorem is given in [16] which yields some other results, like small gaps between consecutive primes in arithmetic progressions and improved upper estimates for the quantity

$$\Delta_r = \liminf_{n\to\infty} \frac{p_{n+r} - p_n}{\log p_n}. \tag{3.41}$$

4. Sketch of the proof of Theorem 2.14

The most crucial idea in the proof of Theorem 2.14 is that we will change the weights and instead of the original normalized weights (cf. (2.15)).

$$a_n = \left(\sum_{d \le R, d \mid P_{\mathcal{H}}(n)} \mu(d) \left(\frac{\log(R/d)}{\log R} \right)^{k+\ell} \right)^2, \ P_{\mathcal{H}}(n) = \prod_{i=1}^{k} (n+h_i), \ \ell = \left[\frac{\sqrt{k}}{2} \right] \tag{4.1}$$

we will work with the new weight ($n \sim N$)

$$a'_n = \begin{cases} a_n & \text{if } P^-\big(P_{\mathcal{H}}(n)\big) > N^{\delta}, \\ 0 & \text{otherwise,} \end{cases} \tag{4.2}$$

where δ will be a fixed small positive constant with $\varepsilon < \varepsilon_0(\eta)$, $k > k_0(\eta, \varepsilon)$, $\delta < \delta_0(k, \eta, \varepsilon)$, $R = N^{(\vartheta-\varepsilon)/2}$ and we consider primes in intervals of length

$$H = \eta \log N \tag{4.3}$$

as indicated in (2.2).

As mentioned at the end of Section 2 the sum of weights $a^*_{n,\mathcal{H}}$ with $\mathcal{P}_{\mathcal{H}}(n)$ having at least one small prime divisor not exceeding N^{δ} is negligible and we have (2.25) with a constant $C = C(k)$, i.e.

$$\begin{gathered} 0 \le \sum_{n\sim N} (a_n - a'_n) = \sum_{\substack{n\sim N \\ P^-(P_{\mathcal{H}}(n)) \le N^{\delta}}} a_n \le C\delta \sum_{n\sim N} a_n, \\ \sum_{\substack{n\sim N \\ P^-(P_{\mathcal{H}}(n)) \le N^{\delta}}} a_n \chi_{\mathcal{P}}(n+h) \log(n+h) \le C\delta \sum_{n\sim N} a_n \chi_{\mathcal{P}}(n+h) \log(n+h). \end{gathered} \tag{4.4}$$

These are Lemmas 4 and 5 of [26].

The other tool is Gallagher's Theorem [11], according to which for k fixed, $H \to \infty$

$$\sum_{\substack{\mathcal{H}\subset[1,H]\\|\mathcal{H}|=k}} \mathfrak{S}(\mathcal{H}) \sim \frac{H^k}{k!}. \tag{4.5}$$

Let further (for a more detailed proof see [17] and [18])

$$\pi(n,H) := \pi(n+H) - \pi(N), \;\; \Theta(n) := \begin{cases} \log n & \text{if } n \in \mathcal{P}, \\ 0 & \text{otherwise,} \end{cases} \tag{4.6}$$

$$\Theta(n,H) := \sum_{h=1}^{H} \Theta(n+h)$$

$$M := \sum_{\substack{p_j\sim N\\ p_{j+1}-p_j\le H}} 1, \;\; Q(N,H) := \sum_{\substack{n\sim N\\ \pi(n,H)>1}} 1 \le HM + O\left(Ne^{-c\sqrt{\log N}}\right), \tag{4.7}$$

and consider now instead of (3.19) the modified quantity

$$S'(h,\mathcal{H}) = \sum_{n\sim N} a'_n \Theta(n+h). \tag{4.8}$$

The substitution of a_n by a'_n will just slightly change the corresponding value of $S'(\mathcal{H})$ and $S'(h,\mathcal{H})$ respectively, to

$$S'(\mathcal{H}) = \sum_{n\sim N} a'_n = (1+O(\delta))S(\mathcal{H}), \tag{4.9}$$

$$S'(h,\mathcal{H}) = \sum_{n\sim N} a'_n \Theta(n+h) = (1+O(\delta))S(h,\mathcal{H}) \tag{4.10}$$

compared with

$$S(h,\mathcal{H}) := \sum_{n\sim N} a_n \Theta(n+h), \tag{4.11}$$

where the asymptotics for the quantity (4.11) are given in (3.25) and (3.33) respectively, and $P(x) = x^{k+\ell}$ in this section.

The crucial change is that in case of $a'_{n,\mathcal{H}} > 0$ all the prime divisors of $\mathcal{P}_{\mathcal{H}}(n)$ are at least N^δ with a fixed small δ, so by (4.1) we have a trivial estimate for it:

$$a'_n \leqslant 2^{\omega(P_{\mathcal{H}}(n))} \leqslant 2^{2k^2/\delta} \ll_{k,\delta} 1. \tag{4.12}$$

On the other hand, in this case we cannot use the simplification of Section 3, that is, to work with a suitably chosen single $\mathcal{H}_k$. Averaging over all $\mathcal{H} \subseteq [1,H]$, $|\mathcal{H}| = k$, with the abbreviations (we take the unconditional case $\vartheta = 1/2$ from now on)

$$\frac{H}{\log R} = \frac{\eta}{\left(\frac{1}{2}-\varepsilon\right)/2} = \eta', \;\; \sum_{\mathcal{H}}^{(k)} = \sum_{\substack{\mathcal{H}\subset[1,H]\\|\mathcal{H}|=k}} \tag{4.13}$$

we obtain from (3.18), using (3.28)–(3.29) and (4.5)

$$\sum_{\mathcal{H}}^{(k)} S'(\mathcal{H}) \sim (1+O(\delta)) \frac{(\eta')^k N C(k,\ell)(2\ell)!}{k!(\ell+1)^2(k+2\ell)!} =: (1+O(\delta))B. \tag{4.14}$$

On the other hand, we have by (3.33) and (4.5)

$$\begin{aligned} &\sum_{\mathcal{H}}^{(k)} \sum_{\substack{n\sim N\\ h\in[1,H]\setminus\mathcal{H}}} a'_n \Theta(n+h) \\ &\sim (k+1)\sum_{\mathcal{H}}^{(k+1)} \mathfrak{S}(\mathcal{H}) \frac{NC(k,\ell)(2\ell)!(1+O(\delta))}{(\log R)^k(\ell+1)^2(k+2\ell)!} := (1+O(\delta))B\eta\log N. \end{aligned} \tag{4.15}$$

Finally, we have by (3.25) and (4.5) with $\ell = \left[\sqrt{k}/2\right]$, $\vartheta = 1/2$

$$\begin{aligned} &\sum_{\mathcal{H}}^{(k)} \sum_{h\in\mathcal{H}} \sum_{n\sim N} a'_n \Theta(n+h) \\ &\sim (1+O(\delta)) \frac{k\eta'^k NC(k,\ell)(2\ell+2)!}{k!(k+2\ell+1)!} \log R \\ &\sim (1+O(\delta))B\left(1-\frac{1}{2(\ell+1)}\right)\left(1-\frac{2\ell+1}{k+2\ell+1}\right)(1-2\varepsilon)\log N. \end{aligned} \tag{4.16}$$

Adding (4.15), (4.16) and subtracting from it (4.14) multiplied by $\log 3N$ we obtain

$$\begin{aligned} &\sum_{\mathcal{H}}^{(k)} \sum_{n\sim N} a'_n\big(\Theta(n,H) - \log 3N\big) \\ &> B\log N\left\{(1-2\varepsilon)\left(1-\frac{C}{\sqrt{k}}\right) + \eta - 1 + O(\delta)\right\} > \frac{\eta}{2} B\log N \end{aligned} \tag{4.17}$$

if, as stated in the introduction of Section 4 (between (4.2) and (4.3)) we fix ε, k, δ with

$$\varepsilon < \varepsilon_0(\eta), \quad k > k_0(\eta,\varepsilon), \quad \delta < \delta_0(k,\eta,\varepsilon). \tag{4.18}$$

Consequently, if (4.18) holds, which we will always assume in the following, then

$$\frac{\eta}{2} B\log N < (1+o(1))\log N \sum_{\substack{n\sim N\\ \pi(n,H)>1}} \pi(n,\mathcal{H}) \sum_{\mathcal{H}}^{(k)} a'_n. \tag{4.19}$$

Introducing the notation

$$T(n,H) := \sum_{\substack{\mathcal{H}\\ P^-\left(P_{\mathcal{H}}(n)\right)>N^\delta}}^{(k)} 1 \tag{4.20}$$

we have by (4.6)–(4.7), (4.12) and Cauchy's inequality

$$\eta B \ll \Bigg(\sum_{\substack{n\sim N\\ \pi(n,\mathcal{H})>1}} 1\Bigg)^{1/2} \Bigg(\sum_{n\sim N} \pi^2(n,H)T(n,H)^2\Bigg)^{1/2} \tag{4.21}$$

$$\ll \left((HM)^{1/2} + O\left(N^{1/2} e^{-c\sqrt{\log N}/2} \right) \right) \left(\sum_{n \sim N} \pi^2(n,H) T(n,H)^2 \right)^{1/2}.$$

Further, we have by Selberg's sieve (Theorem 5.1 of [21] or Theorem 2 in § 2.2.2 of [19]) for any set $\mathcal{H}$ and $\delta < 1/2$

$$\sum_{\substack{n \sim N \\ P^-\left(P_{\mathcal{H}}(n)\right) > R^\delta}} 1 \leqslant \frac{|\mathcal{H}|! \mathfrak{S}(\mathcal{H})}{(\log R^\delta)^{|\mathcal{H}|}} N(1 + o(1)) \quad (R, N \to \infty). \tag{4.22}$$

This implies by Gallagher's Theorem (4.5)

$$\sum_{n \sim N} \pi(n,H)^2 T(n,H)^2 \ll \sum_{1 \le h, h' \le H} \sideset{}{^{(k)}}\sum_{\mathcal{H}_1} \sideset{}{^{(k)}}\sum_{\mathcal{H}_2} \sum_{\substack{n \sim N \\ P^-\left(\mathcal{H}_1 \cup \mathcal{H}_2 \cup \{h\} \cup \{h'\}\right) > N^\delta}} 1 \tag{4.23}$$

$$\ll_k N \sum_{r=k}^{2k+2} \sideset{}{^{(r)}}\sum_{\mathcal{H}_0} \frac{\mathfrak{S}(\mathcal{H}_0)}{(\log R^\delta)^r} \ll_{k,\delta} N \sum_{r=k}^{2k+2} \left(\frac{H}{\log R} \right)^r \ll_{k,\delta} (\eta')^k N.$$

Taking into account the definition of B in (4.14) we obtain from (4.21) and (4.23)

$$\eta (\eta')^{k/2} \ll_{k,\delta} \left(\left(\frac{HM}{N} \right)^{1/2} + e^{-c\sqrt{\log N}/2} \right). \tag{4.24}$$

Consequently,

$$\frac{HM}{N} \gg_{k,\delta,\eta} 1. \tag{4.25}$$

Hence,

$$M \gg_{k,\delta,\eta} \frac{N}{\log N} \gg_{k,\delta,\eta} \pi(2N), \tag{4.26}$$

which proves Theorem 2.14.

It may be shown (see Theorem 2 of [18]) that this is sharp in the sense that the assertion does not remain true if $H = o(\log N)$. The proof uses the Selberg sieve upper bound for prime tuples and Gallagher's result (4.5).

5. Bounded gaps between primes. Zhang's theorem

We recall that in our original work (Theorem 2.11 in Section 2) we showed that $\mathrm{EH}(\vartheta)$ for any $\vartheta > 1/2$ implies $\mathrm{DHL}(k,2)$ for $k > k_0(\vartheta)$, consequently the Bounded Gaps Conjecture. From the proof it is trivial that the condition

$$\max_{a,(a,q)=1} \tag{5.1}$$

in (2.3) can be weakened to

$$\max_{a,(a,q)=1, P_{\mathcal{H}}(a) \equiv 0(q)} \tag{5.2}$$

if we want to show for a specific $\mathcal{H}$ that $n + \mathcal{H}$ contains at least two primes infinitely often. However, in 2008 in a joint work of Y. Motohashi and J. Pintz the following stronger form of Theorem 2.11 was proved, in which the summation in (2.3) can be reduced to smooth moduli. $P^+(n)$ will denote the largest prime factor of n.

Theorem 5.1 ([25]). *If there exist $\delta > 0$, $\vartheta > 1/2$ and an admissible k-tuple $\mathcal{H}$ with $k > k_0(\delta, \vartheta)$ such that for any $\varepsilon > 0$, $A > 0$*

$$\sum_{\substack{q \le N^{\vartheta - \varepsilon} \\ P^+(q) \le N^\delta}} \max_{\substack{a \\ (a,q)=1, q \mid P_{\mathcal{H}}(a)}} \left| \sum_{\substack{p \equiv a(q) \\ p \sim N}} \log p - \frac{N}{\varphi(q)} \right| \le \frac{C(A, \varepsilon) N}{\log^A N} \tag{5.3}$$

holds for $N > N_0(\mathcal{H}, \vartheta, \delta)$, then $n + \mathcal{H}$ contains at least two primes for some $n \sim N$.

Remark 5.2. Zhang proved a version of this result, and it appeared with a different proof in his work [39]. Zhang proved condition (5.3) with the explicit values

$$\vartheta = \frac{1}{2} + \frac{1}{584}, \quad \delta = \frac{1}{1168}, \tag{5.4}$$

which finally led to

Theorem 5.3 ([39]). DHL$(k, 2)$ *is true for $k \geqslant 3.5 \cdot 10^6$ and consequently* $\liminf_{n \to \infty} (p_{n+1} - p_n) \leqslant C = 7 \cdot 10^7$.

His proof of (5.4) uses several deep works of Fouvry, Fouvry–Iwaniec, Bombieri–Friedlander–Iwaniec, Friedlander–Iwaniec, Heath-Brown, which are based on ideas and works of Linnik, Weil, Deligne and Birch–Bombieri concerning the estimate of Kloostermann sums.

The Polymath 8a project of T. Tao [30] introduced many improvements into this procedure (for example to apply instead of the simple weight function $P(x) = x^{k+\ell}$ the optimal Bessel function first used by Conrey, later analyzed in details in [10] together with many improvements in both the Motohashi–Pintz Theorem and in the estimation of Kloostermann sums) and obtained distribution estimates up to level $1/2 + 7/300$, and thus reached

Theorem 5.4 (Polymath 8a). DHL$(k, 2)$ *is true for $k \geqslant 632$ and consequently*

$$\liminf_{n \to \infty} (p_{n+1} - p_n) \leqslant 4680.$$

6. Bounded gaps between primes: The Maynard–Tao theorem

About half a year after the manuscript of Zhang [39], simultaneously and independently, J. Maynard [24] and in his Polymath blogs T. Tao [31] introduced another idea which led to a new, more efficient proof of the Bounded Gaps Conjecture. The main results of Maynard [24] were the following.

Theorem 6.1 (Maynard [24]). DHL$(k, 2)$ *is true for $k \ge 105$, consequently*

$$\liminf_{n \to \infty} (p_{n+1} - p_n) \leqslant 600.$$

Theorem 6.2 (Maynard [24]). *Assuming the Elliott–Halberstam Conjecture,* $\mathrm{DHL}(k,2)$ *is true for* $k \geq 5$*, consequently*

$$\liminf_{n\to\infty}(p_{n+1}-p_n) \leqslant 12.$$

The two surprising aspects of the Maynard–Tao method were that it produced not only pairs but arbitrarily long (finite) blocks of primes in bounded intervals, and for this knowing that (2.3) holds with any fixed $\vartheta > 0$ (however small) would suffice.

The earlier known strongest result of somewhat similar nature was the much weaker one in our work [16]. It asserted for any $r > 0$

$$\Delta_r := \liminf_{n\to\infty} \frac{p_{n+r}-p_n}{\log p_n} \leq e^{-\gamma}\left(\sqrt{r}-1\right)^2. \tag{6.1}$$

Further, under the very deep Elliott–Halberstam Conjecture (see (2.3) with $\vartheta = 1$) we could show [14]

$$\Delta_2 = 0. \tag{6.2}$$

Theorem 6.3 (Maynard–Tao [24]). *We have for any* r

$$\liminf_{n\to\infty}(p_{n+r}-p_n) \ll r^3 e^{4r}. \tag{6.3}$$

The main idea of Maynard and Tao is that the weights are defined instead of

$$a_n = \left(\sum_{\substack{d\leqslant R\\ d\mid P_{\mathcal{H}}(n)}} \mu(d) P\left(\frac{\log R/d}{\log R}\right)\right)^2, \quad \mathcal{P}_{\mathcal{H}}(n) = \prod_{i=1}^{k}(n+h_i) \tag{6.4}$$

in the more general form

$$a_n = \left(\sum_{\substack{d_1\ldots d_k\leqslant R\\ d_i\mid n+h_i}} \mu(d) P\left(\frac{\log d_1}{\log R},\ldots,\frac{\log d_k}{\log R}\right)\right)^2, \tag{6.5}$$

where $P(t_1,\ldots,t_k) := \mathbb{R}^k \to \mathbb{R}$ is a fixed piecewise differentiable function with support on $t_1+t_2+\cdots+t_k \leqslant 1$. The idea of the use of these more general weights goes back to Selberg ([36], p. 245). Similar type of weights were used by Goldston and Yıldırım [12], but due to the special choice of $P(t_1,\ldots,t_k) = \prod\limits_{i=1}^{k}(1-kt_i),\ t_i \leqslant 1/k$, this led only to the result

$$\Delta = \liminf_{n\to\infty} \frac{p_{n+1}-p_n}{\log p_n} \leqslant \frac{1}{4}. \tag{6.6}$$

We remark here that the general choice of $P\left(\frac{\log R/d}{\log R}\right)$ in Section 3 corresponds to the special case of the above with

$$P(t_1,t_2,\ldots,,t_k) = \widetilde{P}(t_1+t_2+\cdots+t_k). \tag{6.7}$$

Another very interesting remark is that in order to show bounded intervals with arbitrarily long finite blocks of primes (with a bound $e^{2r/\vartheta}$ in place of e^{4r}) we do not need the value

$\vartheta = 1/2$, that is, the Bombieri–Vinogradov Theorem, just any value $\vartheta > 0$. So we obtain a numerically slightly weaker form of the existence of arbitrarily long (finite) blocks of primes in bounded invervals even by the use of the first theorem establishing a positive admissible level ϑ for the distribution of primes, due to A. Rényi [33, 34] reached in 1947–48, by the large sieve of Linnik.

Upon further work on the Maynard–Tao method in the Polymath 8b project of Tao, Theorem 6.1 has been improved to

Theorem 6.4 (Polymath 8b project). DHL$(k, 2)$ *is true for* $k \geqslant 50$*, consequently*

$$\liminf_{n\to\infty}(p_{n+1} - p_n) \leq 246.$$

7. De Polignac numbers and some conjectures of Erdős on gaps between consecutive primes

There are various 60–70 years old conjectures of Erdős on which a sharpened version of Zhang's Theorem (or that of Maynard and Tao) combined with other arguments of the second named author can give an answer. Below we give a list of them without proofs which can be found in [28]. The numerical values reflect the stage at the end of Polymath 8A.

Using an argument of the second named author (Lemma 4 in [26]) together with a more general form of the arguments of Theorem 3 of Zhang and its improvement by Tao's project, the following strengthening of Theorem 3 of Zhang can be shown. (Let $P^-(n)$ be the smallest prime factor of n.)

Theorem 7.1 ([28]). *Let* $k \geq 632$*,* $\mathcal{H}$ *an admissible* k*-tuple,* $h_i \ll \log N$*,* $N > N_0(k)$*. Then there are at least*

$$c_1(k, \mathcal{H}) \frac{N}{\log^k N}$$

numbers $n \in [N, 2N)$ *such that* $n + \mathcal{H}$ *contains at least two primes and almost primes in all other components satisfying* $P^-(n + h_i) > N^{c_2(k)}$ *for* $i = 1, 2, \ldots, k$*.*

Remark 7.2. A similar version to the above-mentioned crucial Lemma 4 of [26] appears in the book Opera de Cribro of Friedlander–Iwaniec [9] published also in 2010.

Whereas the original Theorem 3 of Zhang yields only one de Polignac number, by the aid of Theorem 7.1 we can show

Theorem 7.3 ([28]). *There are infinitely many de Polignac numbers. In fact, they have a positive lower density* $> 10^{-7}$.

Theorem 7.4 ([28]). *There exists an ineffective* C *such that we have always at least one de Polignac number between* X *and* $X + C$ *for any* X*. (All gaps between consecutive de Polignac numbers are uniformly bounded.)*

Erdős [6] proved in 1948 the inequality

$$\liminf_{n\to\infty} \frac{d_{n+1}}{d_n} \leq 1 - c_0 < 1 + c_0 \leq \limsup \frac{d_{n+1}}{d_n} \tag{7.1}$$

with a very small positive value c_0 and conjectured that the $\liminf = 0$ and the $\limsup = \infty$.

Theorem 7.5 ([28]). $\liminf\limits_{n\to\infty} \dfrac{d_{n+1}}{d_n} = 0$, $\limsup \dfrac{d_{n+1}}{d_n} = \infty$.

Further, we have even

$$\liminf_{n\to\infty} \frac{d_{n+1}\log n}{d_n} < \infty, \qquad \limsup_{n\to\infty} \frac{d_{n+1}}{d_n \log n} > 0. \tag{7.2}$$

In general it is difficult to show anything for three consecutive differences. However, we can show

Theorem 7.6 ([28]). $\limsup\limits_{n\to\infty} \dfrac{\min(d_{n-1}, d_{n+1})}{d_n (\log n)^c} = \infty$ *with* $c = 1/632$.

Since the Prime Number Theorem implies

$$\frac{1}{N}\sum_{n=1}^{N} \frac{d_n}{\log n} = 1, \tag{7.3}$$

it is interesting to investigate the normalized distribution of the sequence d_n, $d_n/\log n$. Erdős conjectured 60 years ago that the set of limit points,

$$J = \left\{ \frac{d_n}{\log n} \right\}' = [0, \infty], \tag{7.4}$$

but no finite limit point was known until 2005, when we showed $0 \in J$. (We denote by G' the set of limit points of the set G.) This was rather strange since in 1955 Erdős [7] and simultaneously Ricci [35] proved that J has positive Lebesgue measure. A partial answer to the conjecture of Erdős is

Theorem 7.7 ([28]). *There is an (ineffective) constant* c^* *such that*

$$[0, c^*] \subset J. \tag{7.5}$$

The above result raises the question whether considering a finer distribution $d_n/f(n)$ with a monotonically increasing function $f(n) \le \log n$, $f(n) \to \infty$ the same phenomenon is still true. The answer is yes.

Theorem 7.8 ([28]). *Let* $f(n) \le \log n$, $f(n) \to \infty$ *be an increasing function,*

$$J_f = \left\{ \frac{d_n}{f(n)} \right\}'. \tag{7.6}$$

Then there is an (ineffective) constant c_f^* *such that*

$$[0, c_f^*] \subset J_f. \tag{7.7}$$

Zhang's theorem shows the existence of infinitely many generalized twin prime pairs with a difference at most $7 \cdot 10^7$, while the theorem of Green and Tao shows the existence of arbitrarily long (finite) arithmetic progressions in the sequence of primes. A common generalization of these two results is given below. (Let p' denote the prime following p.)

Theorem 7.9 ([28]). *There exists an even* $d \le 4680$ *with the following property. For any* k *there is a* k*-term arithmetic progression of primes such that* $p' = p + d$ *for all elements of the progression.*

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