Journal of Number Theory 131 (2011) 1939–1961

Contents lists available at ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt

On some averages at the zeros of the derivatives of the Riemann zeta-function

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ARTICLE INFO

Article history:
Received 22 December 2010
Accepted 12 April 2011
Available online 14 June 2011
Communicated by J. Brian Conrey

Keywords:
Riemann zeta-function
Derivatives
Averages

ABSTRACT

In this article we study two problems raised by a work of Conrey and Ghosh from 1989. Let $\zeta^{(k)}(s)$ be the k-th derivative of the Riemann zeta-function, and $\chi(s)$ be factor in the functional equation of the Riemann zeta-function. We calculate the average values of $\zeta^{(j)}$ and χ at the nontrivial zeros of $\zeta^{(k)}$.

1. Introduction

Let $s = \sigma + it \in \mathbb{C}$ with $\sigma, t \in \mathbb{R}$. It is known since Riemann and von Mangoldt that the number of zeros of the Riemann zeta-function $\zeta(s)$ in $0 < t < T$ is $\frac{T}{2\pi} \log \frac{T}{2\pi e} + O(\log T)$, and Berndt [1] proved that, for $k \geqslant 1$, the number of zeros of the k-th derivative $\zeta^{(k)}(s)$ in $0 < t < T$ is $\frac{T}{2\pi} \log \frac{T}{4\pi e} + O_k(\log T)$. While studying the zeros of $\zeta^{(k)}(s)$, Conrey and Ghosh [2], assuming the Riemann Hypothesis (RH), expounded and then used the result

$$\sum_{0<\gamma_k<T} \chi(\rho_k) \sim \alpha_k \frac{T}{2\pi} \quad (T \to \infty). \tag{1}$$

Here $k \in \mathbb{Z}^+$, ρ_k denotes a non-real zero of the k-th derivative $\zeta^{(k)}(s)$ and $\gamma_k = \Im \rho_k$,

$$\chi(s) = 2(2\pi)^{s-1} \Gamma(1-s) \sin(s\pi/2) \tag{2}$$

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1 Supported by TÜBİTAK.

0022-314X/$ – see front matter
doi:10.1016/j.jnt.2011.04.007

is the factor in the functional equation

$$\zeta(s) = \chi(s)\zeta(1-s), \tag{3}$$

and

$$\alpha_k := k + 1 - \sum_{r=1}^{k} e^{-z_r} \tag{4}$$

with z_r $(r = 1, \ldots, k)$ being the zeros of

$$P_k(z) := \sum_{j=0}^{k} \frac{z^j}{j!}. \tag{5}$$

Conrey and Ghosh employed (1) in showing that for any $\epsilon > 0$ there are $\gg_\epsilon T$ zeros of $\zeta^{(k)}(s)$ in the region $\frac{1}{2} \leqslant \sigma < \frac{1}{2} + \frac{(1+\epsilon)\log\log T}{\log T}$, $0 < t < T$.

One purpose of this study is to prove (1) by a slightly different approach because we have not been able to verify the error term in the unnumbered formula which precedes formula (20) of [2]. We shall not assume RH. However, this will not make the conclusion on the distribution of zeros of $\zeta^{(k)}(s)$ unconditional because one needs to know that $\zeta^{(k)}(s)$ has at most a finite number of non-real zeros in $\sigma < \frac{1}{2}$ to deduce it using (1), and the work of Levinson and Montgomery [6] shows that this rests essentially on RH. One of the conventions we will use in this paper is to take $\mathbb{N} = \mathbb{Z}^+ \cup \{0\}$. We obtain:

Theorem 1. *For fixed $k \in \mathbb{N}$, we have*

$$\sum_{0<\gamma_k<T} \chi(\rho_k) = -\alpha_k \frac{T}{2\pi} + O_k\left(\frac{T}{\log T}\right) \quad (T \to \infty).$$

Here $\alpha_0 = 1$, since the sum in (4) being void for $k = 0$ is taken to be 0. The discord with (1) arises because the minus sign was not effected while passing to formula (15) of [2] (cf. (25) below).

Our proof first gives, in (37) below,

$$\mathcal{A}_k := -\sum_{u=0}^{\infty}(-1)^u \sum_{v=0}^{k}\binom{k}{v} \sum_{\substack{i_1+\cdots+i_k=u \\ i_1,\ldots,i_k \geqslant 0}} \binom{u}{i_1,\ldots,i_k}$$
$$\times \prod_{w=1}^{k}\left\{(-1)^w w!\binom{k}{w}\right\}^{i_w} \frac{(-1)^v (v+1)!}{(i_1 + 2i_2 + \cdots + ki_k + v)!} \tag{6}$$

as the coefficient of the main term, and to complete the proof we need

Proposition 1.1. *For $k \in \mathbb{N}$, $\mathcal{A}_k = -\alpha_k$.*

As of (6) we employ the convention that if $k = 0$, then for any function f,

$$\sum_{\substack{i_1+\cdots+i_k=u \\ i_1,\ldots,i_k \geqslant 0}} f(i_1, \ldots, i_k, v, m, \ldots) = \begin{cases} f(0, \ldots, 0, v, m, \ldots) & \text{if } u = 0, \\ 0 & \text{if } u \geqslant 1. \end{cases}$$

In [2] Conrey and Ghosh also suggested the problem of estimating the average $\sum_{0<\gamma_k<T} \zeta^{(j)}(\rho_k)$ for any natural numbers j and k. For the case $j = 1$ and $k = 0$, Fujii [3] obtained

$$\sum_{0<\gamma<T} \zeta'(\rho) = \frac{T}{4\pi}\left(\log\frac{T}{2\pi}\right)^2 + (\varsigma_0 - 1)\frac{T}{2\pi}\log\frac{T}{2\pi} + (\varsigma_1 - \varsigma_0)\frac{T}{2\pi} + O\big(T\exp(-C\sqrt{\log T})\big),$$

where C is some positive constant and the ς_i come from the Laurent expansion of $\zeta(s)$ around $s = 1$,

$$\zeta(s) = \frac{1}{s-1} + \sum_{i=0}^{\infty} \varsigma_i (s-1)^i. \tag{7}$$

In this article we prove:

Theorem 2. *Let $k, j \in \mathbb{N}$ be fixed. Then, as $T \to \infty$, we have*

$$\sum_{0<\gamma_k\leqslant T} \zeta^{(j)}(\rho_k) = [j=0]\frac{T}{2\pi}\log\frac{T}{2\pi} + (-1)^j \mathcal{B}(j,k)\frac{T}{2\pi}\left(\log\frac{T}{2\pi}\right)^{j+1} + O_{j,k}\big(T\log^j T\big),$$

where

$$\mathcal{B}(j,k) := -\frac{k+1}{j+1} - j!\sum_{\tau=1}^{k}\frac{e^{-z_\tau}}{z_\tau^{j+1}}P_j(z_\tau) + j!\sum_{\tau=1}^{k}\frac{1}{z_\tau^{j+1}}$$

with the sums over r being void in the case $k = 0$.

Here we have used the Iverson notation that for a statement S, the value of $[S]$ is 1 if S is true, and 0 if S is false.

For some information on the location of the z_r and the estimate

$$\alpha_k \sim -2\Re e^{-(k+1)+\sqrt{2}\varrho_1\sqrt{k+1}+\frac{1-2\varrho_1^2}{3}} \quad \text{as } k \to \infty,$$

where $\varrho_1 \approx -1.35 + i1.99$ is that zero of $\operatorname{erfc}(z) = \frac{2}{\sqrt{\pi}}\int_z^\infty e^{-t^2}\,dt$ which is closest to the origin, we refer the reader to [10] and [11].

2. Preliminaries

In this section we present some lemmas. Our first two lemmas were given by Gonek [4] in the case when m is non-negative. We shall not include their proofs here, the proofs can be obtained by following the arguments in [4] or in their original source Levinson's work [5]. We shall need the extension of their lemmas to the situation when $|m|$ is allowed to tend to infinity sufficiently slowly. In what follows we take a fixed such that $1 < a < 1.9 < \frac{1}{2} + \frac{1}{\log 2}$. (The upper bound on a arises from the need to satisfy the monotonicity requirement in Lemma 4.5 of [9] which is used in the proof of the case when $|r - A| \leqslant \sqrt{A}$ and the symmetric case involving B.) The constants implied by the O-symbols and other constants used in the asymptotic formulas may depend on a, but we do not exhibit this dependence explicitly, in other words, we neglect a dependence. We denote by $A(k)$ a positive number depending on the parameter k. As usual, ϵ denotes a fixed positive number which can be taken to be arbitrarily small. The constants denoted by the same symbol need not have the same value at each occurrence.

Lemma 2.1. *Let A be large and $m \in \mathbb{Z}$ with $|m| = o(\log A)$. We have, for $A \leqslant r \leqslant B \leqslant 2A$,*

$$\int_A^B \exp\left[it \log \frac{t}{re}\right]\left(\frac{t}{2\pi}\right)^{a-\frac{1}{2}}\left(\log \frac{t}{2\pi}\right)^m dt$$
$$= (2\pi)^{1-a} r^a e^{-ir+\frac{\pi i}{4}}\left(\log \frac{r}{2\pi}\right)^m + E(r, A, B)(\log A)^m,$$

while for $r < A$ or $r > B$,

$$\int_A^B \exp\left[it \log \frac{t}{re}\right]\left(\frac{t}{2\pi}\right)^{a-\frac{1}{2}}\left(\log \frac{t}{2\pi}\right)^m dt = K_0^{|m|} E(r, A, B)(\log A)^m.$$

Here

$$E(r, A, B) = O\left(A^{a-\frac{1}{2}}\right) + O\left(\frac{A^{a+\frac{1}{2}}}{|A-r| + A^{\frac{1}{2}}}\right) + O\left(\frac{B^{a+\frac{1}{2}}}{|B-r| + B^{\frac{1}{2}}}\right),$$

the constants implied in the O-terms do not depend on m, K_0 can be taken to be any fixed number > 1 when m is negative and to be 1 when m is non-negative.

Lemma 2.2. *Let $\{b_n\}_{n=1}^{\infty}$ be a sequence of complex numbers such that $b_n \ll n^{\epsilon}$ for any $\epsilon > 0$. Then, for $|m| = o(\log T)$ as $T \to \infty$,*

$$\frac{1}{2\pi}\int_{\frac{T}{2}}^{T} \chi(1-a-it)\left(\log \frac{t}{2\pi}\right)^m \sum_{n=1}^{\infty} \frac{b_n}{n^{a+it}}\, dt$$
$$= \sum_{\frac{T}{4\pi} \leqslant n \leqslant \frac{T}{2\pi}} b_n (\log n)^m + O\left(K_0^{|m|} T^{a-\frac{1}{2}} (\log T)^m\right).$$

Now we consider certain Dirichlet series and the size of their coefficients.

Lemma 2.3. *For $k, i_1, \ldots, i_k, m \in \mathbb{N}$, $\nu \in \{0, 1, \ldots, k\}$, and $\sigma > 1$, define*

$$\sum_{n=1}^{\infty} \frac{b_n(i_1, \ldots, i_k; \nu)}{n^s} := \frac{\zeta^{(\nu+1)}}{\zeta}(s) \prod_{w=1}^{k}\left(\frac{\zeta^{(w)}}{\zeta}(s)\right)^{i_w},$$

and

$$\sum_{n=1}^{\infty} \frac{c_n(i_1, \ldots, i_k; \nu; m)}{n^s} := \frac{\zeta^{(\nu+1)}}{\zeta}(s)\zeta^{(m)}(s) \prod_{w=1}^{k}\left(\frac{\zeta^{(w)}}{\zeta}(s)\right)^{i_w}.$$

We have

$$\left|b_n(i_1, \ldots, i_k; \nu)\right| \leqslant (\log n)^{K+1}, \qquad \left|c_n(i_1, \ldots, i_k; \nu; m)\right| \leqslant (\log n)^{K+m+1},$$

where

$$K := i_1 + 2i_2 + \cdots + ki_k + v. \tag{8}$$

The proof of this lemma is elementary, so it will be omitted. We note that the case $k = 0$ does not cause any notational conflict. In this case the above products are void, and therefore taken to be to 1. Hence when $k = 0$ we understand that $b_n(i_1, \ldots, i_k; v) = -\Lambda(n)$ and

$$c_n(i_1, \ldots, i_k; v; m) = (-1)^m \sum_{d|n} \Lambda(d) \log^m \frac{n}{d}.$$

Next, we estimate the summatory function of the coefficients of the Dirichlet series introduced in Lemma 2.3.

Lemma 2.4. *Let $k, i_1, \ldots, i_k, m \in \mathbb{N}$ and $v \in \{0, 1, \ldots, k\}$. For fixed k, if $i_1 + \cdots + i_k \leqslant \frac{\log x}{\log\log x}$, then as $x \to \infty$ we have*

$$\sum_{n \leqslant x} b_n(i_1, \ldots, i_k; v) = \mathcal{S}(i_1, \ldots, i_k; v) x (\log x)^K + E_b(i_1, \ldots, i_k; v),$$

and

$$\sum_{n \leqslant x} c_n(i_1, \ldots, i_k; v; m) = \mathcal{S}(i_1, \ldots, i_k; v; m) x (\log x)^{K+m+1} + E_c(i_1, \ldots, i_k; v; m),$$

where

$$\mathcal{S}(i_1, \ldots, i_k; v) := \frac{(-1)^{K+1}(v+1)! \prod_{w=1}^k (w!)^{i_w}}{K!},$$

$$\mathcal{S}(i_1, \ldots, i_k; v; m) := \frac{(-1)^{K+m+1}(v+1)! m! \prod_{w=1}^k (w!)^{i_w}}{(K+m+1)!},$$

and

$$E_b(i_1, \ldots, i_k; v) := O\left(A(k)^K \left[(\log x)^{K+2} + \frac{x(\log x)^{K-1}}{(K-1)!} + \frac{x(\log x)^{(\frac{2}{3}+\epsilon)(K+3)}}{e^{\delta_1(k)(\log x)^{\frac{1}{3}-\epsilon}}}\right]\right),$$

$$E_c(i_1, \ldots, i_k; v; m) := O_m\left(A(k)^K \left[(\log x)^{K+m+2} + \frac{x(\log x)^{K+m}}{(K+m)!} + \frac{x(\log x)^{(\frac{2}{3}+\epsilon)(K+m+4)}}{e^{\frac{\delta_1(k)}{2}(\log x)^{\frac{1}{3}-\epsilon}}}\right]\right).$$

Here, we use the convention $(-1)! = 1$ and $i_1 + 2i_2 + \cdots + ki_k = 0$ if $k = 0$.

Proof. We will include here the proof of the part involving the b_n, the proof of the part for the c_n goes along the same lines.

From (7), as $s \to 1$, we see that

$$\frac{\zeta^{(v+1)}}{\zeta}(s) \prod_{w=1}^{k} \left(\frac{\zeta^{(w)}}{\zeta}(s)\right)^{i_w} \ll \left(\frac{A(k)}{|s-1|}\right)^{K+1}. \tag{9}$$

We use the result of Lemma 2.3 and (9) in Perron's formula [9, §3.12] to obtain

$$\sum_{n\leqslant x} b_n(i_1,\ldots,i_k;v) = \frac{1}{2\pi i}\int_{1+\frac{1}{\log x}-ix}^{1+\frac{1}{\log x}+ix} \frac{\zeta^{(v+1)}}{\zeta}(s)\prod_{w=1}^{k}\left(\frac{\zeta^{(w)}}{\zeta}(s)\right)^{i_w}\frac{x^s}{s}\,ds + O\left(\left(A(k)\log x\right)^{K+2}\right). \tag{10}$$

Now we want to shift the line of integration to the left. By the theorem of Vinogradov–Korobov (see [9, §6.19]), we know that there exists a positive absolute constant δ_1 such that $\zeta(s)\neq 0$ throughout the region

$$\sigma \geqslant 1 - \frac{\delta_1}{(\log|t|)^{\frac{2}{3}}(\log\log|t|)^{\frac{1}{3}}}, \quad |t|\geqslant 3,$$

in which one has

$$\frac{\zeta'}{\zeta}(s) \ll \left(\log|t|\right)^{\frac{2}{3}}\left(\log\log|t|\right)^{\frac{1}{3}} \tag{11}$$

(in the proof of the part for the c_n, one also uses the consequence $\zeta(s)\ll(\log|t|)^{\frac{2}{3}}$). If we move the line of integration in (10) to $\sigma = 1-\frac{\delta_1(k)}{(\log x)^{\frac{2}{3}+\epsilon_0}}$, $\delta_1(k)=\frac{\delta_1}{2^k}$, where $\epsilon_0>0$ is a small fixed number, then the zero-free region theorem guarantees that the only pole of the integrand between the vertical lines is at $s=1$, and the residue theorem gives

$$\sum_{n\leqslant x} b_n(i_1,\ldots,i_k;v) = \operatorname*{Res}_{s=1}\left\{\frac{\zeta^{(v+1)}}{\zeta}(s)\prod_{w=1}^{k}\left(\frac{\zeta^{(w)}}{\zeta}(s)\right)^{i_w}\frac{x^s}{s}\right\} - \frac{1}{2\pi i}(I_1+I_2+I_3) + O\left(\left(A(k)\log x\right)^{K+2}\right), \tag{12}$$

where I_1 is the integral over $[1+\frac{1}{\log x}+ix, 1-\frac{\delta_1(k)}{(\log x)^{\frac{2}{3}+\epsilon_0}}+ix)$, I_2 is over $[1-\frac{\delta_1(k)}{(\log x)^{\frac{2}{3}+\epsilon_0}}+ix, 1-\frac{\delta(k)}{(\log x)^{\frac{2}{3}+\epsilon_0}}-ix)$, and I_3 is over $[1-\frac{\delta_1(k)}{(\log x)^{\frac{2}{3}+\epsilon_0}}-ix, 1+\frac{1}{\log x}-ix)$. Using (11) and the identity $\zeta^{(w+1)}/\zeta = (\zeta^{(w)}/\zeta)' + (\zeta^{(w)}/\zeta)(\zeta'/\zeta)$, Cauchy's estimate in a small disc of radius $\asymp 1/(\log|t|)^{\frac{2}{3}}(\log\log|t|)^{\frac{1}{3}}$ centered at $\sigma+it$ gives

$$\frac{\zeta^{(w)}}{\zeta}(s) \ll_w \left(\log|t|\right)^{(\frac{2}{3}+\epsilon_0)w} \quad \left(\sigma\geqslant 1-\frac{\delta_1}{2^w(\log|t|)^{\frac{2}{3}+\epsilon_0}},\ |t|\geqslant 3\right).$$

Hence, on I_1, I_3, and the part of I_2 of distance $\geqslant 3$ from the real line, we have

$$\frac{\zeta^{(w)}}{\zeta}(s) = O_w\left((\log x)^{(\frac{2}{3}+\epsilon_0)w}\right). \tag{13}$$

Since $|\frac{x^s}{s}|\ll 1$ on the horizontal parts of the contour, from (13) we get

$$I_1 \ll \left(A(k)\log x\right)^{(\frac{2}{3}+\epsilon_0)(K+1)}. \tag{14}$$

Clearly this bound is also valid for I_3. For I_2, by symmetry it is enough to consider the part above the real axis which can be split up as $\int_0^3 + \int_3^x$. By (13) the second part is

$$\ll xe^{-\delta_1(k)(\log x)^{\frac{1}{3}-\epsilon_0}} \big(A(k)\log x\big)^{(\frac{2}{3}+\epsilon_0)(K+1)} \int_3^x \frac{dt}{t}$$
$$\ll xe^{-\delta_1(k)(\log x)^{\frac{1}{3}-\epsilon_0}} \big(A(k)\log x\big)^{(\frac{2}{3}+\epsilon_0)(K+3)}.$$

For the first part we use (9) to bound the factors involving the zeta-function, so that this part is

$$\ll xe^{-\delta_1(k)(\log x)^{\frac{1}{3}-\epsilon_0}} \big(A(k)\big)^{K+1} \int_{1-\frac{\delta_1(k)}{(\log x)^{\frac{2}{3}+\epsilon_0}}}^{1-\frac{\delta_1(k)}{(\log x)^{\frac{2}{3}+\epsilon_0}}+3i} \frac{ds}{|s-1|^{K+1}}.$$

This last integral is

$$\ll \int_0^{\frac{\delta_1(k)}{(\log x)^{\frac{2}{3}+\epsilon_0}}} \frac{dt}{(\frac{\delta_1(k)}{(\log x)^{\frac{2}{3}+\epsilon_0}})^{K+1}} + \int_{\frac{\delta_1(k)}{(\log x)^{\frac{2}{3}+\epsilon_0}}}^{3} \frac{dt}{t^{K+1}} \ll \left(\frac{(\log x)^{\frac{2}{3}+\epsilon_0}}{\delta_1(k)}\right)^K.$$

Hence we have

$$I_2 \ll xe^{-\delta_1(k)(\log x)^{\frac{1}{3}-\epsilon_0}} \big(A(k)\log x\big)^{(\frac{2}{3}+\epsilon_0)(K+3)}. \tag{15}$$

Now by (14) and (15), (12) becomes

$$\begin{aligned}\sum_{n\leqslant x} b_n(i_1,\ldots,i_k;\nu) = \operatorname*{Res}_{s=1}&\left\{\frac{\zeta^{(\nu+1)}}{\zeta}(s)\prod_{w=1}^{k}\left(\frac{\zeta^{(w)}}{\zeta}(s)\right)^{i_w}\frac{x^s}{s}\right\}\\ &+O\big(\big(A(k)\log x\big)^{K+2}\big)\\ &+O\big(xe^{-\delta_1(k)(\log x)^{\frac{1}{3}-\epsilon_0}}\big(A(k)\log x\big)^{(\frac{2}{3}+\epsilon_0)(K+3)}\big).\end{aligned} \tag{16}$$

The pole at $s=1$ is of order $K+1$, so this residue is

$$\begin{aligned}&\frac{1}{K!}\frac{d^K}{ds^K}\left\{(s-1)^{K+1}\frac{\zeta^{(\nu+1)}}{\zeta}(s)\prod_{w=1}^{k}\left(\frac{\zeta^{(w)}}{\zeta}(s)\right)^{i_w}\frac{x^s}{s}\right\}_{s=1}\\ &=\frac{1}{K!}\sum_{\substack{j_1+j_2+j_3=K\\ j_1,j_2,j_3\in\mathbb{N}}}\binom{K}{j_1,j_2,j_3}\frac{d^{j_1}}{ds^{j_1}}\left\{(s-1)^{K+1}\frac{\zeta^{(\nu+1)}}{\zeta}(s)\prod_{w=1}^{k}\left(\frac{\zeta^{(w)}}{\zeta}(s)\right)^{i_w}\right\}_{s=1}\\ &\quad\times\frac{d^{j_2}}{ds^{j_2}}\{x^s\}_{s=1}\frac{d^{j_3}}{ds^{j_3}}\left\{\frac{1}{s}\right\}_{s=1}.\end{aligned}$$

Since $j_3 = K - j_1 - j_2$, we can rewrite (16) as

$$\sum_{n\leqslant x} b_n(i_1,\dots,i_k;v)$$
$$= x(-1)^K \sum_{j_1\leqslant K} \frac{(-1)^{j_1}}{j_1!}\frac{d^{j_1}}{ds^{j_1}}\left\{(s-1)^{K+1}\frac{\zeta^{(v+1)}}{\zeta}(s)\prod_{w=1}^{k}\left(\frac{\zeta^{(w)}}{\zeta}(s)\right)^{i_w}\right\}_{s=1}$$
$$\times \sum_{j_2\leqslant K-j_1}\frac{(-1)^{j_2}\log^{j_2}x}{j_2!} + O\left(\left(A(k)\log x\right)^{K+2}\right)$$
$$+ O\left(xe^{-\delta_1(k)(\log x)^{\frac{1}{3}-\epsilon_0}}\left(A(k)\log x\right)^{(\frac{2}{3}+\epsilon_0)(K+3)}\right). \tag{17}$$

If K is 0 or 1, we see that

$$\sum_{n\leqslant x} b_n(i_1,\dots,i_k;v)$$
$$= \begin{cases} -x + O(xe^{-c_1(\log x)^{\frac{1}{3}-\epsilon_0}}) & \text{if } i_1=\dots=i_k=v=0,\\ 2x\log x - 2x + O(xe^{-c_1(\log x)^{\frac{1}{3}-\epsilon_0}}) & \text{if } i_1=\dots=i_k=0,\ v=1,\\ x\log x - x + O(xe^{-c_1(\log x)^{\frac{1}{3}-\epsilon_0}}) & \text{if } i_1=1,\ i_2=\dots=i_k=v=0, \end{cases} \tag{18}$$

for some $c_1 \in (0,1)$. When $K \geqslant 2$ we split the double sum in (17) into three parts: The term with $j_1 = 0$ and $j_2 = K$ which gives the main term, the terms with $j_1 = 0, 1$ and $j_2 = K - 1$, and the remaining terms. So we have

$$\sum_{n\leqslant x} b_n(i_1,\dots,i_k;v)$$
$$= \mathcal{S}(i_1,\dots,i_k;v)x(\log x)^K + \frac{(-1)^K(v+1)!\prod_{w=1}^{k}(w!)^{i_w}}{(K-1)!}x(\log x)^{K-1}$$
$$+ \frac{\frac{d}{ds}\{(s-1)^{K+1}\frac{\zeta^{(v+1)}}{\zeta}(s)\prod_{w=1}^{k}(\frac{\zeta^{(w)}}{\zeta}(s))^{i_w}\}_{s=1}}{(K-1)!}x(\log x)^{K-1}$$
$$+ x(-1)^K \sum_{j_1\leqslant K} \frac{(-1)^{j_1}}{j_1!}\frac{d^{j_1}}{ds^{j_1}}\left\{(s-1)^{K+1}\frac{\zeta^{(v+1)}}{\zeta}(s)\prod_{w=1}^{k}\left(\frac{\zeta^{(w)}}{\zeta}(s)\right)^{i_w}\right\}_{s=1}$$
$$\times \sum_{j_2\leqslant K-\mathfrak{a}(j_1)}\frac{(-1)^{j_2}\log^{j_2}x}{j_2!} + O\left(xe^{-\delta_1(k)(\log x)^{\frac{1}{3}-\epsilon_0}}\left(A(k)\log x\right)^{(\frac{2}{3}+\epsilon_0)(K+3)}\right)$$
$$+ O\left(\left(A(k)\log x\right)^{K+2}\right), \tag{19}$$

where

$$\mathfrak{a}(j_1) = \begin{cases} 2, & \text{if } j_1 = 0 \text{ or } 1,\\ j_1, & \text{otherwise.} \end{cases}$$

Instead of the sum over j_2 in (19), from Taylor remainder theorem we can write

$$\frac{1}{x} - \frac{(-1)^{K-\mathfrak{a}(j_1)+1}(\log x)^{K-\mathfrak{a}(j_1)+1}}{(K-\mathfrak{a}(j_1)+1)!}x^{-\theta}$$

for some $\theta \in [0,1]$. Then the fourth term in the right-hand side of (19) can be bounded as

$$\begin{aligned} &\ll \sum_{j_1 \leqslant K} \frac{1}{j_1!} \left| \frac{d^{j_1}}{ds^{j_1}} \left\{ (s-1)^{K+1} \frac{\zeta^{(v+1)}}{\zeta}(s) \prod_{w=1}^{k} \left(\frac{\zeta^{(w)}}{\zeta}(s) \right)^{i_w} \right\}_{s=1} \right| \\ &\quad + x(\log x)^{K+1} \sum_{j_1 \leqslant K} \frac{(\log x)^{-\mathfrak{a}(j_1)}}{j_1!(K-\mathfrak{a}(j_1)+1)!} \\ &\quad \times \left| \frac{d^{j_1}}{ds^{j_1}} \left\{ (s-1)^{K+1} \frac{\zeta^{(v+1)}}{\zeta}(s) \prod_{w=1}^{k} \left(\frac{\zeta^{(w)}}{\zeta}(s) \right)^{i_w} \right\}_{s=1} \right|. \end{aligned} \tag{20}$$

By Cauchy's estimate on a disk of radius 1 centered at $s=1$ we have

$$\begin{aligned} &\left| \frac{d^{j_1}}{ds^{j_1}} \left\{ (s-1)^{K+1} \frac{\zeta^{(v+1)}}{\zeta}(s) \prod_{w=1}^{k} \left(\frac{\zeta^{(w)}}{\zeta}(s) \right)^{i_w} \right\}_{s=1} \right| \\ &\quad \leqslant j_1! \max_{|s-1|=1} \left| \frac{\zeta^{(w+1)}}{\zeta}(s) \right| \prod_{w=1}^{k} \left| \frac{\zeta^{(w)}}{\zeta}(s) \right|^{i_w} \leqslant j_1! \big(A(k)\big)^K. \end{aligned} \tag{21}$$

Using this, the upper bound in (20) is majorized as

$$\begin{aligned} &\ll \big(A(k)\big)^K \left[\sum_{j_1 \leqslant K} 1 + x(\log x)^{K+1} \sum_{j_1 \leqslant K} \frac{(\log x)^{-\mathfrak{a}(j_1)}}{(K-\mathfrak{a}(j_1)+1)!} \right] \\ &= \big(A(k)\big)^K \left[K + x(\log x)^{K+1} \left(\frac{3(\log x)^{-2}}{(K-1)!} + \sum_{3 \leqslant j_1 \leqslant K} \frac{(\log x)^{-j_1}}{(K-j_1+1)!} \right) \right]. \end{aligned}$$

The condition $i_1 + \cdots + i_k \leqslant \frac{\log x}{\log\log x}$ implies

$$K \leqslant \frac{k \log x}{\log\log x} + k, \tag{22}$$

so if not void the very last series is, for sufficiently large x,

$$\begin{aligned} &= \frac{(\log x)^{-3}}{(K-2)!} \left[1 + \frac{K-2}{\log x} + \frac{(K-2)(K-3)}{(\log x)^2} + \cdots + \frac{(K-2)!}{1!(\log x)^{K-3}} \right] \\ &< \frac{(\log x)^{-3}}{(K-2)!} \left[1 + \frac{2k}{\log\log x} + \left(\frac{2k}{\log\log x} \right)^2 + \cdots \right] \\ &< \frac{2(\log x)^{-3}}{(K-2)!} < \frac{3(\log x)^{-2}}{(K-1)! \log\log x}. \end{aligned}$$

Hence the expression in (20) is

$$\ll \frac{(A(k))^K x(\log x)^{K-1}}{(K-1)!}. \tag{23}$$

By (21), this upper bound also dominates the second and third terms in the right-hand side of (19) with an appropriate $A(k)$, and the proof of the part concerning the b_n is finished.

Applying partial summation to the results of Lemma 2.4, we obtain:

Lemma 2.5. *Let $k, i_1, \dots, i_k, m, r \in \mathbb{N}$ and $\nu \in \{0, 1, \dots, k\}$. For fixed k, if $i_1 + \cdots + i_k \leqslant \frac{\log x}{\log\log x}$, then as $x \to \infty$ we have*

$$\sum_{\frac{x}{2}<n\leqslant x} \frac{b_n(i_1, \dots, i_k; \nu)}{(\log n)^K} = \mathcal{S}(i_1, \dots, i_k; \nu)\frac{x}{2} + O\left(\frac{E_b(i_1, \dots, i_k; \nu)}{(\log x)^K}\right),$$

$$\sum_{\frac{x}{2}<n\leqslant x} \frac{c_n(i_1, \dots, i_k; \nu; m)}{(\log n)^{K-r}} = \mathcal{S}(i_1, \dots, i_k; \nu; m)\frac{x}{2}(\log x)^{r+m+1} + O_{r,m}\left(\frac{E_c(i_1, \dots, i_k; \nu; m)}{(\log x)^{K-r}}\right).$$

3. Proof of Theorem 1

By the residue theorem, for large A, $A < B \leqslant 2A$, we have

$$S_k(A, B) := \sum_{A<\gamma_k\leqslant B} \chi(\rho_k) = \frac{1}{2\pi i}\int_C \chi(s)\frac{\zeta^{(k+1)}}{\zeta^{(k)}}(s)\,ds,$$

for a suitable contour C. For $k = 1$ the work of Titchmarsh [9, Theorem 11.5c], and for $k \geqslant 2$ the work of Spira [7] give the existence of zero-free half-planes $\sigma \geqslant \sigma_k$ for $\zeta^{(k)}(s)$ ($2 < \sigma_1 < 3$, $\sigma_2 < 5$, $\sigma_k \geqslant \frac{7k}{4} + 2$ for $k \geqslant 3$). We can find $\sigma_k' \geqslant \sigma_k$ such that $\frac{\zeta^{(k+1)}}{\zeta^{(k)}}(\sigma_k' + it) \ll_k 1$. For we have $|\zeta^{(k)}(s)| \geqslant \frac{(\log 2)^k}{2^\sigma} - \sum_{n=3}^\infty \frac{(\log n)^k}{n^\sigma} > \frac{(\log 2)^k}{2^{\sigma+1}}$ where the very last inequality holds if σ is large enough. There are no non-real zeros of $\zeta'(s)$ in the left half-plane. In [7] Spira showed that for each $\delta > 0$ there exists an r_k such that $\zeta^{(k)}(s) \neq 0$ in the region defined by $\sigma < -\delta$, $|t| > \delta$, $|s| > r_k$. So let C be the positively oriented rectangle with vertices $\sigma_k' + iA, \sigma_k' + iB, -\delta + iB$ and $-\delta + iA$, with a fixed δ such that $0 < \delta < \frac{1}{8}$ (we will neglect δ-dependence in the constants implied by the O-symbols). We can take the horizontal sides of this rectangle to be a distance $\gg \frac{1}{\log A}$ from any zero of $\zeta^{(k)}(s)$ by adjusting A and B by an amount $\leqslant 1$. From (2) by Stirling's formula we have

$$\chi(s) = \left(\frac{|t|}{2\pi}\right)^{\frac{1}{2}-\sigma} \exp\left(-it\log\frac{|t|}{2\pi e} + \frac{i\pi}{4}\operatorname{sgn}(t)\right)\left(1 + O\left(\frac{1}{|t|}\right)\right) \tag{24}$$

uniformly in $\alpha \leqslant \sigma \leqslant \beta$ and $|t| \geqslant 1$ for any fixed real numbers α and β, and this means adding or deleting $O(\log A)$ number of terms each of size $\ll A^{\frac{1}{2}+\delta}$. Such a choice of A and B ensures that on the horizontal contours the estimate $\frac{\zeta^{(k+1)}}{\zeta^{(k)}}(s) \ll (\log A)^2$ will hold. Then it is easily seen from (24) that the horizontal integrals and the vertical integral from $\sigma_k' + iA$ to $\sigma_k' + iB$ can be bounded trivially and the contribution of these parts is $O_k(A^{\frac{1}{2}+\delta}\log^2 A)$. The integral on the left vertical contour is

$$
\begin{aligned}
\frac{1}{2\pi i}\int_{-\delta+iB}^{-\delta+iA}\chi(s)\frac{\zeta^{(k+1)}}{\zeta^{(k)}}(s)\,ds &= -\frac{1}{2\pi}\int_A^B \chi(-\delta+it)\frac{\zeta^{(k+1)}}{\zeta^{(k)}}(-\delta+it)\,dt \\
&= -\frac{1}{2\pi}\overline{\int_A^B \chi(-\delta-it)\frac{\zeta^{(k+1)}}{\zeta^{(k)}}(-\delta-it)\,dt} \\
&= -\frac{1}{2\pi}\overline{\int_A^B \chi\bigl(1-(1+\delta+it)\bigr)\frac{\zeta^{(k+1)}}{\zeta^{(k)}}\bigl(1-(1+\delta+it)\bigr)\,dt} \\
&= -\frac{1}{2\pi i}\overline{\int_{1+\delta+iA}^{1+\delta+iB}\chi(1-s)\frac{\zeta^{(k+1)}}{\zeta^{(k)}}(1-s)\,ds}.
\end{aligned}
$$

Thus we have

$$
\overline{S_k(A,B)} = -\frac{1}{2\pi i}\int_{1+\delta+iA}^{1+\delta+iB}\chi(1-s)\frac{\zeta^{(k+1)}}{\zeta^{(k)}}(1-s)\,ds + O_k\bigl(A^{\frac{1}{2}+\delta}\log^2 A\bigr). \tag{25}
$$

As in [2], using the estimates

$$
\frac{\chi'}{\chi}(s) = -\log\frac{|t|}{2\pi} + O\left(\frac{1}{|t|}\right), \qquad \left(\frac{d}{ds}\right)^m \frac{\chi'}{\chi}(s) \ll |t|^{-m} \quad \bigl(|t| \geqslant 1\bigr), \tag{26}
$$

we obtain from (3) that

$$
\frac{\zeta^{(k+1)}}{\zeta^{(k)}}(1-s) = -\left(\ell + \frac{G_k'}{G_k}(s,\ell)\right)\left(1 + O\left(\frac{1}{|t|}\right)\right), \tag{27}
$$

where $\ell = \log\frac{|t|}{2\pi}$, $\sigma' \leqslant \sigma \leqslant \sigma''$ for any fixed real numbers σ' and σ'',

$$
G_k(s,z) := \left(z + \frac{d}{dz}\right)^k \zeta(s) = z^k\zeta(s) + kz^{k-1}\zeta'(s) + \cdots + \zeta^{(k)}(s), \tag{28}
$$

and the differentiation in G' is with respect to s. We see that

$$
\frac{G_k'}{G_k}(s,z) = \frac{\sum_{v=0}^k \binom{k}{v}\frac{1}{z^v}\frac{\zeta^{(v+1)}}{\zeta}(s)}{1 + \sum_{w=1}^k \binom{k}{w}\frac{1}{z^w}\frac{\zeta^{(w)}}{\zeta}(s)}. \tag{29}
$$

Since $\frac{\zeta^{(w)}}{\zeta}(s) \ll_w 1$ when $\sigma \geqslant 1+\delta$, we have

$$
\sum_{w=1}^k \binom{k}{w}\frac{1}{\ell^w}\frac{\zeta^{(w)}}{\zeta}(s) \ll_k \frac{1}{\log A}, \tag{30}
$$

and therefore

$$\frac{G_k'}{G_k}(s,\ell) \ll_k 1 \qquad (\sigma \geqslant 1+\delta,\ A \leqslant t \leqslant B). \tag{31}$$

Substituting (27) in (25), and then using Lemma 2.1 and (31) we have

$$\overline{S_k(A,B)} = \frac{1}{2\pi i}\int_{1+\delta+iA}^{1+\delta+iB} \chi(1-s)\frac{G_k'}{G_k}(s,\ell)\,ds + O_k\big(A^{\frac{1}{2}+\delta}\log^2 A\big). \tag{32}$$

Now, we approximate $\frac{G_k'}{G_k}(s,\ell)$ by a Dirichlet series. In the region $\sigma \geqslant 1+\delta$, $A \leqslant t \leqslant B$, for large A, by (30) we can expand the denominator of (29) as a power series,

$$\begin{aligned}\left(1+\sum_{w=1}^{k}\binom{k}{w}\frac{1}{\ell^w}\frac{\zeta^{(w)}}{\zeta}(s)\right)^{-1} &= \sum_{u=0}^{\infty}(-1)^u\left(\sum_{w=1}^{k}\binom{k}{w}\frac{1}{\ell^w}\frac{\zeta^{(w)}}{\zeta}(s)\right)^u \\ &= \sum_{u\leqslant \frac{\log A}{\log\log A}}(-1)^u\left(\sum_{w=1}^{k}\binom{k}{w}\frac{1}{\ell^w}\frac{\zeta^{(w)}}{\zeta}(s)\right)^u + O\left(\frac{1}{A}\right).\end{aligned} \tag{33}$$

From (32), (29) and (33) we have

$$\begin{aligned}\overline{S_k(A,B)} = {} & \sum_{u\leqslant \frac{\log A}{\log\log A}}\sum_{v=0}^{k}(-1)^u\binom{k}{v}\sum_{\substack{i_1+\cdots+i_k=u\\ i_1,\ldots,i_k\geqslant 0}}\binom{u}{i_1,\ldots,i_k}\prod_{w=1}^{k}\binom{k}{w}^{i_w} \\ & \times \frac{1}{2\pi}\int_A^B \frac{\chi(-\delta-it)}{\ell^K}\prod_{w=1}^{k}\left(\frac{\zeta^{(w)}}{\zeta}(1+\delta+it)\right)^{i_w}\frac{\zeta^{(v+1)}}{\zeta}(1+\delta+it)\,dt \\ & + O_k\big(A^{\frac{1}{2}+\delta}\log^2 A\big).\end{aligned} \tag{34}$$

Taking $A=\frac{T}{2}$, $B=T$, and applying Lemma 2.2, we have

$$\begin{aligned}\overline{S_k(T/2,T)} = {} & \sum_{u\leqslant \frac{\log T}{\log\log T}}\sum_{v=0}^{k}(-1)^u\binom{k}{v}\sum_{\substack{i_1+\cdots+i_k=u\\ i_1,\ldots,i_k\geqslant 0}}\binom{u}{i_1,\ldots,i_k}\prod_{w=1}^{k}\binom{k}{w}^{i_w} \\ & \times \sum_{\frac{T}{4\pi}<n\leqslant\frac{T}{2\pi}}\frac{b_n(i_1,\ldots,i_k;v)}{(\log n)^K} + O_k\big(T^{\frac{1}{2}+\delta}\log^2 T\big)\end{aligned}$$

for large T. For the innermost sum above we apply Lemma 2.5 to get

$$\overline{S_k(T/2,T)} = -\frac{T}{4\pi}\left(\sum_{u=0}^{\infty} - \sum_{u>\frac{\log T}{\log\log T}}\right)\Bigg[(-1)^u\sum_{v=0}^{k}\binom{k}{v}\sum_{\substack{i_1+\cdots+i_k=u\\ i_1,\ldots,i_k\geqslant 0}}\binom{u}{i_1,\ldots,i_k}$$

$$\times \prod_{w=1}^{k} \left\{ (-1)^w w! \binom{k}{w} \right\}^{i_w} \frac{(-1)^v (v+1)!}{K!} \Bigg] + E_1 + E_2 + E_3$$
$$+ O_k\big(T^{\frac{1}{2}+\delta} \log^2 T\big), \quad \text{say.} \tag{35}$$

Here, by Lemmas 2.4 and 2.5 and (22), we have

$$\begin{aligned}
E_1 &\ll_k \log^2 T \sum_{u \leqslant \frac{\log T}{\log\log T}} \sum_{\substack{i_1+\cdots+i_k=u \\ i_1,\ldots,i_k \geqslant 0}} \binom{u}{i_1,\ldots,i_k} A(k)^K \\
&\ll_k A(k)^{\frac{\log T}{\log\log T}} \log^2 T \sum_{u \leqslant \frac{\log T}{\log\log T}} \sum_{\substack{i_1+\cdots+i_k=u \\ i_1,\ldots,i_k \geqslant 0}} \binom{u}{i_1,\ldots,i_k} \\
&= A(k)^{\frac{\log T}{\log\log T}} \log^2 T \sum_{u \leqslant \frac{\log T}{\log\log T}} k^u \\
&\ll_k A(k)^{\frac{\log T}{\log\log T}} \log^2 T \\
&\ll_k T^{\epsilon}; \\
E_2 &\ll_k \frac{T}{\log T} \sum_{u \leqslant \frac{\log T}{\log\log T}} \sum_{\substack{i_1+\cdots+i_k=u \\ i_1,\ldots,i_k \geqslant 0}} \binom{u}{i_1,\ldots,i_k} \frac{A(k)^K}{(K-1)!} \\
&\ll_k \frac{T}{\log T} \sum_{u \leqslant \frac{\log T}{\log\log T}} \frac{A(k)^u}{(u-1)!} \sum_{\substack{i_1+\cdots+i_k=u \\ i_1,\ldots,i_k \geqslant 0}} \binom{u}{i_1,\ldots,i_k} \\
&\ll_k \frac{T}{\log T}\big(1 + O\big(e^{A(k)}\big)\big) \\
&\ll_k \frac{T}{\log T}; \\
E_3 &\ll_k \frac{T(\log T)^{2+3\epsilon}}{e^{\delta_1(k)(\log T)^{\frac{1}{3}-\epsilon}}} \sum_{u \leqslant \frac{\log T}{\log\log T}} \sum_{\substack{i_1+\cdots+i_k=u \\ i_1,\ldots,i_k \geqslant 0}} \binom{u}{i_1,\ldots,i_k} \left(\frac{A(k)}{(\log T)^{\frac{1}{3}-\epsilon}}\right)^K \\
&\ll_k \frac{T(\log T)^{2+3\epsilon}}{e^{\delta_1(k)(\log T)^{\frac{1}{3}-\epsilon}}} \sum_{u \leqslant \frac{\log T}{\log\log T}} \left(\frac{A(k)}{(\log T)^{\frac{1}{3}-\epsilon}}\right)^u \sum_{\substack{i_1+\cdots+i_k=u \\ i_1,\ldots,i_k \geqslant 0}} \binom{u}{i_1,\ldots,i_k} \\
&\ll_k \frac{T(\log T)^{2+3\epsilon}}{e^{\delta_1(k)(\log T)^{\frac{1}{3}-\epsilon}}} \frac{1}{1 - A(k)(\log T)^{-\frac{1}{3}+\epsilon}}.
\end{aligned}$$

The sum over $u > \log T / \log\log T$ in (35) is

$$\ll_k T \sum_{u > \frac{\log T}{\log\log T}} \frac{(A(k))^u}{u!} \ll T \frac{(A(k))^{\lceil \log T/\log\log T \rceil}}{\lceil \log T/\log\log T \rceil!} \ll_k T^{\frac{2\log\log\log T}{\log\log T}},$$

by Stirling's formula. Hence (35) can be rewritten as

$$\overline{S_k(T/2,T)} = \Bigg[-\sum_{u=0}^{\infty}(-1)^u \sum_{v=0}^{k}\binom{k}{v} \sum_{\substack{i_1+\cdots+i_k=u\\ i_1,\ldots,i_k\geqslant 0}} \binom{u}{i_1,\ldots,i_k}$$

$$\times \prod_{w=1}^{k}\left\{(-1)^w w!\binom{k}{w}\right\}^{i_w} \frac{(-1)^v(v+1)!}{K!}\Bigg]\frac{T}{4\pi} + O_k\left(\frac{T}{\log T}\right)$$

$$= \frac{\mathcal{A}_k}{2}\frac{T}{2\pi} + O_k\left(\frac{T}{\log T}\right). \tag{36}$$

Thus using (36) also with T replaced by $\frac{T}{2}, \frac{T}{4}, \ldots$ down to almost $\sqrt{T}$, then adding up, and also noting the trivial estimate $S_k(0,\sqrt{T}) \ll T^{\frac{3}{4}+\frac{\delta}{2}}(\log T)^{\frac{1}{2}}$, we obtain

$$\sum_{0<\gamma_k<T} \chi(\rho_k) = \mathcal{A}_k\frac{T}{2\pi} + O_k\left(\frac{T}{\log T}\right) \quad (T\to\infty). \tag{37}$$

4. Proof of Proposition 1.1

For $k=0$ and $k=1$ the assertion is easily verified from the definitions (4) and (6), so let $k\geqslant 2$. The elementary symmetric polynomials $e_n(x_1,\ldots,x_k)$ on k variables realized through

$$\prod_{w=1}^{k}(z-x_w) = \sum_{j=0}^{k}(-1)^j e_j(x_1,\ldots,x_k)z^{k-j}$$

and having the explicit expressions

$$e_0(x_1,\ldots,x_k)=1, \qquad e_1(x_1,\ldots,x_k) = \sum_{1\leqslant i_1\leqslant k} x_{i_1}, \qquad e_2(x_1,\ldots,x_k) = \sum_{1\leqslant i_1<i_2\leqslant k} x_{i_1}x_{i_2},$$

$$\ldots, \qquad e_k(x_1,\ldots,x_k) = \prod_{1\leqslant i\leqslant k} x_i, \qquad e_n(x_1,\ldots,x_k)=0 \quad \text{if } n>k,$$

and the powersums of these k variables, $p_i(x_1,\ldots,x_k)=\sum_{j=1}^{k}x_j^i$, are related through the Newton–Girard formulas

$$ne_n(x_1,\ldots,x_k) = \sum_{i=1}^{n}(-1)^{i-1}e_{n-i}(x_1,\ldots,x_k)p_i(x_1,\ldots,x_k) \quad (n\geqslant 1). \tag{38}$$

We express the powersums in terms of the elementary symmetric polynomials as

$$p_n(x_1,\ldots,x_k) = (-1)^n\sum_{v=0}^{k}(k-v)e_v(x_1,\ldots,x_k) \sum_{\substack{i_1+2i_2+\cdots+ki_k+v=n\\ i_1,\ldots,i_k\geqslant 0}} \binom{i_1+\cdots+i_k}{i_1,\ldots,i_k}$$

$$\times \prod_{w=1}^{k}\big(e_w(x_1,\ldots,x_k)\big)^{i_w}(-1)^{i_1+i_2+\cdots+i_k} \quad (n\geqslant 0). \tag{39}$$

Proof of (39). By carrying out formal expansions we have

$$\sum_{n=0}^{\infty} p_n(x_1,\dots,x_k)t^n = \sum_{n=0}^{\infty}\sum_{w=1}^{k}(x_w t)^n = \sum_{w=1}^{k}\frac{1}{1-x_w t} = \frac{\sum_{m=1}^{k}\prod_{w\in\{1,\dots,k\}\setminus\{m\}}(1-x_w t)}{\prod_{w=1}^{k}(1-x_w t)}$$
$$= \frac{\sum_{m=1}^{k}\sum_{v=0}^{k-1}(-1)^v e_v^*(\{x_1,\dots,x_k\}\setminus\{x_m\})t^v}{1+\sum_{v=1}^{k}(-1)^v e_v(x_1,\dots,x_k)t^v},$$

where

$$e_v^*\big(\{x_1,\dots,x_k\}\setminus\{x_m\}\big) = \sum_{\substack{i_1<\dots<i_v\\ i_1,\dots,i_v\in\{1,\dots,k\}\setminus\{m\}}} x_{i_1}\dots x_{i_v}.$$

First noting

$$\sum_{m=1}^{k} e_v^*\big(\{x_1,\dots,x_k\}\setminus\{x_m\}\big) = (k-v)e_v(x_1,\dots,x_k) \quad (v=0,1,\dots,k-1),$$

then doing a power series expansion to be followed by the substitution $n = i_1+2i_2+\dots+ki_k+v$, we see that

$$\sum_{n=0}^{\infty} p_n(x_1,\dots,x_k)t^n = \frac{\sum_{v=0}^{k}(-1)^v(k-v)e_v(x_1,\dots,x_k)t^v}{1+\sum_{v=1}^{k}(-1)^v e_v(x_1,\dots,x_k)t^v}$$
$$= \sum_{v=0}^{k}(k-v)e_v(x_1,\dots,x_k)\sum_{u=0}^{\infty}(-1)^u \sum_{\substack{i_1+i_2+\dots+i_k=u\\ i_1,\dots,i_k\geqslant 0}}\binom{i_1+\dots+i_k}{i_1,\dots,i_k}$$
$$\times\prod_{w=1}^{k}\big(e_w(x_1,\dots,x_k)\big)^{i_w}(-t)^{i_1+2i_2+\dots+ki_k+v}$$
$$= \sum_{n=0}^{\infty}(-t)^n\sum_{v=0}^{k}(k-v)e_v(x_1,\dots,x_k)\sum_{\substack{i_1+2i_2+\dots+ki_k+v=n\\ i_1,\dots,i_k\geqslant 0}}\binom{i_1+\dots+i_k}{i_1,\dots,i_k}$$
$$\times\prod_{w=1}^{k}\big(e_w(x_1,\dots,x_k)\big)^{i_w}(-1)^{i_1+i_2+\dots+i_k}.$$

Now comparing the coefficients of t^n gives the result. □

For z_r, the roots of $P_k(z)$ defined in (5), we see that

$$e_v(z_1,\dots,z_k) = (-1)^v v!\binom{k}{v} \quad (v=0,\dots,k), \tag{40}$$

and using this in (39) we have

$$p_n(z_1,\dots,z_k)=\sum_{v=0}^{k}(k-v)v!\binom{k}{v}\sum_{\substack{i_1+2i_2+\cdots+ki_k+v=n\\ i_1,\dots,i_k\geqslant 0}}\binom{i_1+\cdots+i_k}{i_1,\dots,i_k}$$
$$\times\prod_{w=1}^{k}\left(w!\binom{k}{w}\right)^{i_w}(-1)^{i_1+i_2+\cdots+i_k}. \tag{41}$$

From the Maclaurin series for the exponential function we write

$$\alpha_k=k+1-\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}p_n(z_1,\dots,z_k),$$

and substituting (40) in this we have

$$\alpha_k=k+1-(k+1)\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\sum_{v=0}^{k}v!\binom{k}{v}\sum_{\substack{i_1+2i_2+\cdots+ki_k+v=n\\ i_1,\dots,i_k\geqslant 0}}\binom{i_1+\cdots+i_k}{i_1,\dots,i_k}$$
$$\times\prod_{w=1}^{k}\left(w!\binom{k}{w}\right)^{i_w}(-1)^{i_1+i_2+\cdots+i_k}$$
$$+\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\sum_{v=0}^{k}(v+1)!\binom{k}{v}\sum_{\substack{i_1+2i_2+\cdots+ki_k+v=n\\ i_1,\dots,i_k\geqslant 0}}\binom{i_1+\cdots+i_k}{i_1,\dots,i_k}$$
$$\times\prod_{w=1}^{k}\left(w!\binom{k}{w}\right)^{i_w}(-1)^{i_1+i_2+\cdots+i_k}.$$

On the other hand, upon the substitution $n=i_1+\cdots+ki_k+v$, the definition of $\mathcal{A}_k$ in (6) can be rewritten in the form

$$\mathcal{A}_k=-\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\sum_{v=0}^{k}\binom{k}{v}(v+1)!\sum_{\substack{i_1+2i_2+\cdots+ki_k+v=n\\ i_1,\dots,i_k\geqslant 0}}\binom{i_1+\cdots+i_k}{i_1,\dots,i_k}$$
$$\times\prod_{w=1}^{k}\left\{w!\binom{k}{w}\right\}^{i_w}(-1)^{i_1+\cdots+i_k}.$$

Thus in order to complete the proof of Proposition 1.1, we need to verify that

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\sum_{v=0}^{k}v!\binom{k}{v}\sum_{\substack{i_1+2i_2+\cdots+ki_k+v=n\\ i_1,\dots,i_k\geqslant 0}}\binom{i_1+\cdots+i_k}{i_1,\dots,i_k}$$
$$\times\prod_{w=1}^{k}\left(w!\binom{k}{w}\right)^{i_w}(-1)^{i_1+i_2+\cdots+i_k}=1. \tag{42}$$

We have

$$\frac{1}{1+\sum_{v=1}^{k} v!\binom{k}{v} z^v}$$
$$= \sum_{n=0}^{\infty} t^n \sum_{\substack{i_1+2i_2+\cdots+ki_k+v=n \\ i_1,\ldots,i_k \geqslant 0}} \binom{i_1+\cdots+i_k}{i_1,\ldots,i_k} \prod_{w=1}^{k} \left(w!\binom{k}{w}\right)^{i_w} (-1)^{i_1+i_2+\cdots+i_k},$$

in which the innermost sum in the left-hand side of (42) occurs as the coefficient of t^n, and by (40) we have

$$\frac{1}{1+\sum_{v=1}^{k} v!\binom{k}{v} z^v} = \frac{1}{\prod_{w=1}^{k}(1-z_w z)} = \prod_{w=1}^{k} \sum_{n=0}^{\infty} (z_w t)^n = \sum_{n=0}^{\infty} t^n \sum_{\substack{i_1+i_2+\cdots+i_k=n \\ i_1,\ldots,i_k \geqslant 0}} \prod_{w=1}^{k} z_w^{i_w}.$$

Comparing the coefficients of t^n in the last two equations, we see that

$$\sum_{\substack{i_1+2i_2+\cdots+ki_k=n \\ i_1,\ldots,i_k \geqslant 0}} \binom{i_1+\cdots+i_k}{i_1,\ldots,i_k} \prod_{w=1}^{k} \left(w!\binom{k}{w}\right)^{i_w} (-1)^{i_1+i_2+\cdots+i_k}$$
$$= \sum_{\substack{i_1+i_2+\cdots+i_k=n \\ i_1,\ldots,i_k \geqslant 0}} \prod_{w=1}^{k} z_w^{i_w} =: h_n(z_1,\ldots,z_k), \tag{43}$$

where h_n is the so-called complete homogeneous symmetric polynomial of degree n of the z_w which satisfies (see [8, p. 450]) for indeterminates $x_1,\ldots,x_k$,

$$h_n(x_1,\ldots,x_k) = \sum_{w=1}^{k} x_w^{n+k-1} \prod_{\substack{1 \leqslant r \leqslant k \\ r \neq w}} (x_w - x_r)^{-1}. \tag{44}$$

Now by (43), the condition (42) is simplified to

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \sum_{v=0}^{\min(k,n)} v!\binom{k}{v} h_{n-v}(z_1,\ldots,z_k) = 0, \tag{45}$$

for the innermost sum in (42) is void if $n < v$. The roots $z_1,\ldots,z_k$ of $P_k(z)$ are distinct, so using (44) in (45), our problem is reduced to verifying that

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \sum_{v=0}^{\min(k,n)} v!\binom{k}{v} \sum_{w=1}^{k} z_w^{n-v+k-1} \prod_{\substack{1 \leqslant r \leqslant k \\ r \neq w}} (z_w - z_r)^{-1} = 0.$$

If $n \geqslant k$, then the left-hand side contains as a factor

$$\sum_{v=0}^{k} v!\binom{k}{v} z_w^{k-v} = k! \sum_{v=0}^{k} \frac{z_w^{k-v}}{(k-v)!} = k! P_k(z_w) = 0, \tag{46}$$

so that the condition is now simplified into

$$\sum_{n=1}^{k-1} \frac{(-1)^n}{n!} \sum_{v=0}^{n} v!\binom{k}{v} \sum_{w=1}^{k} z_w^{n-v+k-1} \prod_{\substack{1\leqslant r\leqslant k\\ r\neq w}} (z_w - z_r)^{-1} = 0.$$

Since $P_k(z_w) = 0$,

$$\sum_{v=0}^{n} v!\binom{k}{v} z_w^{k-v} = -\sum_{v=n+1}^{k} v!\binom{k}{v} z_w^{k-v}, \tag{47}$$

our condition can be cast as

$$\sum_{n=1}^{k-1} \frac{(-1)^n}{n!} \sum_{v=n+1}^{k} v!\binom{k}{v} \sum_{w=1}^{k} z_w^{n-v+k-1} \prod_{\substack{1\leqslant r\leqslant k\\ r\neq w}} (z_w - z_r)^{-1} = 0.$$

Putting $q_n(z) := \sum_{v=n+1}^{k} v!\binom{k}{v} z^{n-v+k}$, where $1 \leqslant n \leqslant k-1$, the condition to be verified takes the form

$$\sum_{n=1}^{k-1} \frac{(-1)^n}{n!} \sum_{w=1}^{k} \frac{q_n(z_w)}{z_w} \prod_{\substack{1\leqslant r\leqslant k\\ r\neq w}} (z_w - z_r)^{-1} = 0. \tag{48}$$

Now $\deg q_n(z) = k-1$, by Lagrange's interpolation formula we have

$$q_n(z) = \sum_{w=1}^{k} q_n(z_w) \prod_{\substack{1\leqslant r\leqslant k\\ r\neq w}} \frac{z - z_r}{z_w - z_r} = \prod_{r=1}^{k}(z - z_r) \sum_{w=1}^{k} \frac{q_n(z_w)}{z - z_w} \prod_{\substack{1\leqslant r\leqslant k\\ r\neq w}} (z_w - z_r)^{-1}.$$

Taking $z = 0$ here, since $q_n(0) = 0$ and the z_r are nonzero, we obtain

$$\sum_{w=1}^{k} \frac{q_n(z_w)}{z_w} \prod_{\substack{1\leqslant r\leqslant k\\ r\neq w}} (z_w - z_r)^{-1} = 0 \quad (1 \leqslant n \leqslant k-1),$$

from which we see that (48) holds. This completes the proof of Proposition 1.1.

5. Proof of Theorem 2

Let $k, j \in \mathbb{N}$. To obtain Theorem 2 we use the residue theorem to write

$$S_{j,k}(A,B) := \sum_{A<\gamma_k\leqslant B} \zeta^{(j)}(\rho_k) = \frac{1}{2\pi i}\int_C \zeta^{(j)}(s)\frac{\zeta^{(k+1)}}{\zeta^{(k)}}(s)\,ds,$$

where C is the same contour as in the proof of Theorem 1 which was described in the first paragraph of Section 3. The integral along the vertical side of real part σ_k' will be $\ll_{j,k} A$. From the estimates

$$\zeta^{(j)}(\sigma+it) \ll_{\epsilon,j} \begin{cases} |t|^{\frac{1}{2}-\sigma+\epsilon} & \text{if } \sigma \leqslant 0, \\ |t|^{\frac{1}{2}(1-\sigma)+\epsilon} & \text{if } 0 \leqslant \sigma \leqslant 1, \\ |t|^{\epsilon} & \text{if } \sigma \geqslant 1, \end{cases}$$

valid with an arbitrarily small fixed $\epsilon > 0$ (see [4]), it follows that the integrals on the horizontal sides of the contour are $\ll A^{\frac{1}{2}+\delta+\epsilon}(\log A)^2$. So we have

$$S_{j,k}(A,B) = \frac{1}{2\pi i}\int_{-\delta+iB}^{-\delta+iA} \zeta^{(j)}(s)\frac{\zeta^{(k+1)}}{\zeta^{(k)}}(s)\,ds + O_{j,k}(A),$$

which in the same way as (25) was obtained can be turned into

$$\overline{S_{j,k}(A,B)} = -\frac{1}{2\pi i}\int_{1+\delta+iA}^{1+\delta+iB} \zeta^{(j)}(1-s)\frac{\zeta^{(k+1)}}{\zeta^{(k)}}(1-s)\,ds + O_{j,k}(A).$$

Upon j-fold differentiation of the functional equation (3), using (26), we have

$$\zeta^{(j)}(1-s) = (-1)^j\chi(1-s)\left(1+O_j\left(\frac{1}{|t|}\right)\right)\sum_{m=0}^{j}\binom{j}{m}\ell^{j-m}\zeta^{(m)}(s) \quad \bigl(|t|>1\bigr),$$

uniformly on any bounded range of σ. Using this and (27), we obtain

$$\begin{aligned}\overline{S_{j,k}(A,B)} &= \frac{(-1)^j}{2\pi i}\sum_{m=0}^{j}\binom{j}{m}\int_{1+\delta+iA}^{1+\delta+iB}\chi(1-s)\ell^{j-m+1}\zeta^{(m)}(s)\,ds \\ &\quad + \frac{(-1)^j}{2\pi i}\sum_{m=0}^{j}\binom{j}{m}\int_{1+\delta+iA}^{1+\delta+iB}\chi(1-s)\ell^{j-m}\frac{G_k'}{G_k}(s,\ell)\zeta^{(m)}(s)\,ds + O_{j,k}(A) \\ &= S' + S'' + O_{j,k}(A), \quad \text{say.}\end{aligned} \tag{49}$$

With $A = \frac{T}{2}$ and $B = T$, Lemma 2.2 gives

$$S' = (-1)^j\sum_{m=0}^{j}\binom{j}{m}(-1)^m\sum_{\frac{T}{4\pi}<n\leqslant\frac{T}{2\pi}}(\log n)^{j+1} + O_j\bigl(T^{\frac{1}{2}+\delta}(\log T)^{j+1}\bigr).$$

The inner sum here does not depend on m, and the outer sum vanishes unless $j=0$, so it follows from Stirling's formula that

$$S' = \begin{cases} O_j(T^{\frac{1}{2}+\delta}(\log T)^{j+1}) & \text{if } j \geqslant 1, \\ [x\log x - x + O(\log x)]_{\frac{T}{4\pi}}^{\frac{T}{2\pi}} + O(T^{\frac{1}{2}+\delta}\log T) & \text{if } j = 0. \end{cases} \tag{50}$$

Using (29) and (33), we have as in (34)

$$S'' = (-1)^j \sum_{m=0}^{j} \binom{j}{m} \sum_{u \leqslant \frac{\log A}{\log\log A}} \sum_{v=0}^{k} (-1)^u \binom{k}{v} \sum_{\substack{i_1+\cdots+i_k=u \\ i_1,\ldots,i_k \geqslant 0}} \binom{u}{i_1,\ldots,i_k} \prod_{w=1}^{k} \binom{k}{w}^{i_w}$$
$$\times \frac{1}{2\pi} \int_A^B \frac{\chi(-\delta-it)}{\ell^{K+m-j}} \prod_{w=1}^{k} \left(\frac{\zeta^{(w)}}{\zeta}(1+\delta+it)\right)^{i_w} \left(\frac{\zeta^{(v+1)}}{\zeta}\zeta^{(m)}\right)(1+\delta+it)\,dt$$
$$+ O_{j,k}\big(A^{\frac{1}{2}+\delta}(\log A)^{j-m}\big).$$

With $A = \frac{T}{2}$ and $B = T$, Lemma 2.2 gives

$$S'' = (-1)^j \sum_{m=0}^{j} \binom{j}{m} \sum_{u \leqslant \frac{\log T}{\log\log T}} \sum_{v=0}^{k} (-1)^u \binom{k}{v} \sum_{\substack{i_1+\cdots+i_k=u \\ i_1,\ldots,i_k \geqslant 0}} \binom{u}{i_1,\ldots,i_k} \prod_{w=1}^{k} \binom{k}{w}^{i_w}$$
$$\times \sum_{\frac{T}{4\pi} < n \leqslant \frac{T}{2\pi}} \frac{c_n(i_1,\ldots,i_k;v;m)}{(\log n)^{K+m-j}} + O_{j,k}\big(T^{\frac{1}{2}+\delta+\epsilon}\big).$$

It follows from Lemma 2.5 that

$$S'' = (-1)^j \frac{T}{4\pi}\left(\log\frac{T}{2\pi}\right)^{j+1} \sum_{u \leqslant \frac{\log T}{\log\log T}} \sum_{m=0}^{j} \binom{j}{m} \sum_{v=0}^{k} (-1)^u \binom{k}{v} \sum_{\substack{i_1+\cdots+i_k=u \\ i_1,\ldots,i_k \geqslant 0}} \binom{u}{i_1,\ldots,i_k}$$
$$\times \prod_{w=1}^{k} \binom{k}{w}^{i_w} \frac{(-1)^{K+m+1} m!(v+1)! \prod_{w=1}^{k}(w!)^{i_w}}{(K+m+1)!} + O_{j,k}\big(T(\log T)^j\big), \tag{51}$$

where the error term is obtained through a calculation similar to that for the E_i after (35). The sum over $u > \log T/\log\log T$ is again small just as in Section 3, so we may let the sum over u extend to ∞. With (50) and (51) plugged in (49), we see that

$$\overline{S_{j,k}(T/2,T)} = [j=0]\frac{T}{4\pi}\log\frac{T}{2\pi} + (-1)^j \frac{T}{4\pi}\left(\log\frac{T}{2\pi}\right)^{j+1}$$
$$\times \sum_{u=0}^{\infty} (-1)^u \sum_{m=0}^{j} \binom{j}{m} \sum_{v=0}^{k} \binom{k}{v} \sum_{\substack{i_1+\cdots+i_k=u \\ i_1,\ldots,i_k \geqslant 0}} \binom{u}{i_1,\ldots,i_k} \prod_{w=1}^{k} \left(w!\binom{k}{w}\right)^{i_w}$$
$$\times \frac{(-1)^{K+m+1} m!(v+1)!}{(K+m+1)!} + O_{j,k}\big(T\log^j T\big).$$

Using this also with T replaced by $\frac{T}{2}, \frac{T}{4}, \ldots$ down to almost $\sqrt{T}$, and adding up gives, in view of the trivial estimate $S_{j,k}(0, \sqrt{T}) \ll T^{\frac{3}{4}+\frac{\delta}{2}+\epsilon}$,

$$\begin{aligned}\sum_{0<\gamma_0\leqslant T} \zeta^{(j)}(\rho_k) &= [j=0]\frac{T}{2\pi}\log\frac{T}{2\pi} + (-1)^j \frac{T}{2\pi}\left(\log\frac{T}{2\pi}\right)^{j+1} \\ &\quad \times \sum_{u=0}^{\infty}(-1)^u \sum_{m=0}^{j}\binom{j}{m}\sum_{v=0}^{k}\binom{k}{v} \sum_{\substack{i_1+\cdots+i_k=u \\ i_1,\ldots,i_k\geqslant 0}} \binom{u}{i_1,\ldots,i_k} \\ &\quad \times \prod_{w=1}^{k}\left((-1)^w w!\binom{k}{w}\right)^{i_w} \frac{(-1)^{v+m+1} m!(v+1)!}{(i_1+2i_2+\cdots+ki_k+v+m+1)!} \\ &\quad + O_{j,k}\big(T\log^j T\big) \\ &= [j=0]\frac{T}{2\pi}\log\frac{T}{2\pi} + (-1)^j\frac{T}{2\pi}\left(\log\frac{T}{2\pi}\right)^{j+1}\mathcal{B}(j,k) \\ &\quad + O_{j,k}\big(T\log^j T\big), \quad \text{say.} \end{aligned} \tag{52}$$

We wish to give an expression for $\mathcal{B}(j,k)$ consisting of finite sums. The procedure is similar to that in Section 4. First we see that

$$\mathcal{B}(j,0) = \sum_{m=0}^{j}\binom{j}{m}\frac{(-1)^{m+1}}{m+1} = \frac{-1}{j+1},$$

which is consistent with the statement of Theorem 2. Now take $k \geqslant 1$, and let $n := i_1 + 2i_2 + \cdots + ki_k + v + m + 1$, so that

$$\begin{aligned}\mathcal{B}(j,k) &= \sum_{m=0}^{j}\binom{j}{m}m!\sum_{n=m+1}^{\infty}\frac{(-1)^n}{n!}\sum_{v=0}^{k}\binom{k}{v}(v+1)! \\ &\quad \times \sum_{\substack{i_1+\cdots+ki_k+v=n-m-1 \\ i_1,\ldots,i_k\geqslant 0}} \binom{i_1+i_2+\cdots+i_k}{i_1,\ldots,i_k}\prod_{w=1}^{k}\left(w!\binom{k}{w}\right)^{i_w}(-1)^{i_1+i_2+\cdots+i_k}.\end{aligned}$$

By (41) and (43) we re-express this as

$$\begin{aligned}\mathcal{B}(j,k) &= \sum_{m=0}^{j}\binom{j}{m}m!\sum_{n=m+1}^{\infty}\frac{(-1)^n}{n!} \\ &\quad \times \left[\sum_{v=0}^{k}\frac{(k+1)!}{(k-v)!}h_{n-m-v-1}(z_1,\ldots,z_k) - p_{n-m-1}(z_1,\ldots,z_k)\right] \\ &= B_1 + B_2, \quad \text{say.}\end{aligned} \tag{53}$$

By (44), we have

$$B_1=\sum_{m=0}^{j}\binom{j}{m}m!\sum_{n=m+1}^{\infty}\frac{(-1)^n}{n!}\sum_{v=0}^{\min(k,n-m-1)}\frac{(k+1)!}{(k-v)!}\sum_{w=1}^{k}z_w^{n-m-v+k-2}\prod_{\substack{1\leqslant r\leqslant k\\ r\neq w}}(z_w-z_r)^{-1}.$$

For $n-m-1\geqslant k$, the last expression contains zero as a factor as in (46), and for $n\leqslant k+m$ we use the idea in (47) to write

$$B_1=-\sum_{m=0}^{j}\binom{j}{m}m!\sum_{n=m+1}^{k+m}\frac{(-1)^n}{n!}\sum_{v=n-m}^{k}\frac{(k+1)!}{(k-v)!}\sum_{w=1}^{k}z_w^{n-m-v+k-2}\prod_{\substack{1\leqslant r\leqslant k\\ r\neq w}}(z_w-z_r)^{-1}.$$

We now put $q_{n,m}(z):=\sum_{v=n-m-1}^{k}v!\binom{k}{v}z^{n-m-v+k-1}$ for $0\leqslant m\leqslant j$, $m+1\leqslant n\leqslant k+m$, so that

$$B_1=-(k+1)\sum_{m=0}^{j}\binom{j}{m}m!\sum_{n=m+1}^{k+m}\frac{(-1)^n}{n!}\sum_{w=1}^{k}\frac{q_{n,m}(z_w)}{z_w}\prod_{\substack{1\leqslant r\leqslant k\\ r\neq w}}(z_w-z_r)^{-1}.$$

Since $\deg q_{n,m}(z)=k-1$, Lagrange's interpolation formula gives

$$q_{n,m}(z)=\prod_{r=1}^{k}(z-z_r)\sum_{w=1}^{k}\frac{q_{n,m}(z_w)}{z-z_w}\prod_{\substack{1\leqslant r\leqslant k\\ r\neq w}}(z_w-z_r)^{-1}$$

as after (48). We see that

$$B_1=\frac{k+1}{k!}\sum_{m=0}^{j}\binom{j}{m}m!\sum_{n=m+1}^{k+m}\frac{(-1)^n}{n!}q_{n,m}(0).$$

Since

$$q_{n,m}(0)=\begin{cases}k! & \text{if } n=m+1,\\ 0 & \text{if } m+1<n\leqslant k+m,\end{cases}$$

we obtain

$$B_1=(k+1)\sum_{m=0}^{j}\binom{j}{m}\frac{(-1)^{m+1}}{m+1}=-\frac{k+1}{j+1}. \tag{54}$$

Next, upon putting $u=m-n$ below, we have

$$B_2=-\sum_{m=0}^{j}\binom{j}{m}m!\sum_{n=m+1}^{\infty}\frac{(-1)^n}{n!}p_{n-m-1}(z_1,\ldots,z_k)$$

$$
\begin{aligned}
&= -\sum_{m=0}^{j}\binom{j}{m}m!\sum_{r=1}^{k}z_r^{-(m+1)}\sum_{n=m+1}^{\infty}\frac{(-1)^n}{n!}z_r^n \\
&= -\sum_{m=0}^{j}\binom{j}{m}m!\sum_{r=1}^{k}z_r^{-(m+1)}\left(e^{-z_r}-\sum_{n=0}^{m}\frac{(-1)^n}{n!}z_r^n\right) \\
&= -j!\sum_{r=1}^{k}\frac{e^{-z_r}}{z_r^{j+1}}P_j(z_r)+\sum_{m=0}^{j}\binom{j}{m}m!\sum_{r=1}^{k}z_r^{-(m+1)}\sum_{n=0}^{m}\frac{(-1)^n}{n!}z_r^n \\
&= -j!\sum_{r=1}^{k}\frac{e^{-z_r}}{z_r^{j+1}}P_j(z_r)+\sum_{u=0}^{j}(-1)^u u!\sum_{m=\ell}^{j}\binom{j}{m}\binom{m}{u}(-1)^m\sum_{r=1}^{k}\frac{1}{z_r^{u+1}} \\
&= -j!\sum_{r=1}^{k}\frac{e^{-z_r}}{z_r^{j+1}}P_j(z_r)+\sum_{u=0}^{j}(-1)^u u!\binom{j}{u}\sum_{m=u}^{j}\binom{j-u}{m-u}(-1)^m\sum_{r=1}^{k}\frac{1}{z_r^{u+1}} \\
&= -j!\sum_{r=1}^{k}\frac{e^{-z_r}}{z_r^{j+1}}P_j(z_r)+j!\sum_{r=1}^{k}z_r^{j+1}. \qquad (55)
\end{aligned}
$$

By (54) and (55), (53) becomes

$$
\mathcal{B}(j,k) = -\frac{k+1}{j+1} - j!\sum_{r=1}^{k}\frac{e^{-z_r}}{z_r^{j+1}}P_j(z_r) + j!\sum_{r=1}^{k}\frac{1}{z_r^{j+1}}, \qquad (56)
$$

and this completes the proof of Theorem 2.

We note that sums of small negative powers of the z_r can easily be calculated using the Newton–Girard formulas (38). As was given in [10], we have

$$
\sum_{r=1}^{k}\frac{1}{z_r^{v}} = \begin{cases} -1 & \text{if } v=1, \\ 0 & \text{if } 2\leqslant v\leqslant k, \\ \frac{1}{k!} & \text{if } v=k+1, \end{cases}
$$

so that the last term in the expression (56) vanishes for $1\leqslant j\leqslant k-1$.

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