# Some Analogues of Pair Correlation of Zeta Zeros

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**Abstract** By modifying Montgomery's calculation of the pair correlation of zeta zeros, we derive analogous results. In this article we work out the correlation of zeta zeros with the relative maxima of the zeta function on the critical line, the pair correlation of these maxima, and the correlation of zeros of two Dirichlet $L$-functions. In each case the relevant Riemann Hypothesis is assumed for obtaining the results. Several auxiliary results necessary for the calculations may be useful in problems about the zeta function.

## 1 Introduction

The Riemann zeta-function $\zeta(s)$, for the complex variable $s = \sigma + it$ with real numbers $\sigma$ and $t$, has all of its so-called non-trivial zeros in the strip $\sigma \in (0,1)$, and the number of these zeros with ordinates in $(0,T]$ is

$$N_\zeta(T) = \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T). \tag{1.1}$$

The Riemann Hypothesis (RH) is the statement that all of these zeros are on the critical line $\sigma = \frac{1}{2}$. In [17] H. L. Montgomery, starting from an explicit formula (given in (2.1) below) in Landau's Handbuch [16] and assuming RH, elicited and defined the expression

$$F_{\zeta,\zeta}(x,T) := \sum_{0<\gamma,\tilde{\gamma}\le T} \frac{4}{4+(\gamma-\tilde{\gamma})^2}\, x^{i(\gamma-\tilde{\gamma})}, \quad (x>0), \tag{1.2}$$

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calculated its values, and broke new ground by the results, conjectures and observations concerning the distribution of zeta zeros on the critical line and relations with the distribution of primes which were deduced therefrom. Here $\gamma$ and $\tilde{\gamma}$ each denote the ordinate of a non-trivial zero of $\zeta(s)$, and $\frac{4}{4+u^2}$ is a suitable weight function. From the definition (1.2) it is clear that $F_{\zeta,\zeta}(x,T) = F_{\zeta,\zeta}(\frac{1}{x},T)$ (so that one can consider only $x \geq 1$), and that

$$F_{\zeta,\zeta}(x,T) = \frac{2}{\pi}\int_{-\infty}^{\infty}\Big|\sum_{0<\gamma\leq T}\frac{x^{i\gamma}}{1+(t-\gamma)^2}\Big|^2 dt \geq 0. \tag{1.3}$$

Montgomery's calculation gave

$$F_{\zeta,\zeta}(x,T) = \frac{T\log^2 T}{2\pi x^2}\big(1+O(\frac{1}{\log T})\big) + \frac{T}{2\pi}\log x + o(T\log T) + O(x\log x), \tag{1.4}$$

as $T \longrightarrow \infty$. In [17] Montgomery expressed his results in a different form which is related to (1.2) by setting

$$x = T^{\alpha},\ T \geq 2,\ \alpha \in \mathbb{R} \tag{1.5}$$

and

$$F(\alpha) = F(\alpha,T) := \left(\frac{T}{2\pi}\log T\right)^{-1}\sum_{0<\gamma,\tilde{\gamma}\leq T} T^{i\alpha(\gamma-\tilde{\gamma})}w(\gamma-\tilde{\gamma}), \tag{1.6}$$

so that the result read

$$F(\alpha) = (1+o(1))T^{-2\alpha}\log T + \alpha + o(1), \tag{1.7}$$

as $T \longrightarrow \infty$, uniformly for $0 \leq \alpha \leq 1$, along with $F(-\alpha) = F(\alpha)$. (With further effort involving the use of a sieve upper bound for prime pairs, Goldston [8] improved the error term $O(x\log x)$ in (1.4) to $O(x)$, as had been pointed out by Montgomery, leading to the uniformity up to $\alpha = 1$; see also [12]).

By convolving $F(\alpha)$ with appropriate kernels, Montgomery deduced that asymptotically at least $\frac{2}{3}$ of zeta zeros are simple, and the existence of infinitely many gaps between consecutive zeta zeros of size at most 0.68 times the average gap. This has since been reduced to 0.51... by several researchers; achieving an upper bound below $\frac{1}{2}$ will have important consequences in number theory (see [15]). By drawing upon conjectural estimates for twin primes, and also upon the $q$-analogue of an expression which comes up in his calculation, Montgomery conjectured that

$$F(\alpha) = 1 + o(1) \tag{1.8}$$

uniformly for $\alpha \geq 1$ in bounded intervals. Trivially $F(\alpha) \leq F(0)$ for any $\alpha$, and by Dirichlet's theorem on Diophantine approximation, there are large $\alpha = \alpha(T)$ such that $F(\alpha)$ becomes arbitrarily close to $F(0) \sim \log T$ (which follows unconditionally from (1.1)), so the restriction to bounded intervals is essential for (1.8). We note that such large values of $F(\alpha)$ can occur over intervals of length at most $O(\frac{1}{\log T})$. This follows from the upper bound in Goldston's (see [9]) estimates that for any $\epsilon > 0$ and sufficiently large $T$

$$\int_{c-1}^{c+1} F(\alpha)\,d\alpha \geq \frac{2}{3} - \epsilon, \quad \int_{c}^{c+1} F(\alpha)\,d\alpha \leq \frac{8}{3} + \epsilon, \tag{1.9}$$

uniformly for any real number $c$ (which may be any function of $T$). From the conjecture (1.8), Montgomery deduced the so-called pair correlation conjecture

$$\sum_{\substack{0<\gamma,\tilde{\gamma}\leq T \\ \frac{2\pi\alpha}{\log T}\leq\gamma-\tilde{\gamma}\leq\frac{2\pi\beta}{\log T}}} 1 \quad \sim \quad \left(\int_{\alpha}^{\beta} 1 - (\frac{\sin \pi u}{\pi u})^2\,du + \delta(\alpha,\beta)\right)\frac{T}{2\pi}\log T, \tag{1.10}$$

as $T \to \infty$. Here $\delta(\alpha,\beta) = 1$ if $0 \in [\alpha,\beta]$, $\delta(\alpha,\beta) = 0$ otherwise. As for the converse direction, Goldston [10] showed that (1.10) implies $\int_{\alpha}^{\alpha+\delta} F(\beta,T)\,d\beta \sim \delta$ for any fixed numbers $\alpha \geq 1$ and $\delta > 0$, i.e. that (1.8) holds on average. From either (1.8) or (1.10), it follows that almost all zeta zeros are simple. F. J. Dyson remarked to Montgomery that the integrand in (1.10) occurs as the pair correlation function of the eigenvalues of random complex Hermitian matrices in the Gaussian Unitary Ensemble, and Montgomery interpreted this being in accordance with the view (which legend dates back to Hilbert and Pólya, see [20]) that there is a yet undiscovered linear operator whose eigenvalues characterize the zeros of the Riemann zeta-function. Montgomery's work [17] has been very influential for the developments in the theory of the Riemann zeta-function (and other zeta functions as well) and formed the basis of the connections with random matrix theory. For a comprehensive survey on this topic and relations with the distribution of primes we refer the reader to Goldston's article [11].

*Brief statement of results:* In this article we modify Montgomery's argument in such a way that we can obtain some analogues of the pair correlation of zeta zeros. We begin with a brief sketch of Montgomery's method. Then in §3 we present an alternative method for correlating zeta zeros to any sequence of real numbers. The first application, in §4, is to Montgomery's original problem. (The contents of §3 and §4 were already announced in [24]). Upon some preliminaries concerning $\chi(s)$ (defined in (1.12)) and the relative maxima of $\zeta(s)$ on the critical line in §5 and concerning the iterated convolutions of the von Mangoldt function in §6, the method of §3 is applied in §7 to the correlation of zeta zeros with the maxima of zeta on the critical line. Corresponding to $F(\alpha) \sim \alpha$ for $\epsilon \leq \alpha \leq 1-\epsilon$ from (1.7), we find in Theorem 1 in §7 that $F_{\zeta,Z_1}(\alpha) \sim \alpha - 2\alpha^2$, $(\epsilon \leq \alpha \leq 1-\epsilon)$. Here $Z_1(s)$, defined in §5, is a function which vanishes at the relative maxima of $\zeta(s)$ on the critical line. For this purpose a Landau-Gonek type formula involving the zeros of $Z_1(s)$ is developed in §7. In Theorem 2 in §8 we obtain the values of the pair correlation of the maxima $F_{Z_1,Z_1}(\alpha)$ for $0 \leq \alpha \leq 1-\epsilon$, to be followed by values of $F_{\zeta\cdot Z_1,\zeta\cdot Z_1}(\alpha)$, and then deduce some results on small gaps. In §9 we adapt our alternative argument to obtain in

Theorem 3 the correlation between the zeros of two Dirichlet $L$-functions. Finally in §10 we mention some possible ways to continue this research.

*Some basic formulae:* Before embarking on the calculations we recall that the Riemann zeta-function satisfies the functional equation

$$\zeta(s) = \chi(s)\zeta(1-s), \tag{1.11}$$

where

$$\chi(s) = 2(2\pi)^{s-1}\Gamma(1-s)\sin(\frac{\pi s}{2}) = \pi^{s-\frac{1}{2}}\frac{\Gamma(\frac{1-s}{2})}{\Gamma(\frac{s}{2})}. \tag{1.12}$$

Logarithmic differentiation of the functional equation gives

$$\frac{\zeta'}{\zeta}(s) = \frac{\chi'}{\chi}(s) - \frac{\zeta'}{\zeta}(1-s). \tag{1.13}$$

From (1.12) and some basic properties of the $\Gamma$-function, we see that

$$\frac{\chi'}{\chi}(s) = -\frac{\Gamma'}{\Gamma}(s) + \frac{\pi}{2}\tan\frac{s\pi}{2} + \log 2\pi = -\frac{\Gamma'}{\Gamma}(1-s) + \frac{\pi}{2}\cot\frac{s\pi}{2} + \log 2\pi. \tag{1.14}$$

Since

$$\frac{\Gamma'}{\Gamma}(s) = \log s - \frac{1}{2s} + O\Big(\frac{1}{|s|^2}\Big), \quad (|s| \ge \delta > 0, |\arg s| \le \pi - \delta), \tag{1.15}$$

where $\delta > 0$ is arbitrarily small but fixed, we have

$$\frac{\chi'}{\chi}(s) = -\log\frac{|s|}{2\pi} + O\Big(\frac{1}{|s|}\Big), \quad (|t| \ge 1, \text{ say; } \sigma_1 \le \sigma \le \sigma_2), \tag{1.16}$$

where $\sigma_1 < \sigma_2$ are any fixed real numbers. Under the same conditions one also has

$$\Big(\frac{d}{ds}\Big)^n \frac{\chi'}{\chi}(s) \ll |s|^{-n}, \quad (n \ge 1). \tag{1.17}$$

Some further facts concerning $\zeta(s)$ and $\chi(s)$ will be given in §5 and §6.

*Notation:* Throughout this paper, $T$ will denote a large number tending to $\infty$, $\epsilon$ will be a fixed positive number which can be taken arbitrarily small and which may not have the same value each time it appears, and letters with indices such as $C_j$ will denote sufficiently large positive constants of fixed value. We write $\mathbb{N} = \mathbb{Z}^+ \cup \{0\}$. We will employ the Iverson notation that for a statement $S$, $[S] = 1$ if $S$ is true, and $[S] = 0$ if $S$ is false.

## 2 A sketch of Montgomery's derivation

The explicit formula from Landau's book [16] is that, if $x > 1$ and $x \neq p^k$ ($p$ prime, and $k$ a positive integer), then

$$\sum_{n\le x}\frac{\Lambda(n)}{n^s} = -\frac{\zeta'}{\zeta}(s) + \frac{x^{1-s}}{1-s} - \sum_{\rho}\frac{x^{\rho-s}}{\rho-s} + \sum_{r=1}^{\infty}\frac{x^{-2r-s}}{2r+s} \tag{2.1}$$

provided $s \neq 1$, $s \neq \rho$, $s \neq -2r$. Here $\rho$ runs through the non-trivial zeros of $\zeta(s)$, and the series over $\rho$ is convergent subject to the interpretation $\lim_{U\to\infty}\sum_{|\Im\rho|\le U}\frac{x^{\rho-s}}{\rho-s}$. Upon assuming RH, Montgomery deduced that

$$\begin{aligned}(2\sigma-1)\sum_{\gamma}\frac{x^{i\gamma}}{(\sigma-\frac12)^2+(t-\gamma)^2} &= -x^{-\frac12}\Big(\sum_{n\le x}\Lambda(n)(\frac{x}{n})^{1-\sigma+it} + \sum_{n>x}\Lambda(n)(\frac{x}{n})^{\sigma+it}\Big)\\ &\quad -\frac{\zeta'}{\zeta}(1-\sigma+it)x^{\frac12-\sigma+it} + \frac{(2\sigma-1)x^{\frac12}}{(\sigma-1+it)(\sigma-it)}\\ &\quad -x^{-\frac12}\sum_{r=1}^{\infty}\frac{(2\sigma-1)x^{-2r}}{(\sigma-1-it-2r)(\sigma+it+2r)},\end{aligned} \tag{2.2}$$

valid for $\sigma > 1$, and all $x \ge 1$. In this formula Montgomery used (1.13)–(1.16) to replace $\frac{\zeta'}{\zeta}(1-\sigma+it)$ by $-\frac{\zeta'}{\zeta}(\sigma-it) - \log(|t|+2) + O(1) = -\log(|t|+2) + O_\sigma(1)$ (for $s$ in a fixed strip in $\sigma > 1$), and he put upper-bound estimates for the last two terms of (2.2). Montgomery took $\sigma = \frac32$, squared the modulus of both sides, and then integrated both sides over $t$ from 0 to $T$. From the expression thus arising from the left-hand side of (2.2), he discarded those $\gamma \notin [0,T]$ within an error of $O(\log^3 T)$, and in order to evaluate the integral he extended the range of integration to $\int_{-\infty}^{\infty}$ within an even smaller error. In this way the left-hand side of (2.2) led to the expression (1.2). To carry out the integration of the square of the series involving $\Lambda(n)$ coming from the right-hand side of (2.2), Montgomery resorted to Parseval's formula for Dirichlet series from [18]:

$$\text{If } \sum_{n=1}^{\infty} n|a_n|^2 \text{ converges, then } \int_0^T\Big|\sum_n a_n n^{-it}\Big|^2 dt = \sum_n |a_n|^2(T+O(n)),$$

The result of this calculation was (1.4) (or (1.7)).

## 3 A modified approach

With $\sigma = \frac52$, (2.2) reads

$$\sum_{\gamma}\frac{4x^{i\gamma}}{4+(t-\gamma)^2}=-x^{-\frac{1}{2}}\Big(\sum_{n\le x}\Lambda(n)(\frac{x}{n})^{-\frac{3}{2}+it}+\sum_{n>x}\Lambda(n)(\frac{x}{n})^{\frac{5}{2}+it}\Big)$$
$$-\frac{\zeta'}{\zeta}(-\frac{3}{2}+it)x^{-2+it}+\frac{4x^{\frac{1}{2}}}{(\frac{3}{2}+it)(\frac{5}{2}-it)}-4x^{-\frac{1}{2}}\sum_{r=1}^{\infty}\frac{x^{-2r}}{(\frac{3}{2}-2r-it)(\frac{5}{2}+2r+it)}.$$

Using

$$\frac{\zeta'}{\zeta}(1-\sigma+it)=-\frac{\zeta'}{\zeta}(\sigma-it)+\log\pi+\frac{1}{(\sigma-it)(1-\sigma+it)}$$
$$-\frac{1}{2}\Big(\frac{\Gamma'}{\Gamma}(\frac{3-\sigma+it}{2})+\frac{\Gamma'}{\Gamma}(1+\frac{\sigma-it}{2})\Big) \tag{3.1}$$

which follows from the formulas in Chapter 12 of [2], with $\sigma=\frac{5}{2}$, we have

$$\sum_{\gamma}\frac{4x^{i(\gamma-t)}}{4+(\gamma-t)^2}=-x^{-2}\sum_{n\le x}\Lambda(n)n^{\frac{3}{2}-it}-x^{2}\sum_{n>x}\Lambda(n)n^{-\frac{5}{2}-it}$$
$$+x^{-2}\Big(\frac{\zeta'}{\zeta}(\frac{5}{2}-it)-\log\pi\Big)+\frac{x^{-2}}{2}\Big(\frac{\Gamma'}{\Gamma}(\frac{1}{4}+\frac{it}{2})+\frac{\Gamma'}{\Gamma}(\frac{9}{4}-\frac{it}{2})\Big) \tag{3.2}$$
$$+\frac{x^{-2}}{(\frac{5}{2}-it)(\frac{3}{2}-it)}+\frac{4x^{\frac{1}{2}-it}}{(\frac{3}{2}+it)(\frac{5}{2}-it)}-4x^{-\frac{1}{2}-it}\sum_{r=1}^{\infty}\frac{x^{-2r}}{(\frac{3}{2}-2r-it)(\frac{5}{2}+2r+it)}.$$

Using (1.15), and doing elementary estimations, we can simplify (3.2) into

$$\sum_{\gamma}\frac{4x^{i(\gamma-t)}}{4+(\gamma-t)^2}=-x^{-2}\sum_{n\le x}\Lambda(n)n^{\frac{3}{2}-it}-x^{2}\sum_{n>x}\Lambda(n)n^{-\frac{5}{2}-it} \tag{3.3}$$
$$+x^{-2}\big(\log(|t|+3)+O(1)\big)+O\Big(\frac{x^{\frac{1}{2}}}{(|t|+1)^2}\Big)+O\Big(\frac{x^{-\frac{5}{2}}}{|t|+1}\Big),\quad (x\ge 1).$$

When $t$ is let to run through a set we sum both sides of (3.3) over this set of values of $t$. This will be feasible if one can calculate the sums over $t$, in particular $\sum_{t}p^{-iat}$ where $p$ is a prime and $a$ is a natural number.

## 4 Pair correlation of zeta zeros

We apply our method first to the quantity (1.2) considered by Montgomery. So, letting $t$ run through those ordinates $\tilde{\gamma}$ of the zeros of the Riemann zeta-function which are in the interval $(0,T]$, from (3.3) we have

$$\sum_{\gamma}\sum_{0<\tilde{\gamma}\le T}\frac{4x^{i(\gamma-\tilde{\gamma})}}{4+(\gamma-\tilde{\gamma})^2} = -x^{-2}\sum_{n\le x}\Lambda(n)n^{\frac{3}{2}}\sum_{0<\tilde{\gamma}\le T}n^{-i\tilde{\gamma}} - x^2\sum_{n>x}\frac{\Lambda(n)}{n^{\frac{5}{2}}}\sum_{0<\tilde{\gamma}\le T}n^{-i\tilde{\gamma}}$$
$$+ x^{-2}\sum_{0<\tilde{\gamma}\le T}\big(\log\tilde{\gamma}+O(1)\big)+O\Big(x^{\frac{1}{2}}\sum_{0<\tilde{\gamma}\le T}\frac{1}{\tilde{\gamma}^2}\Big)+O\Big(x^{-\frac{5}{2}}\sum_{0<\tilde{\gamma}\le T}\frac{1}{\tilde{\gamma}}\Big) \qquad (4.1)$$

for $x \ge 1$. From the count of zeta zeros with ordinates in $(0,T]$ given in (1.1), we easily estimate the last terms of (4.1) and re-state (4.1) as

$$\sum_{\gamma}\sum_{0<\tilde{\gamma}\le T}\frac{4x^{i(\gamma-\tilde{\gamma})}}{4+(\gamma-\tilde{\gamma})^2} = - x^{-2}\sum_{n\le x}\Lambda(n)n^{\frac{3}{2}}\sum_{0<\tilde{\gamma}\le T}n^{-i\tilde{\gamma}} - x^2\sum_{n>x}\frac{\Lambda(n)}{n^{\frac{5}{2}}}\sum_{0<\tilde{\gamma}\le T}n^{-i\tilde{\gamma}}$$
$$+\frac{T\log^2 T}{2\pi x^2}\big(1+O(\frac{1}{\log T})\big)+O(x^{\frac{1}{2}}). \qquad (4.2)$$

In the sum on the left-hand side we can exclude those $\gamma \notin [0,T]$ within an error of $O(\log^3 T)$, by a standard calculation based upon (1.1).

We now have recourse to Gonek's [13] uniform version of Landau's formula

$$\sum_{0<\gamma\le T} y^{\rho} = -\frac{T}{2\pi}\Lambda(y)+O(y\log 2yT\log\log 3y) \qquad (4.3)$$
$$+O\big(\log y\min\big(T,\frac{y}{\langle y\rangle}\big)\big)+O\Big(\log 2T\min\big(T,\frac{1}{\log y}\big)\Big)$$

for $y, T > 1$, where $\rho$ denotes a complex zero of $\zeta(s)$ and $\langle y\rangle$ denotes the distance from $y$ to the nearest prime power other than $y$ itself. This formula is unconditional. Assuming RH and using (4.3) for the inner sum occurring in the first two terms of the right-hand side of (4.2), the contribution from the first term of the right-hand side of (4.3) is

$$\frac{T}{2\pi}\Big\{x^{-2}\sum_{n\le x}n\Lambda(n)^2+x^2\sum_{n>x}\frac{\Lambda(n)^2}{n^3}\Big\}$$
$$=\frac{T}{2\pi}\Big\{x^{-2}\big(\frac{x^2}{2}\log x-\frac{x^2}{4}+O(x^{\frac{3}{2}}\log^3 2x)\big)+x^2\big(\frac{\log x}{2x^2}+\frac{1}{4x^2}+O(x^{-\frac{5}{2}}\log^3 2x)\big)\Big\}$$
$$=\frac{T}{2\pi}\log x+O(Tx^{-\frac{1}{2}}\log^3 2x). \qquad (4.4)$$

Here we have calculated the sums from

$$\sum_{n\le x}\Lambda(n)=x+O(x^{\frac{1}{2}}\log^2 2x) \qquad (4.5)$$

which is an expression of the prime number theorem under RH. The first error term in (4.3) contributes

$$\ll x^{-2}\sum_{n\le x} n^2\Lambda(n)\log 2nT\log\log 3n + x^2\sum_{n>x}\frac{\Lambda(n)}{n^2}\log 2nT\log\log 3n$$
$$\ll x\log 2xT\log\log 3x. \tag{4.6}$$

The second error term in (4.3) contributes

$$\ll \frac{1}{x^2}\sum_{n\le x} n\Lambda(n)\log n\min(T,n) + x^2\sum_{n>x}\frac{\Lambda(n)}{n^3}\log n\min(T,n)$$
$$\ll \begin{cases} x\log 2x + \frac{x^2}{T}\log T & \text{if} \quad x\le T,\\ T\log x & \text{if} \quad x>T.\end{cases} \tag{4.7}$$

Finally the last error term of (4.3) contributes

$$\ll \frac{1}{x^2}\sum_{n\le x} n\Lambda(n)\frac{\log 2T}{\log n} + x^2\sum_{n>x}\frac{\Lambda(n)}{n^3}\frac{\log 2T}{\log n} \ll \frac{\log T}{\log 2x}. \tag{4.8}$$

Combining the above we find that

$$F_{\zeta,\zeta}(x,T) = \frac{T\log^2 T}{2\pi x^2}\big(1+O(\frac{1}{\log T})\big) + \frac{T}{2\pi}\log x \tag{4.9}$$
$$+ O(x\log(xT)\log\log 3x) + O\big(\frac{T\log^3 2x}{x^{\frac{1}{2}}}\big)$$

as $T\to\infty$. This gives an asymptotic result for $1\le x = o\big(\frac{T}{\log\log T}\big)$.

## 5 Preliminaries concerning the relative maxima of $|\zeta(\frac{1}{2}+it)|$

We shall be concerned with the function

$$Z_1(s) := \zeta'(s) - \frac{1}{2}\frac{\chi'}{\chi}(s)\zeta(s) \tag{5.1}$$

which was introduced by Conrey and Ghosh [1]. We easily see the relation

$$|Z_1(\frac{1}{2}+it)| = |Z'(t)| \tag{5.2}$$

with Hardy's $Z$-function

$$Z(t) = \big(\chi(\frac{1}{2}+it)\big)^{-\frac{1}{2}}\zeta(\frac{1}{2}+it), \tag{5.3}$$

which is real for real $t$ and satisfies $|Z(t)| = |\zeta(\frac{1}{2}+it)|$. Taking derivatives of both sides of (5.3), and observing that $\frac{\chi'}{\chi}(\frac{1}{2}+it)$ is a real number, we see that

$$\frac{Z'}{Z}(t) = \Im\,\frac{\zeta'}{\zeta}(\frac{1}{2} - it). \tag{5.4}$$

Taking imaginary parts in the partial fraction formula (from Chapter 12 of [2])

$$\frac{\zeta'}{\zeta}(s) = -1 - \frac{\gamma_e}{2} + \log 2\pi - \frac{1}{s-1} - \frac{1}{2}\frac{\Gamma'}{\Gamma}(\frac{s}{2}+1) + \sum_{\rho}\Big(\frac{1}{s-\rho} + \frac{1}{\rho}\Big) \tag{5.5}$$

(here $\gamma_e$ denotes Euler's constant), we have

$$\Im\,\frac{\zeta'}{\zeta}(\frac{1}{2} - it) = -\frac{t}{\frac{1}{4}+t^2} - \frac{1}{2}\,\Im\,\frac{\Gamma'}{\Gamma}(\frac{5}{4} - \frac{it}{2}) + \Im\sum_{\rho}\frac{1}{\frac{1}{2} - it - \rho}. \tag{5.6}$$

Now we assume RH, and use (1.15), to obtain

$$\frac{Z'}{Z}(t) = \Im\frac{\zeta'}{\zeta}(\frac{1}{2} - it) = \frac{\pi}{4} - O\big(\frac{1}{t}\big) + \sum_{\gamma>0}\frac{2t}{t^2-\gamma^2}. \tag{5.7}$$

This formula reveals that as $\frac{1}{2} + it$ crawls up on the critical line from a zero of $\zeta(s)$ to the next zero, there will be only one point where $Z'(t) = 0$. Hence we obtain that on RH, the zeros of $Z'(t)$ are interlaced with the zeros of $Z(t)$ (the case of possible multiple zeros of $Z(t)$ can also be included in this statement), and $|\zeta(\frac{1}{2} + it)|$ has exactly one maximum between consecutive zeta zeros in the upper half-plane (with the symmetrical configuration in the lower half-plane).

We shall need to know the zeros and poles of $Z_1(s)$, but before that let us consider the same for $\chi(s)$ and its derivative. From (1.12), since the $\Gamma$-function never vanishes, it is clear that the zeros of $\chi(s)$ can only be those zeros of $\sin(\frac{\pi s}{2})$ which are not cancelled by the poles of $\Gamma(1-s)$, i.e.

$$\{s;\, \chi(s) = 0\} = \{0, -2, -4, \ldots\}. \tag{5.8}$$

Similarly, we see

$$\{s;\, s \text{ is a pole of } \chi(s)\} = \{1, 3, 5, \ldots\}, \tag{5.9}$$

and that all zeros and poles of $\chi(s)$ are simple. By Rolle's theorem there are the real zeros of $\chi'(s)$ between the zeros of $\chi(s)$. A little computational examining using (1.14) reveals that

$$\begin{aligned}&\{s \in \mathbb{R};\, \chi'(s) = 0\} = \{\kappa_\ell, 1-\kappa_\ell; \ell \in \mathbb{Z}^+\},\ \kappa_1 \in (-\frac{1}{2}, 0),\ \kappa_2 \in (-3,-2),\\ &\kappa_3 \in (-5,-4),\ \kappa_\ell \in (-2\ell, -2\ell+1) \text{ for } \ell \geq 4,\ \kappa_\ell \to -2\ell \text{ as } \ell \to \infty.\end{aligned} \tag{5.10}$$

These zeros are all simple and the symmetry with respect to the point $\frac{1}{2}$ is due to

$$\frac{\chi'}{\chi}(s) = \frac{\chi'}{\chi}(1-s) \tag{5.11}$$

which is obvious from the functional equation of $\zeta(s)$. To look for non-real zeros of $\chi'(s)$, we use

$$\frac{\Gamma'}{\Gamma}(s) = -\frac{1}{s} - \gamma_e + \sum_{n=1}^{\infty}\Big(\frac{1}{n} - \frac{1}{s+n}\Big), \quad (-s \notin \mathbb{N}) \tag{5.12}$$

in

$$\frac{\chi'}{\chi}(s) = \log\pi - \frac{1}{2}\frac{\Gamma'}{\Gamma}(\frac{s}{2}) - \frac{1}{2}\frac{\Gamma'}{\Gamma}(\frac{1-s}{2}), \tag{5.13}$$

to see that

$$\Im\,\frac{\chi'}{\chi}(\sigma+it) = \frac{t}{4}\sum_{n=0}^{\infty}\Big(\frac{1}{(\frac{1-\sigma}{2}+n)^2+(\frac{t}{2})^2} - \frac{1}{(\frac{\sigma}{2}+n)^2+(\frac{t}{2})^2}\Big) \tag{5.14}$$

except at the poles. Thus the zeros of $\chi'(s)$ can occur either when $t = 0$ (and these have already been identified), or when $\sigma = \frac{1}{2}$. It is customary to write

$$\chi(\frac{1}{2}+it) = e^{-2i\vartheta(t)}, \quad (\vartheta(t)\in\mathbb{R}), \tag{5.15}$$

where

$$\vartheta(t) = \Im\log\Gamma(\frac{1}{4}+\frac{it}{2}) - \frac{t}{2}\log\pi = \frac{t}{2}\log(\frac{t}{2\pi}) - \frac{t}{2} - \frac{\pi}{8} + \frac{1}{48t} + \frac{7}{5760t^3} + \cdots \tag{5.16}$$

(see §6.5 of [4]). From (5.15), we see

$$\frac{\chi'}{\chi}(\frac{1}{2}+it) = -2\vartheta'(t)\in\mathbb{R}, \tag{5.17}$$

so that, by (5.12) and (5.13) we have

$$2\vartheta'(t) = -\Re\,\frac{\chi'}{\chi}(\frac{1}{2}+it) = -(\log\pi+\gamma_e) - \frac{1}{\frac{1}{4}+t^2} + \sum_{n=1}^{\infty}\Big(\frac{1}{n} - \frac{\frac{1}{4}+n}{(\frac{1}{4}+n)^2+(\frac{t}{2})^2}\Big). \tag{5.18}$$

This implies

$$2\vartheta''(t) = \sum_{n=0}^{\infty}\frac{(\frac{1}{4}+n)\frac{t}{2}}{\big((\frac{1}{4}+n)^2+(\frac{t}{2})^2\big)^2} > 0, \quad (t>0), \tag{5.19}$$

i.e. $\vartheta'(t)$ is strictly increasing for $t > 0$. We find from (1.14) that

$$2\vartheta'(0) = -\frac{\chi'}{\chi}(\frac{1}{2}) = -(\gamma_e + \log 4 + \frac{\pi}{2} + \log 2\pi) \approx -5.37\ldots, \tag{5.20}$$

and we know from §6.5 of [4] that

$$\vartheta'(t) > 0, \quad (t \geq 10). \tag{5.21}$$

We deduce that

$$\chi'(s) = 0,\ s \notin \mathbb{R} \implies s = \frac{1}{2} \pm i\tau, \text{for a unique } \tau \in (0, 10) \tag{5.22}$$

($\tau$ must be near $2\pi$, since from (5.16) we see $2\vartheta'(t) = \log(\frac{t}{2\pi}) + O(\frac{1}{t^2})$).

Hall's paper [14] contains thorough information about the zeros and poles of $Z_1(s)$ (Hall denotes our $Z_1(s)$ by $F(s)$). We see that

$$\{s;\, s \text{ is a pole of } Z_1(s)\} = \{0, 1, 3, 5, 7, \ldots\} \tag{5.23}$$

(the pole at $s = 1$ is a double pole coming from the simple poles of $\zeta(s)$ and of $\chi(s)$ at $s = 1$, the simple pole at $s = 0$ arises from $\chi(0) = 0$, and the simple poles at $s = 3, 5, 7, \ldots$ come from the poles of $\chi(s)$ at these points). Hall, assuming RH, showed that all non-real (which will be termed as non-trivial) zeros of $Z_1(s)$ are on the critical line, so that, assuming RH, the non-real points where $Z_1(s)$ vanishes are the relative maxima of $|\zeta(\frac{1}{2} + it)|$, and the multiple zeta zeros if ever these exist. Some information about these have already been given in the beginning of this section. Hall also proved assuming RH that the number of the zeros of $Z_1(s)$ (counted according to multiplicity) with ordinates in $(0, T]$, $N_{Z_1}(T)$, is

$$N_{Z_1}(T) = N_\zeta(T) - \frac{1}{2}\operatorname{sgn}\frac{Z'}{Z}(t) + \frac{3}{2}, \tag{5.24}$$

provided that $T$ is not the ordinate of a zero of $\zeta(s)$ or of $Z_1(s)$. There are real zeros of $Z_1(s)$, all of them simple zeros. These may be termed as the trivial zeros of $Z_1(s)$. The trivial zeros are placed symmetrically with respect to the point $\frac{1}{2}$ because of the functional equation

$$Z_1(s) = -\chi(s) Z_1(1 - s), \tag{5.25}$$

which follows from (5.1) and (1.11). We have

$$\begin{aligned} &\{s;\, s \text{ is a trivial zero of } Z_1(s)\} = \{z_0, z_\ell, 1 - z_\ell;\, \ell \in \mathbb{Z}^+\}, \\ &z_0 = \frac{1}{2},\ z_1 \in (3, 5), z_2 \in (5, 7),\ z_\ell \in (2\ell + 2, 2\ell + 3) \text{ for } \ell \geq 3. \end{aligned} \tag{5.26}$$

Since $Z_1(s)$ doesn't have an associated Dirichlet series, we will need a Dirichlet polynomial which approximates $\frac{Z_1'}{Z_1}(s)$ in $\sigma > 1$, $|t| \geq t_0$. From (5.1) we have

$$\frac{Z_1'}{Z_1}(s) = \frac{\frac{\zeta'}{\zeta}(s) - \frac{2}{\frac{\chi'}{\chi}(s)}\frac{\zeta''}{\zeta}(s) + \frac{\left(\frac{\chi'}{\chi}(s)\right)'}{\frac{\chi'}{\chi}(s)}}{1 - \frac{2}{\frac{\chi'}{\chi}(s)}\frac{\zeta'}{\zeta}(s)}. \tag{5.27}$$

Here we use the estimates (1.16), (1.17), and

$$\frac{\zeta^{(j)}}{\zeta}(s) \ll (\log\log|t|)^j, \quad (j = 1,2;\, \sigma \geq 1,\, |t| \geq t_0), \tag{5.28}$$

which depends on RH and follows from Corollary 13.14 of [19] by a standard application of Cauchy's estimate, so that we can write

$$\frac{Z_1'}{Z_1}(s) = \frac{\frac{\zeta'}{\zeta}(s) + \frac{2}{\log\frac{|t|}{2\pi}}\frac{\zeta''}{\zeta}(s)}{1 + \frac{2}{\log\frac{|t|}{2\pi}}\frac{\zeta'}{\zeta}(s)} + O\left(\frac{1}{|t|\log|t|}\right). \tag{5.29}$$

Expanding the denominator as geometric series, we see that

$$\frac{1}{1 + \frac{2}{\log\frac{|t|}{2\pi}}\frac{\zeta'}{\zeta}(s)} = \sum_{k=0}^{\lfloor \frac{\log|t|}{\log\log|t|} \rfloor} \left(-\frac{2}{\log\frac{|t|}{2\pi}}\frac{\zeta'}{\zeta}(s)\right)^k + O\left(|t|^{-1}\exp\left(\frac{C_1\log|t|\log\log\log|t|}{\log\log|t|}\right)\right). \tag{5.30}$$

Now, for $k \in \mathbb{N}$ and $\sigma > 1$, we write

$$\left(-\frac{\zeta'}{\zeta}(s)\right)^k =: \sum_{n=1}^{\infty}\frac{\Lambda^{(k)}(n)}{n^s}, \tag{5.31}$$

$$\left(-\frac{\zeta'}{\zeta}(s)\right)^k \frac{\zeta''}{\zeta}(s) =: \sum_{n=1}^{\infty}\frac{\lambda^{(k)}(n)}{n^s}. \tag{5.32}$$

Using (5.30) - (5.32) in (5.29) gives the following Dirichlet polynomial approximation (the series begins with $m = 2$ since $\Lambda^{*(k+1)}(1) = \lambda^{*(k)}(1) = 0$).

**Proposition 1** *Assume RH. In the region $1 < \sigma \leq \sigma_0$, $|t| \geq t_0$, we have*

$$\frac{Z_1'}{Z_1}(s) = \sum_{m=2}^{\infty}\frac{1}{m^s}\sum_{k=0}^{\lfloor \frac{\log|t|}{\log\log|t|} \rfloor}\left(\frac{2}{\log\frac{|t|}{2\pi}}\right)^k\left(\frac{2}{\log\frac{|t|}{2\pi}}\lambda^{(k)}(m) - \Lambda^{(k+1)}(m)\right) + O\left(|t|^{-1}\exp\left(\frac{C_1\log|t|\log\log\log|t|}{\log\log|t|}\right)\right). \tag{5.33}$$

*Remark:* We were led to the approximation (5.33) because the idea of replacing $-\frac{\chi'}{\chi}(s)$ by $\log\frac{T}{2\pi}$ in (5.1) and then developing an approximating Dirichlet series, as was done in [1], ends up giving a weaker (valid only for $x$ up to $T^{\frac{2}{3}-\epsilon}$) result than Theorem 1 in §7. Unconditionally, we have $\frac{\zeta'}{\zeta}(s) \ll_j (\log t)^{\frac{2}{3}}(\log\log|t|)^{\frac{1}{3}}$, $(\sigma \geq 1, |t| \geq t_0)$ at our disposal (see [21], §6.19), causing the error term in (5.33) to become $O(|t|^{-\frac{1}{3}+\epsilon})$, and this in turn also leads to a smaller range in Theorem 1. Anyhow, results of the type we are pursuing make more sense when RH is assumed.

## 6 Preliminaries concerning the iterated convolutions of the von Mangoldt function

We shall need in §7 and §8 some properties of the arithmetic functions $\Lambda^{(k)}(n)$ and $\lambda^{(k)}(n)$. We note that the article [5] by Farmer, Gonek and Lee contains some related results. First, proceeding inductively and denoting Dirichlet convolution by $*$, we see the trivial bound, for $k \geq 1$,

$$\Lambda^{(k)}(n) = \Lambda^{(k-1)} * \Lambda(n) = \sum_{d|n} \Lambda(d)\Lambda^{(k-1)}\left(\frac{n}{d}\right) \leq (\log n)^{k-1} \sum_{d|n} \Lambda(d) = (\log n)^k. \tag{6.1}$$

Next, using (6.1), we see that

$$\lambda^{(k)}(n) \leq (\log n)^{k+2}, \qquad (k = 0, 1, 2, \ldots), \tag{6.2}$$

by dint of

$$\lambda^{(k)} = \Lambda^{(k)} * \Lambda_2, \text{ where } \frac{\zeta''}{\zeta}(s) =: \sum_{n=1}^{\infty} \frac{\Lambda_2(n)}{n^s} \ (\Re s > 1), \text{ and } \sum_{d|n} \Lambda_2(d) = (\log n)^2. \tag{6.3}$$

Note that by comparing coefficients in $\frac{\zeta''}{\zeta}(s) = \left(\frac{\zeta'}{\zeta}(s)\right)' + \left(\frac{\zeta'}{\zeta}(s)\right)^2$ we have

$$\lambda^{(0)}(n) = \Lambda_2(n) = \Lambda(n)\log n + \Lambda^{(2)}(n) = \Lambda(n)\log n + \sum_{d|n} \Lambda(d)\Lambda(\frac{n}{d}). \tag{6.4}$$

It is easy to calculate that for $a \in \mathbb{Z}^+$ and $p$ a prime number,

$$\Lambda_2(p^a) = (2a-1)(\log p)^2 \tag{6.5}$$

$$\Lambda^{(k)}(p^a) = \binom{a-1}{k-1}(\log p)^k, \tag{6.6}$$

$$\lambda^{(k)}(p^a) = \left[2\binom{a}{k+1} - \binom{a-1}{k}\right](\log p)^{k+2}. \tag{6.7}$$

Here we are employing a convention that for $u$, $v$ integers which are $\geq -1$,

$$\binom{r}{s} := \begin{cases} 1 & \text{if} \quad r = s, \\ 0 & \text{if} \quad s = -1 \text{ and } r \geq 0, \\ 0 & \text{if} \quad r < s, \\ \dfrac{r!}{s!(r-s)!} & \text{otherwise.} \end{cases} \tag{6.8}$$

Now we give two lemmas which will be used in §7.

**Lemma 1** *Let $k \in \mathbb{Z}^+$. We have, for $N \geq 1$,*

$$\sum_{n \leq \frac{N}{2}} \frac{\Lambda^{(k)}(N-n)}{n} \ll (\log 2N)^k \frac{(\log\log 3N + 7)_k}{(k-1)!}. \tag{6.9}$$

*The same bound holds with $\Lambda^{(k)}(N+n)$ in place of $\Lambda^{(k)}(N-n)$. Here*

$$(K)_k := \prod_{j=0}^{k-1} (K+j), \tag{6.10}$$

*with the convention $(K)_0 := 1$.*

The $k = 1$ case of this lemma was shown by Gonek [13] within his proof of (4.3).
*Proof:* Let

$$S(x) := \left| \left\{ N - x < n \leq N : \Lambda^{(k)}(n) \neq 0 \right\} \right|.$$

By (6.1) we can write

$$\sum_{n \leq \frac{N}{2}} \frac{\Lambda^{(k)}(N-n)}{n} \leq (\log 2N)^k \int_{1^-}^{\frac{N}{2}} \frac{dS(x)}{x} = (\log 2N)^k \left\{ \frac{S(x)}{x} \Big|_{1^-}^{\frac{N}{2}} + \int_{1^-}^{\frac{N}{2}} \frac{S(x)}{x^2} dx \right\}$$

We now need an upper bound for $S(x)$. To this end we quote the following result of Tudesq [22]:

$$\sum_{\substack{y < n \leq y+x \\ \omega(n) = j}} 1 \ll \frac{x}{\log 2x} \frac{(\log\log 3x + 6)_{j-1}}{(j-1)!},$$

where $1 \leq x \leq y$, $j \in \mathbb{Z}^+$ and the constant implied by $\ll$ is absolute. Employing this we have

$$S(x) \leq \sum_{j=1}^{k} \sum_{\substack{N - x < n \leq N \\ \omega(n) = j}} 1 \ll \frac{x}{\log 2x} \sum_{j=1}^{k} \frac{(\log\log 3x + 6)_{j-1}}{(j-1)!} = \frac{x}{\log 2x} \frac{(\log\log 3x + 7)_{k-1}}{(k-1)!},$$

since

$$\sum_{j=0}^{k} \frac{(r)_j}{j!} = \frac{(r+1)_k}{k!}, \quad (k \in \mathbb{N}, r \in \mathbb{R}). \tag{6.11}$$

This upper bound for $S(x)$ gives the claimed result. The sum having $\Lambda^{(k)}(N+n)$ in its summand can be treated in the same manner. □

By the same line of reasoning we have

**Lemma 2** *Let $k \in \mathbb{N}$. We have for $N \geq 1$*

$$\sum_{n \leq \frac{N}{2}} \frac{\lambda^{(k)}(N-n)}{n} \ll (\log 2N)^{k+2} \frac{(\log\log 3N+7)_{k+2}}{(k+1)!}. \tag{6.12}$$

*The same bound holds with $\lambda^{(k)}(N+n)$ in place of $\lambda^{(k)}(N-n)$.*

The rest of this section will be used in §8. We define

$$\sum_{n=1}^{\infty} \frac{\Delta(k,\iota;n)}{n^s} := \left(-\frac{\zeta'}{\zeta}(s)\right)^k \left(\frac{\zeta''}{\zeta}(s)\right)^{\iota}, \quad \iota = 0 \text{ or } 1 \tag{6.13}$$

so that (cf. (5.31)-(5.32))

$$\Delta(k,0;n) = \Lambda^{(k)}(n), \quad \Delta(k,1;n) = \lambda^{(k)}(n). \tag{6.14}$$

We are going to study the sum $\sum_{n \leq x} \Delta(k_1,\iota_1,n)\Delta(k_2,\iota_2,n)$. The result will be Proposition 2, and its corollary will be needed in §8. We begin with the following Lemmas.

**Lemma 3** *For prime $p$ and $k \in \mathbb{N}$, we have*

$$\Delta(k,\iota;pn) = \begin{cases} k(\log p)\Delta(k-1,\iota;n) & \\ \quad +[\iota=1](\log p)\,[2\Delta(k+1,0;n) + (\log p)\Delta(k,0;n)] & \text{if } (p,n)=1, \\ O\left((k+1)(\log p)(\log pn)^{k+2\iota-1}\right) & \text{if } p|n. \end{cases} \tag{6.15}$$

*Proof:* The $k=0$ case can be verified easily, so let $k \geq 1$. First take $\iota = 0$. For $(p,m)=1$, by (6.14) and (6.6) we have

$$\begin{aligned} \Delta(k,0;p^a m) = \Lambda^{(k)}(p^a m) &= \sum_{r=1}^{\min(a,k)} \binom{k}{r} \Lambda^{(r)}(p^a)\Lambda^{(k-r)}(m) \\ &= \sum_{r=1}^{k} \binom{k}{r}\binom{a-1}{r-1} (\log p)^r \Lambda^{(k-r)}(m). \end{aligned} \tag{6.16}$$

With $a=1$ this reads, for $(p,n)=1$,

$$\Delta(k,0;pn) = \Lambda^{(k)}(pn) = k(\log p)\Lambda^{(k-1)}(n) = k(\log p)\Delta(k-1,0;n)$$

which is the statement of (6.15) for this case. Using $\binom{a-1}{r-1} \le \frac{a^{r-1}}{(r-1)!}$ in (6.16), by (6.1) we have

$$\Delta(k,0;p^a m) = \Lambda^{(k)}(p^a m) \le \sum_{r=1}^{k} \frac{k}{r}\binom{k-1}{r-1}\frac{a^{r-1}}{(r-1)!}(\log p)^r(\log m)^{k-r}$$
$$\le k(\log p)\sum_{r=1}^{k}\binom{k-1}{r-1}(\log p^a)^{r-1}(\log m)^{(k-1)-(r-1)} = k(\log p)(\log p^a m)^{k-1}.$$

This completes the case $\iota = 0$ and $p \mid n$.

Now we consider the case $\iota = 1$. For $(p,n) = 1$, we have

$$\Delta(k,1;pn) = \lambda^{(k)}(pn) = \sum_{d|pn}\Lambda^{(k)}(\frac{pn}{d})\Lambda_2(d) = \sum_{d|n}\Lambda^{(k)}(\frac{pn}{d})\Lambda_2(d) + \sum_{d|n}\Lambda^{(k)}(\frac{n}{d})\Lambda_2(pd)$$

In the last member of this equation, applying the already proved $\iota = 0$ case to $\Lambda^{(k)}$ in the first sum, and using (6.4) in the second sum we obtain

$$\Delta(k,\iota;pn) = k(\log p)\sum_{d|n}\Lambda^{(k-1)}(\frac{n}{d})\Lambda_2(d) + 2(\log p)\sum_{\substack{d|n\\ d>1}}\Lambda^{(k)}(\frac{n}{d})\Lambda(d) + (\log p)^2\Lambda^{(k)}(n),$$

which gives the statement of (6.15) for this case.

For the last case with $p \mid n$, let $n = p^r n'$, $(n',p) = 1$, $r \in \mathbb{Z}^+$. Observe that

$$\Delta(k,1;pn) = \lambda^{(k)}(pn) = \sum_{d|pn}\Lambda^{(k)}(\frac{pn}{d})\Lambda_2(d)$$
$$= \sum_{d|n}\Lambda^{(k)}(\frac{pn}{d})\Lambda_2(d) + \sum_{\substack{d'|n'\\ d'>1}}\Lambda^{(k)}(\frac{n'}{d'})\Lambda_2(p^{r+1}d') + \Lambda^{(k)}(n')\Lambda_2(p^{r+1}).$$

Using (6.1), (6.5),

$$\Lambda_2(p^{r+1}d') = 2(\log p)\Lambda(d'), \quad (d' > 1, (d',p) = 1) \tag{6.17}$$

which follows from (6.4), and the case $\iota = 0$, we obtain

$$\Delta(k,1;pn) \le k(\log p)(\log pn)^{k-1}\sum_{d|n}\Lambda_2(d)$$
$$+ 2(\log p)(\log n')^k\sum_{d'|n'}\Lambda(d') + (2r+1)(\log p)^2(\log n')^k.$$

Now the assertion in the case $\iota = 1$ and $(p,n) > 1$ is immediately seen. □

We now begin examining the sum $\sum_{n\le \frac{x}{b}} \Delta(k,\iota;bn)$, where $b$ is either 1 or a prime number $p \le x$ (clearly, the sum is void if $b > x$), for $k \le \frac{\log x}{\log 2}$ (if $k > \frac{\log x}{\log 2}$, then $\Delta(k,\iota;bn) = 0$). The results will form

**Lemma 4** *Let $b$ be 1 or any prime number $p \le x$. For $1 + [b \neq 1] \le k + \iota \le \frac{\log x}{\log 2}$, there exists a constant $C_2 > 1$ such that*

$$\sum_{n\le \frac{x}{b}} \Delta(k,\iota;bn) \le \frac{C_2^k}{(k+2\iota-1)!}\frac{x}{b}(\log x)^{k+2\iota-1}. \tag{6.18}$$

*Proof:* First we take $b = 1$. We have

$$\sum_{n\le x} \Delta(k,\iota;n) \sim \frac{2^\iota x(\log x)^{k+2\iota-1}}{(k+2\iota-1)!}, (x\to\infty, \text{any fixed } k\in\mathbb{N}, \iota\in\{0,1\}, k+\iota\ge 1) \tag{6.19}$$

as a generalization of the prime number theorem, since

$$\mathrm{Res}_{s=1}\left[\left(-\frac{\zeta'}{\zeta}(s)\right)^k\left(\frac{\zeta''}{\zeta}(s)\right)^\iota \frac{x^s}{s}\right] \sim \frac{2^\iota x(\log x)^{k+2\iota-1}}{(k+2\iota-1)!}, (k\in\mathbb{N}, \iota\in\{0,1\}, k+\iota\ge 1) \tag{6.20}$$

(the details of the proof of (6.19) are similar to the zeta-function based proof of the prime number theorem as in [2]). However, we will also need to work with $k$ not fixed. We want to show that

$$\sum_{n\le x} \Delta(k,\iota;n) \le \frac{C_3^k}{(k+2\iota-1)!} x(\log x)^{k+2\iota-1} \tag{6.21}$$

with some absolute constant $C_3 > 1$ by doing induction on $k$. The induction basis cases are covered in (6.19). Starting from the definition in (6.13) we see that

$$\begin{aligned}\sum_{n\le x}\Delta(k+1,\iota;n) &= \sum_{n\le x}\sum_{d|n}\Delta(k,\iota;\frac{n}{d})\Lambda(d) = \sum_{d\le x}\Lambda(d)\sum_{e\le \frac{x}{d}}\Delta(k,\iota;e)\\ &\le \frac{C_3^k x}{(k+2\iota-1)!}\sum_{d\le x}\frac{\Lambda(d)}{d}\left(\log\frac{x}{d}\right)^{k+2\iota-1} \le \frac{C_3^{k+1}}{(k+2\iota)!}x(\log x)^{k+2\iota},\end{aligned} \tag{6.22}$$

where we have used

$$\sum_{n\le x}\frac{\Lambda(n)}{n}\left(\log\frac{x}{n}\right)^j = \frac{(\log x)^{j+1}}{j+1} + O\left((\log x)^j\right), \quad (j\in\mathbb{N}) \tag{6.23}$$

which can be seen from Mertens's classical result via partial summation (this result is included in Lemma 5.5 of [5]). This proves (6.21).

Next, taking $b = p$ a prime number, we examine the sum $\sum_{n\le\frac{x}{p}} \Delta(k,\iota;pn)$ Let $\ell$ be the unique integer such that $p^\ell \le x < p^{\ell+1}$. First we examine the cases $k+\iota = 1$ which turn out to be special. For $(k,\iota) = (1,0)$, the sum is $\sum_{n\le\frac{x}{p}} \Lambda(pn)$. The non-zero summands are from $n = 1, p, p^2, \ldots, p^{\ell-1}$, and the sum has value $\ell \log p \in (\frac{1}{2}\log x, \log x]$, so that

$$\sum_{n\le\frac{x}{p}} \Delta(1,0;pn) = \sum_{n\le\frac{x}{p}} \Lambda(pn) \le \log x. \tag{6.24}$$

For $(k,\iota) = (0,1)$, the sum is $\sum_{n\le\frac{x}{p}} \Lambda_2(pn)$. The non-zero summands are from $n = 1, p, p^2, \ldots, p^{\ell-1}$ with total contribution $\ell^2(\log p)^2 \in (\frac{1}{4}(\log x)^2, (\log x)^2]$, and from $n = p^r p_1^{r_1} \le \dfrac{x}{p}$ with a prime $p_1 \ne p$ and $r \ge 0, r_1 \ge 1$. By

$$\Lambda_2(p_1^{a_1} p_2^{a_2}) = 2\log p_1 \log p_2, \quad (a_1, a_2 \ge 1), \tag{6.25}$$

which is seen from (6.4), and

$$\theta(x) := \sum_{\substack{p\le x\\ p:\,\text{prime}}} \log p \le C_4 x, \quad (x > 0) \tag{6.26}$$

(with $C_4 = 1 + \frac{1}{36260}$ according to Dusart [3]), we have

$$\begin{aligned}\sum_{\substack{p^r p_1^{r_1} \le \frac{x}{p}\\ r\ge 0, r_1\ge 1}} \Lambda_2(p^{r+1} p_1^{r_1}) &\le 2\log p \sum_{r\ge 0}\sum_{r_1\ge 1} \sum_{p_1 \le \left(\frac{x}{p^{r+1}}\right)^{\frac{1}{r_1}}} \log p_1 \\ &\le 2C_4 \log p \sum_{r\ge 0}\sum_{r_1\ge 1} \left(\frac{x}{p^{r+1}}\right)^{\frac{1}{r_1}} \le C_5 \log p \sum_{r\ge 0} \frac{x}{p^{r+1}} \le C_6 \frac{x}{p}\log p, \end{aligned} \tag{6.27}$$

where we have made use of $\left(\frac{x}{p^{r+1}}\right)^{\frac{1}{r_1}} \ge 2$ for non-void sums so that $r_1$ can be at most $\frac{\log \frac{x}{p^{r+1}}}{\log 2}$. We have chosen to use the majorization (6.26) because if we try to evaluate all of the sums asymptotically we will get quantities containing the factor $\log\frac{x}{p}$ and this will necessitate examining the results according to the size of $p$ relative to $x$ and thus further complicate the matter. So, we express the result for this case as

$$\sum_{n\le\frac{x}{p}} \Delta(0,1;pn) = \sum_{n\le\frac{x}{p}} \Lambda_2(pn) \le C_7 \frac{x}{p}\log p + (\log x)^2. \tag{6.28}$$

We wish to obtain an inequality of the type (6.21) by induction on $k+\iota$, but from (6.24) and (6.28) we see that in the $k+\iota = 1$ cases the power of $\log x$ won't match

the power in (6.21) if $\frac{x}{\log x} = o(p)$, $p \leq x$. So, for the induction basis we take $k + \iota = 2$ which has the two cases $(k, \iota) = (2, 0)$ and $(k, \iota) = (1, 1)$.
For $(k, \iota) = (2, 0)$, the sum is $\sum_{n \leq \frac{x}{p}} \Lambda^{(2)}(pn)$, differing from $\sum_{n \leq \frac{x}{p}} \Lambda_2(pn)$ only in the contribution from $pn = p, p^2, \ldots, p^\ell$. This contribution, $\frac{\ell(\ell-1)}{2}(\log p)^2$, vanishes for $p > \sqrt{x}$. Hence, from (6.28) we have

$$\sum_{n \leq \frac{x}{p}} \Delta(2, 0; pn) = \sum_{n \leq \frac{x}{p}} \Lambda^{(2)}(pn) \leq C_8 \frac{x}{p} \log x. \tag{6.29}$$

Now take $(k, \iota) = (1, 1)$ in which case the sum is $\sum_{n \leq \frac{x}{p}} \lambda^{(1)}(pn)$. The non-zero summands are from $n = p, p^2, \ldots, p^{\ell-1}$ with total contribution $\frac{\ell(\ell-1)(2\ell-1)}{6}(\log p)^3$ by (6.7), and from $n = p^r p_1^{r_1}, (r \geq 0, r_1 \geq 1)$ and $n = p^r p_1^{r_1} p_2^{r_2}, (r \geq 0, r_1, r_2 \geq 1)$ where $p$, $p_1$ and $p_2$ are distinct primes. We have, by (6.3), (6.5) and (6.25),

$$\lambda^{(1)}(p^{r+1} p_1^{r_1}) = (4r + 1)(\log p)^2(\log p_1) + (4r_1 - 3)(\log p_1)^2(\log p) \tag{6.30}$$

$$\lambda^{(1)}(p^{r+1} p_1^{r_1} p_2^{r_2}) = 6(\log p)(\log p_1)(\log p_2). \tag{6.31}$$

Then, estimating similarly to (6.27), we have

$$\begin{aligned}
\sum_{\substack{p^r p_1^{r_1} \leq \frac{x}{p} \\ r \geq 0, r_1 \geq 1}} \lambda^{(1)}(p^{r+1} p_1^{r_1}) &\leq (\log p)^2 \sum_{r \geq 0} (4r + 1) \sum_{r \geq 1} \sum_{p_1 \leq \left(\frac{x}{p^{r+1}}\right)^{\frac{1}{r_1}}} \log p_1 \\
&\quad + (\log p) \sum_{r \geq 0} \sum_{r_1 \geq 1} (4r_1 - 3) \sum_{p_1 \leq \left(\frac{x}{p^{r+1}}\right)^{\frac{1}{r_1}}} (\log p_1)^2 \\
&\leq C_4 (\log p)^2 \sum_{r \geq 0} (4r + 1) \sum_{r \geq 1} \left(\frac{x}{p^{r+1}}\right)^{\frac{1}{r_1}} \\
&\quad + C_4 (\log p) \sum_{r \geq 0} \sum_{r_1 \geq 1} (4r_1 - 3) \left(\frac{x}{p^{r+1}}\right)^{\frac{1}{r_1}} \log \left(\frac{x}{p^{r+1}}\right)^{\frac{1}{r_1}} \\
&\leq C_9 (\log p)^2 \sum_{r \geq 0} (4r + 1) \frac{x}{p^{r+1}} + C_{10} (\log p)(\log x) \sum_{r \geq 0} \sum_{r_1 \geq 1} \frac{4r_1 - 3}{r_1} \left(\frac{x}{p^{r+1}}\right)^{\frac{1}{r_1}} \\
&\leq C_{11} \frac{x}{p} (\log p)(\log x),
\end{aligned}$$

and

$$\sum_{\substack{p^r p_1^{r_1} p_2^{r_2} \le \frac{x}{p} \\ r\ge 0, r_1, r_2 \ge 1}} \lambda^{(1)}(p^{r+1}p_1^{r_1}p_2^{r_2}) \le 3(\log p) \sum_{\substack{r\ge 0\\ r_1\ge 1}} \sum_{p_1 \le \left(\frac{x}{2p^{r+1}}\right)^{\frac{1}{r_1}}} \log p_1 \sum_{r_2\ge 1} \sum_{p_2 \le \left(\frac{x}{p^{r+1}p_1^{r_1}}\right)^{\frac{1}{r_2}}} \log p_2$$
$$\le C_{12}(\log p) \sum_{\substack{r\ge 0\\ r_1\ge 1}} \sum_{p_1 \le \left(\frac{x}{2p^{r+1}}\right)^{\frac{1}{r_1}}} \log p_1 \sum_{r_2\ge 1} \left(\frac{x}{p^{r+1}p_1^{r_1}}\right)^{\frac{1}{r_2}}$$
$$\le C_{13} x(\log p) \sum_{r\ge 0} \frac{1}{p^{r+1}} \sum_{r_1\ge 1} \sum_{p_1 \le \left(\frac{x}{2p^{r+1}}\right)^{\frac{1}{r_1}}} \frac{\log p_1}{p_1^{r_1}}$$
$$\le C_{14} x(\log p) \sum_{r\ge 0} \frac{1}{p^{r+1}} \sum_{p_1 \le \frac{x}{2p^{r+1}}} (\log p_1) \sum_{r_1\ge 1} \frac{1}{p_1^{r_1}} \le C_{15} \frac{x}{p} (\log p)(\log x).$$

Putting together these calculations, we have for the case $(k,\iota) = (1,1)$,

$$\sum_{n\le \frac{x}{p}} \Delta(1,1;pn) = \sum_{n\le \frac{x}{p}} \lambda^{(1)}(pn) \le C_{16} \frac{x}{p} (\log x)^2. \tag{6.32}$$

Formulas (6.29) and (6.32) are of the same type as (6.21), and this furnishes the induction basis for establishing

$$\sum_{n\le \frac{x}{p}} \Delta(k,\iota;pn) \le \frac{C_{17}^k}{(k+2\iota-1)!} \frac{x}{p} (\log x)^{k+2\iota-1}. \tag{6.33}$$

with some absolute constant $C_{17} > 1$. Letting $pn = de$, we have

$$\begin{aligned}\sum_{n\le \frac{x}{p}} \Delta(k+1,\iota;pn) &= \sum_{n\le \frac{x}{p}} \sum_{d|pn} \Delta(k,\iota;\frac{pn}{d})\Lambda(d)\\ &= \sum_{\substack{d\le \frac{x}{p}\\ (d,p)=1}} \Lambda(d) \sum_{\substack{e\le \frac{x}{d}\\ p|e}} \Delta(k,\iota;e) + \sum_{\substack{d\le x\\ p|d}} \Lambda(d) \sum_{e\le \frac{x}{d}} \Delta(k,\iota;e)\\ &= \sum_{\substack{d\le \frac{x}{p}\\ (d,p)=1}} \Lambda(d) \sum_{e'\le \frac{x}{pd}} \Delta(k,\iota;pe') + \sum_{d'\le \frac{x}{p}} \Lambda(pd') \sum_{e\le \frac{x}{pd'}} \Delta(k,\iota;e).\end{aligned} \tag{6.34}$$

The last two double sums can be majorized by using the induction hypothesis and (6.23). We have

$$\sum_{\substack{d\le \frac{x}{p}\\(d,p)=1}} \Lambda(d) \sum_{e'\le \frac{x}{pd}} \Delta(k,\iota;pe') \le \frac{C_{17}^k}{(k+2\iota-1)!}\frac{x}{p}\sum_{d\le\frac{x}{p}}\frac{\Lambda(d)}{d}\left(\log\frac{x}{d}\right)^{k+2\iota-1}$$

$$\le \frac{C_{17}^k}{(k+2\iota-1)!}\frac{x}{p}\left(\frac{(\log x)^{j+2\iota}}{(k+2\iota)}+O((\log x)^{k+2\iota-1})\right),$$

and, since $\Lambda(pd')\le\Lambda(d')$,

$$\sum_{d'\le\frac{x}{p}}\Lambda(pd')\sum_{e\le\frac{x}{pd'}}\Delta(k,\iota;e)\le\frac{C_{17}^k}{(k+2\iota-1)!}\frac{x}{p}\sum_{d'\le\frac{x}{p}}\frac{\Lambda(pd')}{d'}\left(\log\frac{x}{pd'}\right)^{k+2\iota-1}$$

$$\le \frac{C_{17}^k}{(k+2\iota-1)!}\frac{x}{p}\left(\frac{(\log x)^{k+2\iota}}{(k+2\iota)}+O((\log x)^{k+2\iota-1})\right).$$

Using these in (6.34), the induction step is completed. □

Next, we prove

**Proposition 2** *For* $0\le k_1,k_2\le\frac{\log x}{\log 2}$, $\iota_1,\iota_2=0$ *or* $1$, $k_1+\iota_1$, $k_2+\iota_2\ge 1$, *we have*

$$\begin{aligned}S_{k_1,k_2,\iota_1,\iota_2}(x) &:= \sum_{n\le x}\Delta(k_1,\iota_1;n)\Delta(k_2,\iota_2;n)\\ &= \frac{P(k_1,k_2,\iota_1,\iota_2)}{(k_1+k_2+2\iota_1+2\iota_2-1)!}x(\log x)^{k_1+k_2+2\iota_1+2\iota_2-1}\\ &\quad + O\left(\frac{C_{18}^{k_1+k_2}x(\log x)^{k_1+k_2+2\iota_1+2\iota_2-2}}{(\max\{k_1,k_2\})!}\right),\end{aligned} \tag{6.35}$$

*where*

$$P(k_1,k_2,\iota_1,\iota_2) := \begin{cases} k_1! & \text{if } \iota_1=\iota_2=0,\ k_1=k_2,\\ 2\cdot(\max\{k_1,k_2\})! & \text{if } \iota_1-\iota_2=\pm1,\ k_1-k_2=\mp1,\\ (\max\{k_1,k_2\})! & \text{if } \iota_1-\iota_2=\pm1,\ k_1-k_2=\mp2,\\ (k_1+2)!+4\cdot(k_1+1)!+2\cdot k_1! & \text{if } \iota_1=\iota_2=1,\ k_1=k_2,\\ 2\cdot(\max\{k_1,k_2\}+1)! & \text{if } \iota_1=\iota_2=1,\ k_1-k_2=\pm1,\\ 0 & \text{otherwise}.\end{cases} \tag{6.36}$$

*Proof:* We will do induction on $k_1+k_2+\iota_1+\iota_2$, for which the least value is 2, and there are three cases of this to be considered for establishing the induction basis. We have

$$S_{1,1,0,0}(x)=\sum_{n\le x}\Lambda(n)^2=x\log x+O(x), \tag{6.37}$$

$$S_{1,0,0,1}(x)=S_{0,1,1,0}(x)=\sum_{n\le x}\Lambda(n)\Lambda_2(n) \tag{6.38}$$
$$=\sum_{n\le x}\Big[\Lambda(n)^2\log n+\Lambda(n)\sum_{d|n}\Lambda(d)\Lambda(\frac{n}{d})\Big]=x(\log x)^2+O(x\log x),$$

since the first term in brackets contributes $x(\log x)^2+O(x\log x)$, and in the second term only proper prime power values of $n$ contribute and this contribution is $O(x^{\frac{1}{2}+\epsilon})$. The final case for the induction basis is

$$S_{0,0,1,1}(x)=\sum_{n\le x}\lambda^{(0)}(n)^2=\sum_{n\le x}\Lambda_2(n)^2$$
$$=\sum_{n\le x}\Big[\Lambda(n)^2(\log n)^2+2\Lambda(n)\log n\sum_{d|n}\Lambda(d)\Lambda(\frac{n}{d})+\Big(\sum_{d|n}\Lambda(d)\Lambda(\frac{n}{d})\Big)^2\Big].$$

The first term in the summand leads to $x(\log x)^3+O(x(\log x)^2)$. In the second term of the summand, only proper prime power $n$ contribute, so that this term leads to $O(x^{\frac{1}{2}+\epsilon})$. The contribution of proper prime power $n$ in the third term is bounded by the same error term. The third term is 0 if $n$ has more than two distinct prime factors, so it remains to consider the contribution of $n$ having exactly two distinct prime factors in the third term. The set of $n=p_1p_2\le x$, $p_1<p_2$ contributes to the third term (we need not exclude the cases $p_1=p_2$ since these give a negligible contribution, and we take all pairs of $p_1$ and $p_2$ and then halve the sum; we will use this setting also in other cases below)

$$\frac{1}{2}\sum_{p_1p_2\le x}(2\log p_1\log p_2)^2=2\sum_{p_1\le\frac{x}{2}}(\log p_1)^2\sum_{p_2\le\frac{x}{p_1}}(\log p_2)^2$$
$$=2\sum_{p_1\le\frac{x}{2}}(\log p_1)^2\Big(\frac{x}{p_1}\log(\frac{x}{p_1})+O(\frac{x}{p_1})\Big)$$
$$=2x\log x\sum_{p_1\le\frac{x}{2}}\frac{(\log p_1)^2}{p_1}-2x\sum_{p_1\le\frac{x}{2}}\frac{(\log p_1)^3}{p_1}+O(x(\log x)^2)$$
$$=2x\log x\big(\frac{1}{2}(\log x)^2+O(\log x)\big)-2x\Big(\frac{1}{3}(\log x)^3+O\big((\log x)^2\big)\Big)$$
$$=\frac{1}{3}x(\log x)^3+O(x(\log x)^2).$$

The third term also gets contributions from $n=p_1^{r_1}p_2^{r_2}, p_1\ne p_2, r_1\ge 1, r_2\ge 2$, but

$$\sum_{r_1\ge 1}\sum_{p_1^{r_1}\le \frac{x}{4}}(\log p_1)^2\sum_{r_2\ge 2}\sum_{p_2\le \left(\frac{x}{p_1^{r_1}}\right)^{\frac{1}{r_2}}}(\log p_2)^2 \ll \sum_{r_1\ge 1}\sum_{p_1^{r_1}\le \frac{x}{4}}(\log p_1)^2\sum_{r_2\ge 2}\left(\frac{x}{p_1^{r_1}}\right)^{\frac{1}{r_2}}\log x$$

$$\ll x^{\frac{1}{2}}\log x\sum_{r_1\ge 1}\sum_{p_1^{r_1}\le \frac{x}{4}}\frac{(\log p_1)^2}{p_1^{\frac{r_1}{2}}} = x^{\frac{1}{2}}\log x\sum_{p_1\le \frac{x}{4}}\frac{(\log p_1)^2}{p_1^{\frac{1}{2}}} \ll x(\log x)^2.$$

Hence

$$S_{0,0,1,1}(x) = \frac{4}{3}x(\log x)^3 + O(x\log^2 x). \tag{6.39}$$

The formulas (6.37), (6.38) and (6.39) obey (6.35) and (the first, the second and the fourth cases, respectively, of) (6.36), and so the induction basis is established.

In dealing with the cases $k_1 + k_2 + \iota_1 + \iota_2 \ge 3$, we will first consider the cases when one of $k_1$ and $k_2$ is 0, say $k_1 = 0$ so that $\iota_1 = 1$. There are four such cases: $S_{0,1,1,1}(x), S_{0,2,1,0}(x), S_{0,k_2,1,0}$ with $k_2 \ge 3$, and $S_{0,k_2,1,1}(x)$ with $k_2 \ge 2$.

We begin by considering

$$S_{0,1,1,1}(x) = \sum_{n\le x}\lambda^{(0)}(n)\lambda^{(1)}(n) = \sum_{n\le x}\Lambda_2(n)(\Lambda * \Lambda_2)(n).$$

Here only those $n$ which are either proper prime powers or which have exactly two distinct prime factors contribute. Using (6.4)-(6.7), the contribution of proper prime power $n$ is

$$\sum_{p^a\le x}\lambda^{(0)}(p^a)\lambda^{(1)}(p^a) = \sum_{p^a\le x}(2a-1)(a-1)^2(\log p)^5 \ll x^{\frac{1}{2}+\epsilon}.$$

Using (6.30), the contribution of $n = p_1p_2$ terms is seen to be

$$\begin{aligned}
&\frac{1}{2}\sum_{p_1p_2\le x}\lambda^{(0)}(p_1p_2)\lambda^{(1)}(p_1p_2)\\
&= \frac{1}{2}\sum_{p_1p_2\le x}(2\log p_1\log p_2)\Big[\log p_1(\log p_2)^2 + (\log p_1)^2\log p_2\Big]\\
&= 2\sum_{p_1p_2\le x}(\log p_1)^3(\log p_2)^2 = 2\sum_{p_1\le \frac{x}{2}}(\log p_1)^3\sum_{p_2<\frac{x}{p_1}}(\log p_2)^2\\
&= 2\sum_{p_1\le \frac{x}{2}}(\log p_1)^3\Big(\frac{x}{p_1}\log\frac{x}{p_1} + O(\frac{x}{p_1})\Big)\\
&= 2x\log x\sum_{p_1\le \frac{x}{2}}\frac{(\log p_1)^3}{p_1} - 2x\sum_{p_1\le \frac{x}{2}}\frac{(\log p_1)^4}{p_1} + O(x(\log x)^3)\\
&= \frac{1}{6}x(\log x)^4 + O(x(\log x)^3),
\end{aligned}$$

As for the contribution of those $n$ having two distinct prime factors of which at least one is a proper prime power, we have

$$\sum_{\substack{p_1^{r_1}p_2^{r_2}\le x\\ r_1\ge 1, r_2\ge 2}} \lambda^{(0)}(p_1^{r_1}p_2^{r_2})\lambda^{(1)}(p_1^{r_1}p_2^{r_2}) \ll \sum_{\substack{p_1^{r_1}p_2^{r_2}\le x\\ r_1\ge 1, r_2\ge 2}} r_1(\log p_1)^3(\log p_2)^2 + r_2(\log p_1)^2(\log p_2)^3$$

$$\ll \sum_{r_1\ge 1} r_1 \sum_{p_1^{r_1}\le \frac{x}{4}} (\log p_1)^3 \sum_{r_2\ge 2} \sum_{p_2\le \left(\frac{x}{p_1^{r_1}}\right)^{\frac{1}{r_2}}} (\log p_2)^2$$
$$+ \sum_{r_1\ge 1} \sum_{p_1^{r_1}\le \frac{x}{4}} (\log p_1)^2 \sum_{r_2\ge 2} r_2 \sum_{p_2\le \left(\frac{x}{p_1^{r_1}}\right)^{\frac{1}{r_2}}} (\log p_2)^3$$

$$\ll \sum_{r_1\ge 1} r_1 \sum_{p_1^{r_1}\le \frac{x}{4}} (\log p_1)^3 \sum_{r_2\ge 2} \left(\frac{x}{p_1^{r_1}}\right)^{\frac{1}{r_2}} \log\left(\frac{x}{p_1^{r_1}}\right)^{\frac{1}{r_2}}$$
$$+ \sum_{r_1\ge 1} \sum_{p_1^{r_1}\le \frac{x}{4}} (\log p_1)^2 \sum_{r_2\ge 2} r_2 \left(\frac{x}{p_1^{r_1}}\right)^{\frac{1}{r_2}} \left(\log\left(\frac{x}{p_1^{r_1}}\right)^{\frac{1}{r_2}}\right)^2$$

$$\ll x^{\frac{1}{2}}(\log x)\sum_{r_1\ge 1} r_1 \sum_{p_1^{r_1}\le \frac{x}{4}} \frac{(\log p_1)^3}{p_1^{\frac{r_1}{2}}} + x^{\frac{1}{2}}(\log x)^2 \sum_{r_1\ge 1} r_1 \sum_{p_1^{r_1}\le \frac{x}{4}} \frac{(\log p_1)^2}{p_1^{\frac{r_1}{2}}} \ll x(\log x)^3.$$

Hence we obtain

$$S_{0,1,1,1}(x) = \frac{1}{6}x(\log x)^4 + O(x(\log x)^3). \tag{6.40}$$

Next, using (6.4)-(6.6) and (6.25), we consider

$$S_{0,2,1,0}(x) = \sum_{n\le x} \lambda^{(0)}(n)\Lambda^{(2)}(n) = \sum_{n\le x} \Lambda_2(n)\Lambda^{(2)}(n).$$

Only those $n$ having at most two distinct prime factors contribute. We have

$$\sum_{p^r\le x} \Lambda_2(p^r)\Lambda^{(2)}(p^r) = \sum_{p^r\le x} (2r-1)(r-1)(\log p)^2$$
$$\le (\log x)\sum_{r\ge 2} \frac{(2r-1)(r-1)}{r} \sum_{p\le x^{\frac{1}{r}}} \log p \ll (\log x)\sum_{r\ge 2} r x^{\frac{1}{r}} \ll x^{\frac{1}{2}}\log x,$$

and the contribution from $n$ with exactly two distinct prime factors is

$$
\begin{aligned}
&\frac{1}{2}\sum_{p_1^{r_1}p_2^{r_2}\le x}\Lambda_2(p_1^{r_1}p_2^{r_2})\Lambda^{(2)}(p_1^{r_1}p_2^{r_2}) = 2\sum_{p_1^{r_1}p_2^{r_2}\le x}(\log p_1)^2(\log p_2)^2\\
&= 2\sum_{p_1^{r_1}\le\frac{x}{2}}(\log p_1)^2\sum_{r_2\ge 1}\sum_{p_2\le\left(\frac{x}{p_1}\right)^{\frac{1}{r_2}}}(\log p_2)^2\\
&= 2\sum_{p_1^{r_1}\le\frac{x}{2}}(\log p_1)^2\sum_{r_2\ge 1}\Big(\frac{x}{p_1}\Big)^{\frac{1}{r_2}}\log\bigg((\frac{x}{p_1})^{\frac{1}{r_2}}\bigg)+O\Big((\frac{x}{p_1})^{\frac{1}{r_2}}\Big)\\
&= 2\sum_{p_1^{r_1}\le\frac{x}{2}}(\log p_1)^2\log\Big(\frac{x}{p_1^{r_1}}\Big)\sum_{r_2\ge 1}\frac{1}{r_2}\Big(\frac{x}{p_1^{r_1}}\Big)^{\frac{1}{r_2}}+O\Big(x\sum_{p_1^{r_1}\le\frac{x}{2}}\frac{(\log p_1)^2}{p_1^{r_1}}\Big)\\
&= 2\sum_{p_1^{r_1}\le\frac{x}{2}}(\log p_1)^2\log\Big(\frac{x}{p_1^{r_1}}\Big)\Big(\frac{x}{p_1^{r_1}}+O\big((\frac{x}{p_1^{r_1}})^{\frac{1}{2}}\big)\Big)+O(x(\log x)^2)\\
&= 2x\log x\sum_{r_1\ge 1}\sum_{p_1\le\left(\frac{x}{2}\right)^{\frac{1}{r_1}}}\frac{(\log p_1)^2}{p_1^{r_1}}-2x\sum_{r_1\ge 1}\sum_{p_1\le\left(\frac{x}{2}\right)^{\frac{1}{r_1}}}\frac{(\log p_1)^3}{p_1^{r_1}}\\
&\qquad+O\Big(x^{\frac{1}{2}}\sum_{p_1^{r_1}\le\frac{x}{2}}\frac{(\log p_1)^2}{p_1^{\frac{r_1}{2}}}\Big)+O(x(\log x)^2)\\
&= 2x(\log x)\Big(\frac{(\log x)^2}{2}+O(1)\Big)-2x\Big(\frac{(\log x)^3}{3}+O(1)\Big)+O(x(\log x)^2)\\
&= \frac{1}{3}x(\log x)^3+O(x(\log x)^2).
\end{aligned}
$$

Hence we have

$$
S_{0,2,1,0}(x) = \frac{1}{3}x(\log x)^3+O(x(\log x)^2). \tag{6.41}
$$

Now consider $S_{0,k,1,0}(x)=\sum_{n\le x}\lambda^{(0)}(n)\Lambda^{(k)}(n)=\sum_{n\le x}\Lambda_2(n)\Lambda^{(k)}(n), \quad (k\ge 3)$. Because of the presence of $\Lambda_2(n)$ only those $n$ which have at most two distinct prime factors contribute to the sum. Only those prime powers $n=p^r$ with $r\ge k$ will contribute since $\Lambda^{(k)}(p^r)=0$ for $r<k$. We see, by (6.5) and (6.6),

$$
\begin{aligned}
&\sum_{\substack{p^r\le x\\ r\ge k}}\Lambda_2(p^r)\Lambda^{(k)}(p^r)=\sum_{r\ge k}(2r-1)\binom{r-1}{k-1}\sum_{p\le x^{\frac{1}{r}}}(\log p)^{k+2}\\
&\ll\sum_{r\ge k}(2r-1)\binom{r-1}{k-1}\Big(\frac{1}{r}\Big)^{k+1}(\log x)^{k+1}\sum_{p\le x^{\frac{1}{r}}}\log p\ll\sum_{r\ge k}\frac{1}{k!}x^{\frac{1}{r}}(\log x)^{k+1}\\
&\ll\frac{1}{k!}x^{\frac{1}{k}}(\log x)^{k+1}.
\end{aligned}
$$

In calculating the contribution of $n$ with exactly two distinct prime factors we use

$$\Lambda^{(k)}(p_1^{r_1}p_2^{r_2}) = \begin{cases} \sum_{j=1}^{k-1}\binom{k}{j}\binom{r_1-1}{j-1}\binom{r_2-1}{k-j-1}(\log p_1)^j(\log p_2)^{k-j} & \text{for } r_1+r_2\geq k \\ 0 & \text{for } r_1+r_2<k, \end{cases} \tag{6.42}$$

and without loss of generality we take $r_1 \geq \frac{k_2}{2}$. We have

$$\begin{aligned}
&\sum_{p_1^{r_1}p_2^{r_2}\leq x}\Lambda_2(p_1^{r_1}p_2^{r_2})\Lambda^{(k)}(p_1^{r_1}p_2^{r_2}) \\
&\ll \sum_{j=1}^{k-1}\binom{k}{j}\sum_{\substack{r_1\geq \max(j,\frac{k}{2})\\ r_2\geq k-j}}\binom{r_1-1}{j-1}\binom{r_2-1}{k-j-1}\sum_{p_1^{r_1}p_2^{r_2}\leq x}(\log p_1)^{j+1}(\log p_2)^{k-j+1} \\
&\ll \sum_{j=1}^{k-1}\binom{k}{j}\sum_{\substack{r_1\geq \max(j,\frac{k}{2})\\ r_2\geq k-j}}\frac{r_1^{j-1}}{(j-1)!}\frac{r_2^{k-j-1}}{(k-j-1)!}\sum_{p_1^{r_1}\leq\frac{x}{2}}(\log p_1)^{j+1}\sum_{p_2\leq\left(\frac{x}{p_1^{r_1}}\right)^{\frac{1}{r_2}}}(\log p_2)^{k-j+1} \\
&\ll \sum_{j=1}^{k-1}\binom{k}{j}\sum_{\substack{r_1\geq \max(j,\frac{k}{2})\\ r_2\geq k-j}}\frac{r_1^{j-1}}{(j-1)!}\frac{r_2^{k-j-1}}{(k-j-1)!}\sum_{p_1^{r_1}\leq\frac{x}{2}}(\log p_1)^{j+1}\left(\log\left(\frac{x}{p_1^{r_1}}\right)^{\frac{1}{r_2}}\right)^{k-j}\left(\frac{x}{p_1^{r_1}}\right)^{\frac{1}{r_2}} \\
&\ll \sum_{j=1}^{k-1}\binom{k}{j}\sum_{r_1\geq \max(j,\frac{k}{2})}\frac{r_1^{j-1}}{(j-1)!}\sum_{p_1^{r_1}\leq\frac{x}{2}}(\log p_1)^{j+1}\left(\log\left(\frac{x}{p_1^{r_1}}\right)\right)^{k-j}\sum_{r_2\geq k-j}\frac{1}{r_2(k-j-1)!}\left(\frac{x}{p_1^{r_1}}\right)^{\frac{1}{r_2}} \\
&\ll \sum_{j=1}^{k-1}\binom{k}{j}\frac{1}{(k-j)!}\sum_{r_1\geq \max(j,\frac{k}{2})}\frac{r_1^{j-1}}{(j-1)!}\sum_{p_1^{r_1}\leq\frac{x}{2}}(\log p_1)^{j+1}\left(\log\left(\frac{x}{p_1^{r_1}}\right)\right)^{k-j}\left(\frac{x}{p_1^{r_1}}\right)^{\frac{1}{k-j}} \\
&\ll (\log x)^k\sum_{j=1}^{k-1}\binom{k}{j}\frac{1}{(k-j)!j!}\sum_{r_1\geq \max(j,\frac{k}{2})}\frac{j}{r_1}\sum_{p_1\leq\left(\frac{x}{2}\right)^{\frac{1}{r_1}}}(\log p_1)\left(\frac{x}{p_1^{r_1}}\right)^{\frac{1}{k-j}} \\
&\ll \frac{(\log x)^k}{k!}\sum_{r_1\geq\frac{k}{2}}\sum_{p_1\leq\left(\frac{x}{2}\right)^{\frac{1}{r_1}}}(\log p_1)\sum_{j\leq\min(r_1,k-1)}\binom{k}{j}^2\left(\frac{x}{p_1^{r_1}}\right)^{\frac{1}{k-j}} \\
&\ll \max_{1\leq j\leq k-1}\binom{k}{j}^2\frac{(\log x)^k}{k!}\sum_{r_1\geq\frac{k}{2}}\sum_{p_1\leq\left(\frac{x}{2}\right)^{\frac{1}{r_1}}}(\log p_1)\frac{x}{p_1^{r_1}} \ll \frac{C_{19}^k x(\log x)^k}{k!},
\end{aligned}$$

since the sum over $r_1$ starts from $r_1\geq\lfloor\frac{k}{2}\rfloor\geq 2$. Combining these, we obtain

$$S_{0,k_2,1,0}(x) = O\left(\frac{C_{20}^{k_2} x(\log x)^{k_2}}{k_2!}\right), \qquad (k_2 \ge 3). \tag{6.43}$$

By (6.2), this also settles

$$\begin{aligned} S_{0,k_2,1,1}(x) &= \sum_{n\le x} \lambda^{(0)}(n)\lambda^{(k)}(n) = \sum_{n\le x} \Lambda_2(n)\lambda^{(k)}(n) \ll (\log x)^2 \sum_{n\le x} \Lambda_2(n)\Lambda^{(k)}(n) \\ &= O\left(\frac{C_{21}^{k_2} x(\log x)^{k_2+2}}{k_2!}\right), \qquad (k_2 \ge 3), \end{aligned} \tag{6.44}$$

and there remains the case

$$S_{0,2,1,1}(x) = \sum_{n\le x} \lambda^{(0)}(n)\lambda^{(2)}(n) = \sum_{n\le x} \Lambda_2(n)\lambda^{(2)}(n).$$

Only those $n$ with at most two distinct prime factors contribute to the sum. By (6.5) and (6.7), the prime powers contribute

$$\begin{aligned} &\sum_{p^r\le x} \Lambda_2(p^r)\lambda^{(2)}(p^r) = \sum_{p^r\le x} (2r-1)\left(2\binom{r}{3} - \binom{r-1}{2}\right)(\log p)^6 \ll \sum_{r\ge 3} r^4 \sum_{p\le x^{\frac{1}{r}}} (\log p)^6 \\ &\ll \sum_{r\ge 3} r^4 \left(\log\left(x^{\frac{1}{r}}\right)\right)^5 \sum_{p\le x^{\frac{1}{r}}} \log p \ll (\log x)^5 \sum_{r\ge 3} \frac{x^{\frac{1}{r}}}{r} \ll x^{\frac{1}{3}} (\log x)^5. \end{aligned}$$

To estimate the contribution from those $n = p_1^{r_1} p_2^{r_2}$ $p_1 \ne p_2$, first note

$$\begin{aligned} \lambda^{(2)}(p_1^{r_1}p_2^{r_2}) &= (r_1-1)(r_1-2)(\log p_1)^3(\log p_2) + (r_2-1)(r_2-2)(\log p_1)(\log p_2)^3 \\ &\quad + (8r_1r_2 - 3r_1 - 3r_2 - 2)(\log p_1)^2(\log p_2)^2, \end{aligned} \tag{6.45}$$

as can be found from (6.3)-(6.7), so that

$$\begin{aligned} &\sum_{p_1^{r_1}p_2^{r_2}\le x} \Lambda_2(p_1^{r_1}p_2^{r_2})\lambda^{(2)}(p_1^{r_1}p_2^{r_2}) \\ &\ll \sum_{p_1^{r_1}p_2^{r_2}\le x} (r_1-1)(r_1-2)(\log p_1)^4(\log p_2)^2 + (r_2-1)(r_2-2)(\log p_1)^2(\log p_2)^4 \\ &\qquad + (8r_1r_2 - 3r_1 - 3r_2 - 2)(\log p_1)^3(\log p_2)^3. \end{aligned}$$

The first term here contributes

$$\ll \sum_{r_1\ge 2} r_1^2 \sum_{p_1^{r_1}\le \frac{x}{2}} (\log p_1)^4 \sum_{r_2\ge 1} \sum_{p_2\le \left(\frac{x}{p_1^{r_1}}\right)^{\frac{1}{r_2}}} (\log p_2)^2$$

$$\ll \sum_{r_1\ge 2} r_1^2 \sum_{p_1^{r_1}\le \frac{x}{2}} (\log p_1)^4 \sum_{r_2\ge 1} \log\left(\frac{x}{p_1^{r_1}}\right)^{\frac{1}{r_2}} \sum_{p_2\le \left(\frac{x}{p_1^{r_1}}\right)^{\frac{1}{r_2}}} \log p_2$$

$$\ll (\log x) \sum_{r_1\ge 2} r_1^2 \sum_{p_1^{r_1}\le \frac{x}{2}} (\log p_1)^4 \sum_{r_2\ge 1} \frac{\left(\frac{x}{p_1^{r_1}}\right)^{\frac{1}{r_2}}}{r_2} \ll x(\log x) \sum_{r_1\ge 2} r_1^2 \sum_{p_1\le \left(\frac{x}{2}\right)^{\frac{1}{r_1}}} \frac{(\log p_1)^4}{p_1^{r_1}}$$

$$\ll x(\log x) \sum_{r_1\ge 2} r_1^2 \left(\log\left(\frac{x}{2}\right)^{\frac{1}{r_1}}\right)^3 \sum_{p_1\le \left(\frac{x}{2}\right)^{\frac{1}{r_1}}} \frac{\log p_1}{p_1^{r_1}} \ll x(\log x)^4 \sum_{r_1\ge 2} \frac{1}{r_1} \sum_{p_1} \frac{\log p_1}{p_1^{r_1}}$$

$$\ll x(\log x)^4.$$

The treatment of the second term is the same. The third term contributes (since for non-zero contribution we need $r_1 + r_2 \ge 3$, we can take $r_1 \ge 2, r_2 \ge 1$)

$$\ll \sum_{r_1\ge 2} \sum_{p_1^{r_1}\le \frac{x}{2}} (\log p_1)^3 \sum_{r_2\ge 1} (8r_1r_2 - 3r_1 - 3r_2 - 2) \sum_{p_2\le \left(\frac{x}{p_1^{r_1}}\right)^{\frac{1}{r_2}}} (\log p_2)^3$$

$$\ll (\log x)^2 \sum_{r_1\ge 2} \sum_{p_1^{r_1}\le \frac{x}{2}} (\log p_1)^3 \sum_{r_2\ge 1} \frac{8r_1r_2 - 3r_1 - 3r_2 - 2}{r_2^2} \sum_{p_2\le \left(\frac{x}{p_1^{r_1}}\right)^{\frac{1}{r_2}}} \log p_2$$

$$\ll (\log x)^2 \sum_{r_1\ge 2} \sum_{p_1^{r_1}\le \frac{x}{2}} (\log p_1)^3 \sum_{r_2\ge 1} \frac{8r_1r_2 - 3r_1 - 3r_2 - 2}{r_2^2} \left(\frac{x}{p_1^{r_1}}\right)^{\frac{1}{r_2}}$$

$$\ll x(\log x)^2 \sum_{r_1\ge 2} r_1 \sum_{p_1\le \left(\frac{x}{2}\right)^{\frac{1}{r_1}}} \frac{(\log p_1)^3}{p_1^{r_1}}$$

$$\ll x(\log x)^2 \sum_{r_1\ge 2} r_1 \left(\log\left(\frac{x}{2}\right)^{\frac{1}{r_1}}\right)^2 \sum_{p_1\le \left(\frac{x}{2}\right)^{\frac{1}{r_1}}} \frac{\log p_1}{p_1^{r_1}} \ll x(\log x)^4 \sum_{r_1\ge 2} \frac{1}{r_1} \sum_{p_1} \frac{\log p_1}{p_1^{r_1}}$$

$$\ll x(\log x)^4.$$

Hence we have

$$S_{0,2,1,1}(x) \ll x(\log x)^4. \tag{6.46}$$

All of the formulas (6.40), (6.41), (6.43), (6.44) and (6.46) conform to the claim of Proposition 2, so the treatment of the cases $k_1 + k_2 + \iota_1 + \iota_2 \geq 3$ with one of $k_1$ and $k_2$ being 0 is completed.

From now on we take $k_1 + k_2 + \iota_1 + \iota_2 \geq 3$ with $1 \leq k_1 \leq k_2$. Since

$$\Delta(k_2, \iota_2; n) = \sum_{p^r | n} (\log p) \Delta(k_2 - 1, \iota_2; \frac{n}{p^r}),$$

where $p$ denotes a prime, writing $n = ep^r$, we see

$$\begin{aligned}
S_{k_1,k_2,\iota_1,\iota_2}(x) &= \sum_{p^r \leq x} (\log p) \sum_{e \leq \frac{x}{p^r}} \Delta(k_1, \iota_1; ep^r) \Delta(k_2 - 1, \iota_2; e) \\
&= \sum_{p \leq x} (\log p) \sum_{e \leq \frac{x}{p}} \Delta(k_1, \iota_1; ep) \Delta(k_2 - 1, \iota_2; e) \\
&\quad + \sum_{\substack{p^r \leq x \\ r \geq 2}} (\log p) \sum_{e \leq \frac{x}{p^r}} \Delta(k_1, \iota_1; ep^r) \Delta(k_2 - 1, \iota_2; e) \\
&= \sum_{p \leq x} (\log p) \sum_{e \leq \frac{x}{p}} \Delta(k_1, \iota_1; ep) \Delta(k_2 - 1, \iota_2; e) + \Sigma_8, \text{ say,} \\
&= \sum_{p \leq x} (\log p) \sum_{\substack{e \leq \frac{x}{p} \\ (e,p)=1}} \Delta(k_1, \iota_1; ep) \Delta(k_2 - 1, \iota_2; e) \\
&\quad + \sum_{p \leq x} (\log p) \sum_{\substack{e \leq \frac{x}{p} \\ (e,p)=p}} \Delta(k_1, \iota_1; ep) \Delta(k_2 - 1, \iota_2; e) + \Sigma_8 \\
&= \sum_{p \leq x} (\log p) \sum_{\substack{e \leq \frac{x}{p} \\ (e,p)=1}} \Delta(k_1, \iota_1; ep) \Delta(k_2 - 1, \iota_2; e) + \Sigma_7 + \Sigma_8, \text{ say,}
\end{aligned} \tag{6.47}$$

Now we use Lemma 3 for $\Delta(k_1, \iota_1; ep)$, and have

$$
\begin{aligned}
&\sum_{p\le x}(\log p)\sum_{\substack{e\le \frac{x}{p}\\(e,p)=1}}\Delta(k_1,\iota_1;ep)\Delta(k_2-1,\iota_2;e)\\
&\quad= k_1\sum_{p\le x}(\log p)^2\sum_{\substack{e\le \frac{x}{p}\\(e,p)=1}}\Delta(k_1-1,\iota_1;ep)\Delta(k_2-1,\iota_2;e)\\
&\qquad+2[\iota_1=1]\sum_{p\le x}(\log p)^2\sum_{\substack{e\le \frac{x}{p}\\(e,p)=1}}\Delta(k_1+1,0;e)\Delta(k_2-1,\iota_2;e)\\
&\qquad+[\iota_1=1]\sum_{p\le x}(\log p)^3\sum_{\substack{e\le \frac{x}{p}\\(e,p)=1}}\Delta(k_1,0;e)\Delta(k_2-1,\iota_2;e)\\
&\quad= k_1\sum_{p\le x}(\log p)^2\sum_{e\le \frac{x}{p}}\Delta(k_1-1,\iota_1;e)\Delta(k_2-1,\iota_2;e)\\
&\qquad+2[\iota_1=1]\sum_{p\le x}(\log p)^2\sum_{e\le \frac{x}{p}}\Delta(k_1+1,0;e)\Delta(k_2-1,\iota_2;e)\\
&\qquad+[\iota_1=1]\sum_{p\le x}(\log p)^3\sum_{e\le \frac{x}{p}}\Delta(k_1,0;e)\Delta(k_2-1,\iota_2;e)\\
&\qquad-k_1\sum_{p\le x}(\log p)^2\sum_{\substack{e\le \frac{x}{p}\\(e,p)=p}}\Delta(k_1-1,\iota_1;e)\Delta(k_2-1,\iota_2;e)\\
&\qquad-2[\iota_1=1]\sum_{p\le x}(\log p)^2\sum_{\substack{e\le \frac{x}{p}\\(e,p)=p}}\Delta(k_1+1,0;e)\Delta(k_2-1,\iota_2;e)\\
&\qquad-[\iota_1=1]\sum_{p\le x}(\log p)^3\sum_{\substack{e\le \frac{x}{p}\\(e,p)=p}}\Delta(k_1,0;e)\Delta(k_2-1,\iota_2;e)\\
&\quad=\Sigma_1+\Sigma_2+\Sigma_3-\Sigma_4-\Sigma_5-\Sigma_6,\ \text{say,}
\end{aligned}\tag{6.48}
$$

so that

$$
S_{k_1,k_2,\iota_1,\iota_2}(x)=\Sigma_1+\Sigma_2+\Sigma_3-\Sigma_4-\Sigma_5-\Sigma_6+\Sigma_7+\Sigma_8.\tag{6.49}
$$

Applying the induction hypothesis to $\Sigma_1, \Sigma_2$ and $\Sigma_3$ we have

$$
\begin{aligned}
\Sigma_1 &= k_1 \sum_{p\le x} (\log p)^2 \sum_{e\le \frac{x}{p}} \Delta(k_1-1,\iota_1;e)\Delta(k_2-1,\iota_2;e)\\
&= x\frac{k_1 P(k_1-1,k_2-1,\iota_1,\iota_2)}{(k_1+k_2+2\iota_1+2\iota_2-3)!}\sum_{p\le x}\frac{(\log p)^2}{p}\left(\log\frac{x}{p}\right)^{k_1+k_2+2\iota_1+2\iota_2-3}\\
&\quad + O\left(\frac{xC_{18}^{k_1+k_2-2}}{\max(k_1-1,k_2-1)!}\sum_{p\le x}\frac{(\log p)^2}{p}\left(\log\frac{x}{p}\right)^{k_1+k_2+2\iota_1+2\iota_2-4}\right),\\
\Sigma_2 &= 2[\iota_1=1]\sum_{p\le x}(\log p)^2 \sum_{\substack{e\le \frac{x}{p}\\ (e,p)=1}} \Delta(k_1+1,0;e)\Delta(k_2-1,\iota_2;e)\\
&= 2[\iota_1=1]x\frac{P(k_1+1,k_2-1,0,\iota_2)}{(k_1+k_2+2\iota_1+2\iota_2-3)!}\sum_{p\le x}\frac{(\log p)^2}{p}\left(\log\frac{x}{p}\right)^{k_1+k_2+2\iota_1+2\iota_2-3}\\
&\quad + O\left([\iota_1=1]\frac{xC_{18}^{k_1+k_2}}{\max(k_1+1,k_2-1)!}\sum_{p\le x}\frac{(\log p)^2}{p}\left(\log\frac{x}{p}\right)^{k_1+k_2+2\iota_1+2\iota_2-4}\right),\\
\Sigma_3 &= [\iota_1=1]\sum_{p\le x}(\log p)^3\sum_{e\le\frac{x}{p}}\Delta(k_1,0;e)\Delta(k_2-1,\iota_2;e)\\
&= [\iota_1=1]x\frac{P(k_1,k_2-1,0,\iota_2)}{(k_1+k_2+2\iota_1+2\iota_2-4)!}\sum_{p\le x}\frac{(\log p)^3}{p}\left(\log\frac{x}{p}\right)^{k_1+k_2+2\iota_1+2\iota_2-4}\\
&\quad + O\left([\iota_1=1]\frac{xC_{18}^{k_1+k_2-1}}{\max(k_1,k_2-1)!}\sum_{p\le x}\frac{(\log p)^3}{p}\left(\log\frac{x}{p}\right)^{k_1+k_2+2\iota_1+2\iota_2-5}\right). \qquad (6.50)
\end{aligned}
$$

We must mention that some care is needed in ensuring the applicability of the induction hypothesis. In these applications we need to check whether the hypothesis of Proposition 2 is satisfied. Specifically, for $\Sigma_1$ we need to have $k_1-1+\iota_1\ge 1$, $k_2-1+\iota_2\ge 1$. If these are not satisfied, then either $k_2+\iota_2=1$ or $k_1+\iota_1=1$. Recall that we are working under the condition $k_1+k_2+\iota_1+\iota_2\ge 3$ with $1\le k_1\le k_2$. If $k_2+\iota_2=1$, then $k_2=1, \iota_2=0$, and therefore $k_1=1$, $\iota_1=1$. So we encounter

$$
\begin{aligned}
S_{1,1,1,0}(x) &= \sum_{n\le x}\lambda^{(1)}(n)\Lambda(n) = \sum_{r\ge 2}(r-1)^2\sum_{p\le x^{\frac{1}{r}}}(\log p)^4\\
&\le \sum_{r\ge 2}(r-1)^2\left(\log(x^{\frac{1}{r}})\right)^3\sum_{p\le x^{\frac{1}{r}}}\log p \ll (\log x)^3\sum_{r\ge 2}\frac{1}{r}\ll x^{\frac{1}{2}}(\log x)^3, \qquad (6.51)
\end{aligned}
$$

which is compatible with what (6.35)-(6.36) written for $(k_1,k_2,\iota_1,\iota_2)=(1,1,1,0)$ gives, namely

$$
S_{1,1,1,0}(x) = 0 + O\big(C_{18}^2 x(\log x)^2\big).
$$

If $k_1+\iota_1=1$, then $k_1=1, \iota_1=0$ and $k_2+\iota_2\geq 2$. If $k_2=1$, then $\iota_2=1$ too, and we encounter $S_{1,1,0,1}(x)$ which is the same as $S_{1,1,1,0}(x)$ just settled in (6.51). If $k_2\geq 2$, then $\iota_2=0$ or $\iota_2=1$, and we encounter $S_{1,k_2,0,0}(x)$ or $S_{1,k_2,0,1}(x)$. Now, with $k\geq 2$, we have

$$S_{1,k,0,0}(x)=\sum_{n\leq x}\Lambda(n)\Lambda^{(k)}(n)=\sum_{r\geq k}\binom{r-1}{k-1}\sum_{p\leq x^{\frac{1}{r}}}(\log p)^{k+1} \tag{6.52}$$

$$\leq \sum_{r\geq k}\binom{r-1}{k-1}\Big(\log(x^{\frac{1}{r}})\Big)^k\sum_{p\leq x^{\frac{1}{r}}}(\log p)\ll\frac{(\log x)^k}{(k-1)!}\sum_{r\geq k}\frac{1}{r}x^{\frac{1}{r}}\ll\frac{x^{\frac{1}{k}}(\log x)^k}{k!},$$

and this is compatible with

$$S_{1,k,0,0}(x)=0+O\left(\frac{C_{18}^{k+1}(\log x)^{k-1}}{k!}\right),\quad (k\geq 2)$$

which is (6.35)-(6.36) written for $(k_1,k_2,\iota_1,\iota_2)=(1,k,0,0)$. Similarly, for $k\geq 2$,

$$S_{1,k,0,1}(x)=\sum_{n\leq x}\Lambda(n)\lambda^{(k)}(n)=\sum_{r\geq k+1}\binom{r-1}{k}\frac{2r-k-1}{k+1}\sum_{p\leq x^{\frac{1}{r}}}(\log p)^{k+3}$$

$$\leq\sum_{r\geq k+1}\binom{r-1}{k}\frac{2r-k-1}{k+1}\Big(\log(x^{\frac{1}{r}})\Big)^{k+2}\sum_{p\leq x^{\frac{1}{r}}}(\log p)\ll\frac{(\log x)^{k+2}}{(k+1)!}\sum_{r\geq k+1}\frac{1}{r}x^{\frac{1}{r}}$$

$$\ll\frac{x^{\frac{1}{k+1}}(\log x)^{k+2}}{(k+2)!}, \tag{6.53}$$

which is compatible with

$$S_{1,k,0,1}(x)=0+O\left(\frac{C_{18}^{k+1}(\log x)^{k+1}}{k!}\right),\quad (k\geq 2)$$

obtained by writing (6.35)-(6.36) for $(k_1,k_2,\iota_1,\iota_2)=(1,k,0,1)$. An examination of $\Sigma_2$ and $\Sigma_3$ shows that the only new situation to check is for $\Sigma_2$ with $k_2=1,\iota_2=0$, in which case $k_1=1$, and the sum to be checked is $S_{2,0,0,0}(x)$. But, trivially $S_{2,0,0,0}(x)=0$, and this is compatible with $S_{2,0,0,0}(x)=0+O(C_{18}^2x)$ which is (6.35)-(6.36) written with $(k_1,k_2,\iota_1,\iota_2)=(2,0,0,0)$. Hence the expressions in (6.50) are completely justified, and in the few cases where the exponents of $\log\frac{x}{p}$ in the sums in the error terms may be negative the statement of Proposition 2 has been verified independently.

We evaluate the inner sums (over $p$) in (6.50) using a variant of (6.23),

$$\sum_{p\leq x}\frac{(\log p)^k}{p}\left(\log\frac{x}{p}\right)^{\ell}=\frac{(k-1)!\ell!}{(k+\ell)!}(\log x)^{k+\ell}+O\left(\frac{(k-1)!\ell!}{(k+\ell-1)!}(\log x)^{k+\ell-1}\right) \tag{6.54}$$

for $k \geq 1, \ell \geq 0$, which is Lemma 5.4 of [5], and for $\Sigma_1 + \Sigma_2 + \Sigma_3$ we use

$$\begin{aligned} k_1 P(k_1 - 1, k_2 - 1, \iota_1, \iota_2) + 2[\iota_1 = 1]\big(P(k_1 + 1, k_2 - 1, 0, \iota_2) + P(k_1, k_2 - 1, 0, \iota_2)\big) \\ = P(k_1, k_2, \iota_1, \iota_2) \end{aligned} \tag{6.55}$$

which can be verified from the definition of $P(k_1, k_2, \iota_1, \iota_2)$. So we obtain

$$\begin{aligned} &\Sigma_1 + \Sigma_2 + \Sigma_3 \\ &= \frac{P(k_1, k_2, \iota_1, \iota_2)}{(k_1 + k_2 + 2\iota_1 + 2\iota_2 - 1)!} x(\log x)^{k_1 + k_2 + 2\iota_1 + 2\iota_2 - 1} \cdot \left(1 + O\left(\frac{k_2}{\log x}\right)\right) \\ &\qquad + O\left(\frac{C_{18}^{k_1 + k_2} x(\log x)^{k_1 + k_2 + 2\iota_1 + 2\iota_2 - 2}}{(k_2 + 1)!} \cdot \big(1 + \frac{k_2}{\log x}\big)\right) \\ &= \frac{P(k_1, k_2, \iota_1, \iota_2)}{(k_1 + k_2 + 2\iota_1 + 2\iota_2 - 1)!} x(\log x)^{k_1 + k_2 + 2\iota_1 + 2\iota_2 - 1} \\ &\qquad + O\left(\frac{C_{18}^{k_1 + k_2} x(\log x)^{k_1 + k_2 + 2\iota_1 + 2\iota_2 - 2}}{(\max\{k_1, k_2\})!}\right). \end{aligned} \tag{6.56}$$

Here, in writing the middle member of the formula, we have invoked $k_2 \geq k_1$ which was taken as a convention, and in passing to the last line we have used

$$\frac{k_2 P(k_1, k_2, \iota_1, \iota_2)}{(k_1 + k_2 + 2\iota_1 + 2\iota_2 - 1)!} \ll \frac{k_2 + 1)!}{(2k_2 - 1)!},$$

in case the left-hand side is not 0, which is seen from (6.36).

It remains to show that $\Sigma_4, \ldots, \Sigma_8$ which were defined in (6.47) and (6.48) can all be included in the error term of (6.35). In $\Sigma_4$ letting $e = e'p$, and then using Lemmas 3 and 4, we have

$$\begin{aligned} \Sigma_4 &= k_1 \sum_{p \leq x} (\log p)^2 \sum_{\substack{e \leq \frac{x}{p} \\ (e,p) = p}} \Delta(k_1 - 1, \iota_1; e) \Delta(k_2 - 1, \iota_2; e) \\ &\ll k_1^2 (\log x)^{k_1 + 2\iota_1 - 2} \sum_{p \leq x} (\log p)^3 \sum_{e' \leq \frac{x}{p^2}} \Delta(k_2 - 1, \iota_2; pe') \\ &\ll \frac{C_2^{k_2 - 1} k_1^2}{(k_2 + 2\iota_2 - 2)!} x(\log x)^{k_1 + k_2 + 2\iota_1 + 2\iota_2 - 4} \sum_{p \leq x} \frac{(\log p)^3}{p^2} \\ &\ll \frac{C_{18}^{k_1 + k_2} x(\log x)^{k_1 + k_2 + 2\iota_1 + 2\iota_2 - 2}}{k_2!} \end{aligned} \tag{6.57}$$

provided that $C_{18}$ is chosen sufficiently large. It is easily seen that $\Sigma_5 - \Sigma_8$ are handled similarly. This completes the proof of Proposition 2. □

Applying summation by parts to the result of Proposition 2, we immediately obtain

**Proposition 3**

$$\sum_{n\le x}\Delta(k_1,\iota_1;n)\Delta(k_2,\iota_2;n)n=\frac{P(k_1,k_2,\iota_1,\iota_2)x^2(\log x)^{k_1+k_2+2\iota_1+2\iota_2-1}}{2(k_1+k_2+2\iota_1+2\iota_2-1)!}+O\left(\frac{C_{18}^{k_1+k_2}x^2(\log x)^{k_1+k_2+2\iota_1+2\iota_2-2}}{(\max\{k_1,k_2\})!}\right); \tag{6.58}$$

$$\sum_{n>x}\frac{\Delta(k_1,\iota_1;n)\Delta(\ell,\iota_2;n)}{n^3}=\frac{P(k_1,k_2,\iota_1,\iota_2)(\log x)^{k_1+k_2+2\iota_1+2\iota_2-1}}{2x^2(k_1+k_2+2\iota_1+2\iota_2-1)!}+O\left(\frac{C_{18}^{k_1+k_2}(\log x)^{k_1+k_2+2\iota_1+2\iota_2-2}}{x^2(\max\{k_1,k_2\})!}\right). \tag{6.59}$$

## 7 Correlation of zeta zeros with the relative maxima of $|\zeta(\frac{1}{2}+it)|$

Assuming RH, let $\varrho=\frac{1}{2}+i\upsilon$, $\upsilon>0$, run through the non-trivial zeros of $Z_1(s)$. For correlations of zeta zeros with the relative maxima of the Riemann zeta-function on the critical line we set out from (3.3) by writing

$$\sum_{\gamma}\sum_{\frac{T}{2}<\upsilon\le T}\frac{4x^{i(\gamma-\upsilon)}}{4+(\gamma-\upsilon)^2} \tag{7.1}$$
$$=-x^{-2}\sum_{n\le x}\Lambda(n)n^2\sum_{\frac{T}{2}<\upsilon\le T}n^{-\varrho}-x^2\sum_{n>x}\Lambda(n)n^{-2}\sum_{\frac{T}{2}<\upsilon\le T}n^{-\varrho}$$
$$+\frac{T\log^2T}{4\pi x^2}\left(1+O\left(\frac{1}{\log T}\right)\right)+O\Big(\frac{x^{\frac{1}{2}}\log T}{T}\Big)+O(x^{-\frac{5}{2}}\log T)$$

for $x\ge 1$ and $T\ge T_0$. We see that we need to calculate

$$\sum_{\frac{T}{2}<\upsilon\le T}n^{-\varrho}\quad \text{for } n=p^a,\ p:\text{prime},\ a\in\mathbb{Z}^+. \tag{7.2}$$

Just as in the pair correlation of zeta zeros we do not hope to get an asymptotic result for $x>T$, so that if we assume $x\le T^M$, then for the tail of the sum over $n>x$ in (7.1) we use

$$x^2\sum_{n>T^M}\Lambda(n)n^{-2}\sum_{\frac{T}{2}<\upsilon\le T}n^{-\varrho}\ll x^2T\log T\sum_{n>T^M}\frac{\Lambda(n)}{n^{\frac{5}{2}}}\ll x^2T^{1-\frac{3M}{2}}\log T. \tag{7.3}$$

We will see towards the end of this calculation that any fixed $M>\frac{4}{3}$ can be employed. Upon this simplification we write

$$\sum_{\gamma}\sum_{\frac{T}{2}<\upsilon\le T}\frac{4x^{i(\gamma-\upsilon)}}{4+(\gamma-\upsilon)^2} \tag{7.4}$$
$$= -x^{-2}\sum_{n\le x}\Lambda(n)n^2\sum_{\frac{T}{2}<\upsilon\le T}n^{-\varrho} - x^2\sum_{x<n\le T^M}\Lambda(n)n^{-2}\sum_{\frac{T}{2}<\upsilon\le T}n^{-\varrho}$$
$$+\frac{T\log^2 T}{4\pi x^2}\left(1+O\left(\frac{1}{\log T}\right)\right)+O\left(\frac{x^{\frac{1}{2}}\log T}{T}\right)+O\left(\frac{\log T}{x^{\frac{5}{2}}}\right)+O\left(x^2T^{1-\frac{3M}{2}}\log T\right)$$

for $1\le x<T^M$ and $T\ge T_0$.

We are now going to calculate, more generally than (7.2), the expression

$$\sum_{\frac{T}{2}<\upsilon\le T} n^{-\varrho}\left(\frac{\chi'}{\chi}(\varrho+b)\right)^{-k} \quad \text{for } 2\le n\le T^M,\, 0\le k\le \lfloor\frac{\log T}{\log\log T}\rfloor,\, b=-2 \text{ or } 2 \tag{7.5}$$

(where $n$ and $k$ are integers) for later use in §8; the special case of $n$ a prime power and $k=0$ is (7.2). The result will be formula (7.26) in Proposition 4 below.

To calculate (7.5) we will use

$$\sum_{\frac{T}{2}<\upsilon\le T} n^{-\varrho}\left(\frac{\chi'}{\chi}(\varrho+b)\right)^{-k} = \frac{1}{2\pi i}\oint_C \frac{Z_1'}{Z_1}(s)\left(\frac{\chi'}{\chi}(s+b)\right)^{-k} n^{-s}\,ds$$
$$+O\left(n^{-\frac{1}{2}}(\log T)^{1-k}\right), \tag{7.6}$$

where the contour $C$ is the counterclockwise traversed rectangle having vertices at $-\delta+it_r, 1+\delta+it_r$, $(r=1,2)$, with

$$\delta=\frac{1}{\log T},\ t_r=\frac{rT}{2}+O(1),\ (r=1,2), \tag{7.7}$$

chosen such that the horizontal sides of $C$ are a distance $\gg \frac{1}{\log T}$ away from the zeros of $Z_1(s)$. The error term in (7.6) is the effect of such a choice estimated trivially. Just as the estimate for $\frac{\zeta'}{\zeta}(s)$ (see Chapter 17 of [2]), from an argument based upon the representation of $\frac{Z_1'}{Z_1}(s)$ as a sum of partial fractions (which is similar to that for $\frac{\zeta'}{\zeta}(s)$ since the count of zeros is almost the same) we see

$$\frac{Z_1'}{Z_1}(\sigma+it_r)=O(\log^2 T),\quad (-1\le\sigma\le 2), \tag{7.8}$$

so that the integrals along the horizontal sides of $C$ satisfy

$$\int_{-\delta\pm t_r}^{1+\delta\pm t_r}\frac{Z_1'}{Z_1}(s)\left(\frac{\chi'}{\chi}(s+b)\right)^{-k}n^{-s}\,ds \ll \frac{n^{\delta}(\log T)^{2-k}}{\log n}\ll\frac{(\log T)^{2-k}}{\log n}. \tag{7.9}$$

The integral along the right vertical side of $C$ becomes, upon using (5.33) and (1.16),

$$\frac{1}{2\pi i}\int_{1+\delta+it_1}^{1+\delta+it_2}\frac{Z_1'}{Z_1}(s)\left(\frac{\chi'}{\chi}(s+b)\right)^{-k}n^{-s}\,ds \tag{7.10}$$
$$=-\frac{(-1)^k}{2\pi}\sum_{\ell=0}^{\lfloor\frac{\log T}{\log\log T}\rfloor}2^{\ell}\sum_{m=2}^{\infty}\frac{\Lambda^{(\ell+1)}(m)}{(mn)^{1+\delta}}\int_{t_1}^{t_2}\frac{(mn)^{-it}}{(\log\frac{t}{2\pi})^{k+\ell}}\,dt$$
$$+\frac{(-1)^k}{2\pi}\sum_{\ell=0}^{\lfloor\frac{\log T}{\log\log T}\rfloor}2^{\ell+1}\sum_{m=2}^{\infty}\frac{\lambda^{(\ell)}(m)}{(mn)^{1+\delta}}\int_{t_1}^{t_2}\frac{(mn)^{-it}}{(\log\frac{t}{2\pi})^{k+\ell+1}}\,dt$$
$$+O\left(\frac{k}{n}\int_{t_1}^{t_2}\frac{\left|\frac{Z_1'}{Z_1}(1+\delta+it)\right|}{t\left(\log\frac{t}{2\pi}\right)^{k+1}}\right)dt+O\left(\frac{(\log T)^{-k}}{n}\exp\left(\frac{C_{22}\log T\log\log\log T}{\log\log T}\right)\right).$$

By (5.27)-(5.29),

$$\left|\frac{Z_1'}{Z_1}(1+\delta+it)\right|\ll\log\log|t|,\quad(|t|\geq t_0), \tag{7.11}$$

so the first error term in (7.10) is trivially $\ll\dfrac{k\log\log T}{n(\log T)^{k+1}}\ll\dfrac{(\log T)^{-k}}{n}$. The integrals here are handled by the following lemma.

**Lemma 5** *For $A$ large,*

$$\int_A^B w^{it}\frac{dt}{(\log\frac{t}{2\pi})^k}=\begin{cases}O\left(\frac{1}{|\log w|(\log A)^k}\right) & \text{if } w\neq 1,\\ \frac{B-A}{\left(\log\frac{A}{2\pi}\right)^k}+O\left(\frac{kA}{(\log A)^{k+1}}\right) & \text{if } w=1,\end{cases} \tag{7.12}$$

*where $A<B\leq 2A$, $w>0$, $k\in\mathbb{N}$, $k=o(\log A)$. The constants implied in the $O$-terms are absolute.*

*Proof:* We recall Lemma 4.3 of [21]: Let $F(x)$, $G(x)$ be real functions, $G(x)/F'(x)$ monotonic, and $|F'(x)/G(x)|\geq m>0$ throughout the interval $[a,b]$. Then

$$\left|\int_A^B G(x)e^{iF(x)}dx\right|\leq\frac{4}{m}.$$

Using this, if $w\neq 1$, we have

$$\left|\int_A^B w^{it}\frac{dt}{(\log\frac{t}{2\pi})^k}\right|\leq\frac{4}{|\log w|\left(\log\frac{A}{2\pi}\right)^k}.$$

By Bernoulli's inequality, we have

$$\frac{1}{\left(\log\frac{A}{2\pi}\right)^k}=\frac{1}{(\log A)^k\left(1-\frac{\log 2\pi}{\log A}\right)^k}\le\frac{1}{(\log A)^k\left(1-k\frac{\log 2\pi}{\log A}\right)}\ll\frac{1}{(\log A)^k},$$

since $k=o(\log A)$, which completes the proof of the case $w\neq 1$.

If $w=1$, then integration by parts gives

$$\begin{aligned}\int_A^B\frac{dt}{\left(\log\frac{t}{2\pi}\right)^k}&=\frac{t}{\left(\log\frac{t}{2\pi}\right)^k}\Bigg|_A^B+k\int_A^B\frac{dt}{\left(\log\frac{t}{2\pi}\right)^{k+1}}\\&=\frac{B}{\left(\log\frac{B}{2\pi}\right)^k}-\frac{A}{\left(\log\frac{A}{2\pi}\right)^k}+O\left(\frac{k(B-A)}{\left(\log\frac{A}{2\pi}\right)^{k+1}}\right).\end{aligned}$$

Since $A<B\le 2A$, the last error term is $\ll\frac{kA}{(\log\frac{A}{2\pi})^{k+1}}\ll\frac{kA}{(\log A)^{k+1}}$. To finish the proof we simply observe that $\frac{1}{(\log\frac{B}{2\pi})^k}=\frac{1}{(\log\frac{A}{2\pi})^k}+O\left(\frac{k}{(\log A)^{k+1}}\right)$ by the mean value theorem. □

By Lemma 5, and

$$\frac{\zeta^{(j)}}{\zeta}(s)\sim\frac{(-1)^j j!}{(s-1)^j},\quad (s\to 1;\ j\in\mathbb{Z}^+)\tag{7.13}$$

employed in (5.31), we see that

$$\begin{aligned}&\sum_{\ell=0}^{\lfloor\frac{\log T}{\log\log T}\rfloor}2^\ell\sum_{m=2}^\infty\frac{\Lambda^{(\ell+1)}(m)}{(mn)^{1+\delta}}\int_{t_1}^{t_2}\frac{(mn)^{-it}}{\left(\log\frac{t}{2\pi}\right)^{k+\ell}}\,dt\\&\quad\ll\sum_{\ell=0}^{\lfloor\frac{\log T}{\log\log T}\rfloor}2^\ell\sum_{m=2}^\infty\frac{\Lambda^{(\ell+1)}(m)}{(mn)^{1+\delta}}\frac{1}{(\log mn)(\log T)^{k+\ell}}\\&\quad\ll\frac{(\log T)^{-k}}{n^{1+\delta}\log n}\sum_{\ell=0}^{\lfloor\frac{\log T}{\log\log T}\rfloor}\left(\frac{2}{\log T}\right)^\ell\sum_{m=2}^\infty\frac{\Lambda^{(\ell+1)}(m)}{m^{1+\delta}}\\&\quad\ll\frac{(\log T)^{-k}}{n\log n}\sum_{\ell=0}^{\lfloor\frac{\log T}{\log\log T}\rfloor}\left(\frac{2}{\log T}\right)^\ell\left(-\frac{\zeta'}{\zeta}(1+\delta)\right)^{\ell+1}\\&\quad\ll\frac{(\log T)^{-k}}{\delta n\log n}\sum_{\ell=0}^{\lfloor\frac{\log T}{\log\log T}\rfloor}\left(\frac{3}{\delta\log T}\right)^\ell\ll\frac{(\log T)^{1-k}}{n\log n}\exp\left(\frac{C_{23}\log T}{\log\log T}\right).\end{aligned}\tag{7.14}$$

A similar calculation gives that the same kind of bound holds for the part involving $\lambda^{(\ell)}$ in the right-hand side of (7.10). The bound in (7.14) is smaller than the last error term of (7.10), so we obtain

$$\frac{1}{2\pi i}\int_{1+\delta+it_1}^{1+\delta+it_2}\frac{Z_1'}{Z_1}(s)\left(\frac{\chi'}{\chi}(s+b)\right)^{-k}n^{-s}\,ds \ll \frac{(\log T)^{-k}}{n}\exp\left(\frac{C_{24}\log T\log\log\log T}{\log\log T}\right). \tag{7.15}$$

It remains to calculate the integral along the left vertical side of $C$. In this integral we replace $s$ by $1-s$, using

$$\frac{Z_1'}{Z_1}(s) = \frac{\chi'}{\chi}(s) - \frac{Z_1'}{Z_1}(1-s) \tag{7.16}$$

which follows from (5.25), so that we have

$$\int_{-\delta+it_2}^{-\delta+it_1}\frac{Z_1'}{Z_1}(s)\left(\frac{\chi'}{\chi}(s+b)\right)^{-k}n^{-s}ds = -\int_{1+\delta-it_2}^{1+\delta-it_1}\frac{\chi'}{\chi}(s)\left(\frac{\chi'}{\chi}(1-s+b)\right)^{-k}n^{s-1}ds$$
$$+\int_{1+\delta-it_2}^{1+\delta-it_1}\frac{Z_1'}{Z_1}(s)\left(\frac{\chi'}{\chi}(1-s+b)\right)^{-k}n^{s-1}ds. \tag{7.17}$$

Integrating by parts and then using the estimates (1.16) and (1.17) we see that

$$\begin{aligned}&\int_{1+\delta-it_2}^{1+\delta-it_1}\frac{\chi'}{\chi}(s)\left(\frac{\chi'}{\chi}(1-s+b)\right)^{-k}n^{s-1}\,ds\\ =&\frac{\chi'}{\chi}(s)\left(\frac{\chi'}{\chi}(1-s+b)\right)^{-k}\frac{n^{s-1}}{\log n}\Bigg|_{1+\delta-it_2}^{1+\delta-it_1}\\ &-\frac{1}{\log n}\int_{1+\delta-it_2}^{1+\delta-it_1}\left(\frac{\chi'}{\chi}\right)'(s)\left(\frac{\chi'}{\chi}(1-s+b)\right)^{-k}n^{s-1}\,ds\\ &-\frac{1}{\log n}\int_{1+\delta-it_2}^{1+\delta-it_1}\frac{\chi'}{\chi}(s)\frac{d}{ds}\left\{\left(\frac{\chi'}{\chi}(1-s+b)\right)^{-k}\right\}n^{s-1}ds \ll \frac{(\log T)^{1-k}}{\log n}.\end{aligned} \tag{7.18}$$

Next, letting $s = 1+\delta-it$, and using (7.11), (5.33) and (1.16), we have

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{1+\delta-it_2}^{1+\delta-it_1}\frac{Z_1'}{Z_1}(s)\left(\frac{\chi'}{\chi}(1-s+b)\right)^{-k}n^{s-1}\,ds\\
&\quad=\frac{1}{2\pi i}\int_{1+\delta-iT}^{1+\delta-i\frac{T}{2}}\frac{Z_1'}{Z_1}(s)\left(\frac{\chi'}{\chi}(1-s+b)\right)^{-k}n^{s-1}\,ds+O\left(\frac{\log\log T}{(\log T)^k}\right)\\
&\quad=-\frac{(-1)^k n^{\delta}}{2\pi}\sum_{\ell=0}^{\lfloor\frac{\log T}{\log\log T}\rfloor}2^{\ell}\sum_{m=2}^{\infty}\frac{\Lambda^{(\ell+1)}(m)}{m^{1+\delta}}\int_{\frac{T}{2}}^{T}\left(\frac{m}{n}\right)^{it}\frac{dt}{\left(\log\frac{t}{2\pi}\right)^{k+\ell}}\\
&\qquad+\frac{(-1)^k n^{\delta}}{2\pi}\sum_{\ell=0}^{\lfloor\frac{\log T}{\log\log T}\rfloor}2^{\ell+1}\sum_{m=2}^{\infty}\frac{\lambda^{(\ell)}(m)}{m^{1+\delta}}\int_{\frac{T}{2}}^{T}\left(\frac{m}{n}\right)^{it}\frac{dt}{\left(\log\frac{t}{2\pi}\right)^{k+\ell+1}}\\
&\qquad+O\left(k\int_{\frac{T}{2}}^{T}\left|\frac{Z_1'}{Z_1}(1+\delta-it)\right|\left(\log\frac{t}{2\pi}\right)^{-k-1}\frac{dt}{t}\right)\\
&\qquad+O\left((\log T)^{-k}\exp\left(\frac{C_{25}\log T\log\log\log T}{\log\log T}\right)\right). \qquad (7.19)
\end{aligned}$$

Putting together our findings as of (7.6), we can write

$$\begin{aligned}
&\sum_{\frac{T}{2}<\upsilon\le T}n^{-\varrho}\left(\frac{\chi'}{\chi}(\varrho+b)\right)^{-k}\\
&\quad=-\frac{(-1)^k n^{\delta}}{2\pi}\sum_{\ell=0}^{\lfloor\frac{\log T}{\log\log T}\rfloor}2^{\ell}\sum_{m=2}^{\infty}\frac{\Lambda^{(\ell+1)}(m)}{m^{1+\delta}}\int_{\frac{T}{2}}^{T}\left(\frac{m}{n}\right)^{it}\frac{dt}{\left(\log\frac{t}{2\pi}\right)^{k+\ell}}\\
&\qquad+\frac{(-1)^k n^{\delta}}{2\pi}\sum_{\ell=0}^{\lfloor\frac{\log T}{\log\log T}\rfloor}2^{\ell+1}\sum_{m=2}^{\infty}\frac{\lambda^{(\ell)}(m)}{m^{1+\delta}}\int_{\frac{T}{2}}^{T}\left(\frac{m}{n}\right)^{it}\frac{dt}{\left(\log\frac{t}{2\pi}\right)^{k+\ell+1}}\\
&\qquad+O\left((\log T)^{-k}\exp\left(\frac{C_{26}\log T\log\log\log T}{\log\log T}\right)\right). \qquad (7.20)
\end{aligned}$$

We split the sums over $m$ into three parts: the $m=n$ term, the terms with $0<|m-n|<\frac{n}{2}$, the terms with $|m-n|\ge\frac{n}{2}$. The $m=n$ term gives

$$\frac{(-1)^k}{2\pi n}\int_{\frac{T}{2}}^{T}\sum_{\ell=0}^{\lfloor\frac{\log T}{\log\log T}\rfloor}2^{\ell}\left(\frac{2\lambda^{(\ell)(n)}}{(\log\frac{t}{2\pi})^{k+\ell+1}}-\frac{\Lambda^{(\ell+1)}(n)}{(\log\frac{t}{2\pi})^{k+\ell}}\right)dt, \qquad (7.21)$$

and it will be seen that this is the main term. By Lemma 5, the contribution from the $m\ne n$ terms to (7.20) is

$$\ll\frac{n^{\delta}}{(\log T)^k}\sum_{\ell=0}^{\lfloor\frac{\log T}{\log\log T}\rfloor}\left(\frac{2}{\log T}\right)^{\ell}\sum_{m\ne n}\frac{\Lambda^{(\ell+1)}(m)+\frac{2}{\log T}\lambda^{(\ell)}(m)}{m^{1+\delta}|\log\frac{m}{n}|}. \qquad (7.22)$$

For $m \le \frac{n}{2}$ and $m \ge \frac{3n}{2}$, we have $|\log \frac{m}{n}| \gg 1$, so estimating similarly to (7.14), the contribution from the terms with $|m-n| \ge \frac{n}{2}$ is seen to be

$$\ll (\log T)^{-k} \exp\left(\frac{C_{27}\log T}{\log\log T}\right). \tag{7.23}$$

For the terms with $0 < |m-n| < \frac{n}{2}$ we use $|\log \frac{m}{n}| \gg \frac{|n-m|}{m}$, so the contribution of these terms to (7.20) is

$$\ll (\log T)^{-k} \sum_{\ell=0}^{\lfloor \frac{\log T}{\log\log T}\rfloor} \left(\frac{2}{\log T}\right)^{\ell} \sum_{0<|m-n|<\frac{n}{2}} \frac{\Lambda^{(\ell+1)}(m) + \frac{2}{\log T}\lambda^{(\ell)}(m)}{|m-n|}. \tag{7.24}$$

Estimating the inner sum by Lemmas 1 and 2, we see that (7.24) is majorized by

$$\begin{aligned}
&(\log T)^{-k} \sum_{\ell=0}^{\lfloor \frac{\log T}{\log\log T}\rfloor} \left(\frac{2\log 2n}{\log T}\right)^{\ell+1} \left\{\frac{(\log T)(\log\log 3n+7)_{\ell+1}}{\ell!} + \frac{(\log 2n)(\log\log 3n+7)_{\ell+2}}{(\ell+1)!}\right\}\\
&\ll (\log T)^{1-k} \exp\left(C_{28}\frac{\log T}{\log\log T}\right) \sum_{\ell=0}^{\lfloor \frac{\log T}{\log\log T}\rfloor} \frac{(\log\log 3n+9)_{\ell}}{\ell!}\\
&\ll (\log T)^{-k} \exp\left(C_{29}\frac{\log T}{\log\log T}\right) \frac{(\log\log 3n+10)_{\lfloor \frac{\log T}{\log\log T}\rfloor}}{\lfloor \frac{\log T}{\log\log T}\rfloor !}\\
&= (\log T)^{-k} \exp\left(C_{29}\frac{\log T}{\log\log T}\right) \exp\left(\sum_{j=\lfloor \log\log 3n\rfloor+10}^{\lfloor \frac{\log T}{\log\log T}\rfloor} \log\left(1+\frac{\log\log 3n+9}{j}\right)\right)\\
&\qquad \times \prod_{1\le j\le \lfloor \log\log 3n\rfloor+9} \left(1+\frac{\log\log 3n+9}{j}\right)\\
&\le (\log T)^{-k} \exp\left(C_{29}\frac{\log T}{\log\log T}\right) \exp\left((\log\log 3n+9) \sum_{j=\lfloor \log\log 3n\rfloor+10}^{\lfloor \frac{\log T}{\log\log T}\rfloor} \frac{1}{j}\right)\\
&\qquad \times (\log\log 3n+10)^{\lfloor \log\log 3n\rfloor+9}\\
&\le (\log T)^{-k} \exp\left(C_{30}\frac{\log T}{\log\log T}\right),
\end{aligned} \tag{7.25}$$

where we have used (6.11) and that $n \le T^M$.

Putting the estimates of (7.21)-(7.25) in (7.20) we obtain

**Proposition 4** *Assume RH. Suppose that $T \ge T_0$, $2 \le n \le T^M$, where we will take $M > 1$ to be a suitable fixed number. Let $k \in \mathbb{N}$, $0 \le k \le \lfloor \frac{\log T}{\log\log T}\rfloor$ and $b = -2$ or $2$. Let $\varrho = \frac{1}{2} + i\upsilon$ run through the zeros of $Z_1(s)$. We have*

$$\sum_{\frac{T}{2}<\upsilon\le T} n^{-\varrho}\left(\frac{\chi'}{\chi}(\varrho+b)\right)^{-k} = \frac{(-1)^k}{2\pi n}\int_{\frac{T}{2}}^{T}\sum_{\ell=0}^{\lfloor\frac{\log T}{\log\log T}\rfloor} 2^{\ell}\left(\frac{2\lambda^{(\ell)}(n)}{(\log\frac{t}{2\pi})^{k+\ell+1}} - \frac{\Lambda^{(\ell+1)}(n)}{(\log\frac{t}{2\pi})^{k+\ell}}\right)dt$$
$$+ O\left((\log T)^{-k}\exp\left(\frac{C_{31}\log T\log\log\log T}{\log\log T}\right)\right). \qquad (7.26)$$

From this result we deduce the estimate for (7.2) to be used in this section.

**Corollary 1** *Under the conditions of Proposition 4, with* $2 \le p^a \le T^M$*, where* $p$ *is a prime number and* $a \in \mathbb{Z}^+$*, we have*

$$\sum_{\frac{T}{2}<\upsilon\le T}(p^a)^{-\varrho} = \frac{T\log p}{4\pi p^a}\left(\left(1+\frac{2\log p}{\log T}\right)^a - 2\right)\cdot\left(1+O(\frac{1}{\log T})\right)$$
$$+ O\left(\exp\left(\frac{C_{32}\log T\log\log\log T}{\log\log T}\right)\right). \qquad (7.27)$$

*Proof:* We write (7.26) with $k = 0$. By (6.6) - (6.8), the last term of the summation is with $\ell = \min(a-1, \lfloor\frac{\log T}{\log\log T}\rfloor)$. Then we rearrange the terms of the summation with respect to the power of $\log\frac{t}{2\pi}$. Using the values in (6.6) and (6.7), in the case $a \le \lfloor\frac{\log T}{\log\log T}\rfloor + 1$ the main term coming from (7.26) reduces to

$$\frac{\log p}{2\pi p^a}\Big[-T + \int_{\frac{T}{2}}^{T}\left(1+\frac{2\log p}{\log\frac{t}{2\pi}}\right)^a dt\Big],$$

from which it is straightforward to get to (7.27). The case $a > \lfloor\frac{\log T}{\log\log T}\rfloor + 1$ differs from the previous case within an error

$$\ll \frac{T\log p}{p^a}\sum_{\ell=\lfloor\frac{\log T}{\log\log T}\rfloor}^{a-1}\Big[\binom{a-1}{\ell}\Big(\frac{2\log p}{\log\frac{T}{2\pi}}\Big)^{\ell} + 2\binom{a}{\ell+1}\Big(\frac{2\log p}{\log\frac{T}{2\pi}}\Big)^{\ell+1}$$
$$+\binom{a-1}{\ell}\Big(\frac{2\log p}{\log\frac{T}{2\pi}}\Big)^{\ell+1}\Big]$$
$$\ll \frac{T\log p}{p^a}\sum_{\ell=\lfloor\frac{\log T}{\log\log T}\rfloor}^{a-1}\binom{a}{\ell+1}\left(\frac{C_{33}\log p}{\log T}\right)^{\ell}.$$

Employing the inequality $\binom{a}{\ell+1} \le \Big(\frac{ae}{\ell+1}\Big)^{\ell+1}$, which is a simple consequence of Stirling's formula, the last quantity can be simplified to

$$\frac{T\log p^a}{p^a}\sum_{\ell=\lfloor\frac{\log T}{\log\log T}\rfloor}^{a-1}\Big(\frac{C_{34}a\log p}{\ell\log T}\Big)^{\ell} \ll \frac{T\log p^a}{p^a}\sum_{\ell=\lfloor\frac{\log T}{\log\log T}\rfloor}^{a-1}\Big(\frac{C_{35}}{\ell}\Big)^{\ell}$$
$$\ll \frac{T\log p^a}{p^a}\Big(\frac{C_{36}\log\log T}{\log T}\Big)^{\frac{\log T}{\log\log T}} \ll p^{-a}\exp\Big(\frac{C_{37}\log T\log\log\log T}{\log\log T}\Big),$$

and this is contained in an error term of the type as in (7.27) (if necessary by replacing $C_{32}$ by a larger constant). □

We now resume the calculation for the correlation of zeta zeros with the zeta maxima on the critical line starting with using (7.27) in (7.4). From the sum over $n \le x$ we have

$$\begin{aligned}-x^{-2}\sum_{n\le x}\Lambda(n)n^2\sum_{\frac{T}{2}<\upsilon\le T}n^{-\varrho} &= \frac{T}{4\pi x^2}\sum_{p\le x}p(\log p)^2\Big(1-\frac{2\log p}{\log T}\Big)\Big(1+O(\frac{1}{\log T})\Big)\\ &+O\left(\frac{T}{x^2}\sum_{\substack{p^a\le x\\ a\ge 2}}p^a(\log p)^2\Big(1+\frac{2\log p}{\log T}\Big)^a\right) \qquad (7.28)\\ &+O\left(\frac{1}{x^2}\exp\Big(\frac{C_{38}\log T\log\log\log T}{\log\log T}\Big)\sum_{n\le x}\Lambda(n)n^2\right),\end{aligned}$$

and from the sum over $x < n \le T^M$ we have

$$\begin{aligned}-x^{2}\sum_{x<n\le T^M}\frac{\Lambda(n)}{n^2}\sum_{\frac{T}{2}<\upsilon\le T}n^{-\varrho} &= \frac{Tx^2}{4\pi}\sum_{x<p\le T^M}\frac{(\log p)^2}{p^3}\Big(1-\frac{2\log p}{\log T}\Big)\Big(1+O(\frac{1}{\log T})\Big)\\ &+O\left(Tx^2\sum_{\substack{x<p^a\le T^M\\ a\ge 2}}\frac{(\log p)^2}{p^{3a}}\Big(1+\frac{2\log p}{\log T}\Big)^a\right) \qquad (7.29)\\ &+O\left(x^2\exp\Big(\frac{C_{38}\log T\log\log\log T}{\log\log T}\Big)\sum_{x<n\le T^M}\frac{\Lambda(n)}{n^2}\right).\end{aligned}$$

Now we use some consequences of the prime number theorem obtained by partial summation. The contribution of the last error terms of (7.28) and (7.29) is

$$\ll x\exp\Big(\frac{C_{38}\log T\log\log\log T}{\log\log T}\Big). \qquad (7.30)$$

For the main terms of (7.28) and (7.29) we use the following which are obtained from (4.5) by partial summation:

$$\sum_{p\le x} p\log^2 p = \frac{x^2\log x}{2} - \frac{x^2}{4} + O(x^{\frac{3}{2}}\log^3 2x)$$

$$\sum_{p\le x} p\log^3 p = \frac{x^2\log^2 x}{2} - \frac{x^2\log x}{2} + \frac{x^2}{4} + O(x^{\frac{3}{2}}\log^4 2x)$$

$$\sum_{x<p<T^M} \frac{\log^2 p}{p^3} = \frac{\log x}{2x^2} - \frac{1}{4x^2} + O\Big(\frac{\log^3 2x}{x^{\frac{5}{2}}}\Big) + O\Big(\frac{\log T}{T^{2M}}\Big) \tag{7.31}$$

$$\sum_{x<p<T^M} \frac{\log^3 p}{p^3} = \frac{\log^2 x}{2x^2} + \frac{\log x}{x^2} + \frac{1}{4x^2} + O\Big(\frac{\log^4 2x}{x^{\frac{5}{2}}}\Big) + O\Big(\frac{\log^2 T}{T^{2M}}\Big)$$

Then, the main term of the sums on the right-hand side of (7.4) is seen to be

$$\frac{T}{4\pi}\Big(\log x - \frac{2\log^2 x}{\log T} - \frac{1}{2} - \frac{\log x}{\log T}\Big)\Big(1 + O(\frac{1}{\log T})\Big) \tag{7.32}$$
$$+ O(Tx^{-\frac{1}{2}}\log^3 2x) + O(x^2 T^{1-2M}\log T).$$

Just by comparing (7.30) and (7.32) we see that we are forced to put a restriction such as $x \le T^{1-\epsilon}$ in order to have an asymptotic estimate. It remains to bound the first error terms in (7.28) and (7.29). We have

$$\Big(1 + \frac{2\log p}{\log T}\Big)^a < e^{\frac{2a\log p}{\log T}} \le e^{2M},$$

$$\sum_{\substack{p^a\le x\\ a\ge 2}} p^a(\log p)^2 \le \sum_{a\ge 2} x\log(x^{\frac{1}{a}}) \sum_{p\le x^{\frac{1}{a}}} \log p \ll x\log x \sum_{a\ge 2}\frac{x^{\frac{1}{a}}}{a} \ll x^{\frac{3}{2}}\log 2x,$$

$$\sum_{\substack{x<p^a\le T^M\\ a\ge 2}} \frac{(\log p)^2}{p^{3a}} \ll \sum_{a\ge 2}\frac{\log(x^{\frac{1}{a}})}{x^{3-\frac{2}{a}}} \sum_{p\ge x^{\frac{1}{a}}} \frac{\log p}{p^2} \ll \frac{\log x}{x^3}\sum_{a\ge 2}\frac{x^{\frac{1}{a}}}{a} \ll \frac{\log x}{x^{\frac{5}{2}}}, \tag{7.33}$$

so that the first error terms in (7.28) and (7.29) are

$$\ll Tx^{-\frac{1}{2}}\log 2x. \tag{7.34}$$

Combining (7.4), (7.30), (7.32), (7.34), and taking a fixed $M > \frac{4}{3}$, we have

$$\sum_{\gamma}\sum_{\frac{T}{2}<\upsilon\le T} \frac{4x^{i(\gamma-\upsilon)}}{4+(\gamma-\upsilon)^2} \tag{7.35}$$
$$= \frac{T\log x}{4\pi}\left(1 - \frac{2\log x}{\log T}\right) + O(T) + \frac{T\log^2 T}{4\pi x^2}\left(1 + O\left(\frac{1}{\log T}\right)\right)$$

for $1 \le x < T^{1-\epsilon}$ as $T \to \infty$.

Now we write the version of (7.35) for each case with $\frac{T}{2^{k+1}} < \upsilon \le \frac{T}{2^k}$, $k = 0, 1, \ldots, \lfloor \frac{4 \log\log T}{\log 2} \rfloor$, and add up the results. For $0 < \upsilon < \frac{T}{\log^4 T}$ we use the trivial estimate

$$\sum_{\gamma} \sum_{0<\upsilon\le\frac{T}{\log^4 T}} \frac{4x^{i(\gamma-\upsilon)}}{4+(\gamma-\upsilon)^2} \ll \frac{T}{\log^4 T} \cdot \log^2 T \ll \frac{T}{\log^2 T}. \tag{7.36}$$

Then, in the same way as for (4.2), we discard the terms with $\gamma \notin (0, T]$ within an error of $O(\log^3 T)$. Thus we obtain

**Theorem 1** *Assume RH. Let $\frac{1}{2} + i\gamma$ and $\frac{1}{2} + i\upsilon$ run through the critical zeros of $\zeta(s)$ and of $Z_1(s)$ respectively. We have, uniformly for $1 \le x \le T^{1-\epsilon}$,*

$$\begin{aligned} F_{\zeta, Z_1}(x, T) &:= \sum_{0<\gamma\le T} \sum_{0<\upsilon\le T} \frac{4x^{i(\gamma-\upsilon)}}{4+(\gamma-\upsilon)^2} \\ &= \frac{T\log x}{2\pi}\left(1 - \frac{2\log x}{\log T}\right) + \frac{T\log^2 T}{2\pi x^2}\left(1 + O\left(\frac{1}{\log T}\right)\right) + O(T), \end{aligned} \tag{7.37}$$

*as $T \to \infty$, in which case $F_{\zeta, Z_1}(x, T)$ is asymptotically real.*

*Remark:* Since the main term of $F_{\zeta, Z_1}(x, T)$ has a sign change in $1 \le x \le T^{1-\epsilon}$, we can't deduce conclusions as Montgomery did using the positivity of $F_{\zeta,\zeta}(x, T)$.

## 8 Pair correlation of zeros of $Z_1(s)$

We shall first derive an explicit formula involving the zeros of $Z_1(s)$, an analogue of (2.2), which relates a sum over the non-trivial zeros $\varrho = \frac{1}{2} + i\upsilon$ of $Z_1(s)$ to a sum over involving arithmetical functions. We begin by considering the integral

$$\oint_{\mathcal{R}} \frac{Z_1'}{Z_1}(w + \frac{1}{2}) K(w, s) x^w \, dw, \quad (x \ge 1), \tag{8.1}$$

along the rectangular contour $\mathcal{R}$ with vertices at $c \pm iY$, $-U \pm +iY$, with

$$\begin{aligned} &c = \frac{1}{2} + \epsilon_0, \quad (\epsilon_0 > 0 \text{ can be arbitrarily small but fixed}), \\ &U = 2k + 1, \quad (k: \text{ large natural number}), \quad \frac{U}{4} \ge \frac{Y}{2} \ge |t| \\ &w = u + iy,\, u, y \in \mathbb{R}, \quad \sigma \in [\frac{3}{2}, \frac{11}{4}] \end{aligned} \tag{8.2}$$

(the restriction on $\sigma$ is just for convenience, we can take $\sigma > 1 + \epsilon$; later on we will use our explicit formula with $\sigma = \frac{5}{2}$), and

$$K(w,s) := \frac{2\sigma - 1}{(w-(s-\frac{1}{2}))(w-(\frac{1}{2}-\overline{s}))}. \tag{8.3}$$

The kernel $K(w,s)$ was introduced by Farmer, Gonek and Lee in [5] in their derivation of an explicit formula involving the zeros of $\xi'(s)$, where $\xi(s) = \frac{s(s-1)}{2}\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s)$. The integrand is regular on the contour of integration. By (5.26) the left vertical side of $\mathcal{R}$ is at a distance $\geq \frac{1}{2}$ from any zero of $Z_1(w+\frac{1}{2})$. By (7.16), we have

$$\frac{Z_1'}{Z_1}(-2k-\frac{1}{2}+iy) = \frac{\chi'}{\chi}(-2k-\frac{1}{2}+iy) - \frac{Z_1'}{Z_1}(2k+\frac{3}{2}-iy),$$

and by using the last member of (1.16) we see that

$$\frac{\chi'}{\chi}(-2k-\frac{1}{2}+iy) = O(\log|w|).$$

From (5.27) we observe that $\frac{Z_1'}{Z_1}(2k+\frac{3}{2}-iy)$ takes on small values because here the values of $\frac{\zeta'}{\zeta}$ and $\frac{\zeta''}{\zeta}$ are small, and the estimates (1.16) and (1.17) are applicable as can be seen through the middle member of (1.14). Hence we have

$$\frac{Z_1'}{Z_1}(w+\frac{1}{2}) = O(\log|w|), \quad (\Re w = -U = -2k-1), \tag{8.4}$$

and

$$\int_{-U-iY}^{-U+iY} \frac{Z_1'}{Z_1}(w+\frac{1}{2})K(w,s)x^w\,dw \ll \frac{Y\log U}{x^U U^2}. \tag{8.5}$$

The estimate (8.4) also holds on the part $\Re w \leq -1$ of the horizontal sides of $\mathcal{R}$ as can be seen again from (7.16). On the part $-1 \leq \Re w \leq c$ of the horizontal sides, we use (7.8). Thus the horizontal sides contribute

$$\begin{aligned} &\ll \int_c^{-1} \frac{\log^2 Y}{Y^2} x^u\,du + \int_{-1}^{-Y} \frac{\log Y}{Y^2} x^u\,du + \int_{-Y}^{-U} \frac{\log(u^2+Y^2)}{u^2+Y^2} x^u\,du \\ &\ll x^c \frac{\log^2 Y}{Y^2} + \frac{\log Y}{Yx} + x^{-Y}\frac{\log Y}{Y}. \end{aligned} \tag{8.6}$$

If we first let $U \to \infty$, and then let $Y \to \infty$, the integrals along the horizontal and left vertical sides will tend to 0. In the region enclosed by $\mathcal{R}$, the poles of the integrand are all simple poles at the points $w = -\frac{1}{2}, 0, \frac{1}{2}, \frac{1}{2}-\overline{s}, iv$ and at the points $\frac{1}{2}-z_\ell$, $\ell \in \mathbb{Z}^+$. Thus, by the residue theorem we obtain

$$\frac{1}{2\pi i}\int_{(c)}\frac{Z_1'}{Z_1}(w+\frac{1}{2})K(w,s)x^w\,dw = -K\left(-\frac{1}{2},s\right)x^{-\frac{1}{2}}+K(0,s)-2K\left(\frac{1}{2},s\right)x^{\frac{1}{2}}$$
$$-\frac{Z_1'}{Z_1}(1-\overline{s})x^{\frac{1}{2}-\overline{s}}+\sum_{\varrho}K(i\upsilon,s)x^{i\upsilon}+\sum_{\ell=1}^{\infty}K(\frac{1}{2}-z_\ell,s)x^{\frac{1}{2}-z_\ell}, \tag{8.7}$$

where $\int_{(c)}$ denotes that the integral is over the whole line $\Re\, w = c$. To simplify this formula, we use the estimates

$$\left|K\left(-\frac{1}{2},s\right)x^{-\frac{1}{2}}\right| = \frac{(2\sigma-1)x^{-\frac{1}{2}}}{|s(1-\overline{s})|} \ll \frac{x^{-\frac{1}{2}}}{1+t^2},$$
$$\left|K\left(\frac{1}{2},s\right)x^{\frac{1}{2}}\right| = \frac{(2\sigma-1)x^{\frac{1}{2}}}{|(1-s)\overline{s}|} \ll \frac{x^{\frac{1}{2}}}{1+t^2},$$
$$|K(0,s)| = \frac{2\sigma-1}{|s-\frac{1}{2}|^2} \ll \frac{1}{1+t^2},$$

and we majorize the last sum in (8.7), by using the information given in (5.26), as

$$\sum_{\ell=1}^{\infty}K(\frac{1}{2}-z_\ell,s)x^{\frac{1}{2}-z_\ell} \ll \sum_{\ell=1}^{\infty}\frac{x^{\frac{1}{2}-z_\ell}}{\ell^2+t^2} \ll \frac{x^{-\frac{5}{2}}}{1+|t|}.$$

We can now re-write (8.7) as

$$\frac{1}{2\pi i}\int_{(c)}\frac{Z_1'}{Z_1}(w+\frac{1}{2})K(w,s)x^w\,dw = -\sum_{\varrho}\frac{(2\sigma-1)x^{i\upsilon}}{(\sigma-\frac{1}{2})^2+(t-\upsilon)^2}-x^{\frac{1}{2}-\overline{s}}\frac{Z_1'}{Z_1}(1-\overline{s})$$
$$+O\left(\frac{x^{\frac{1}{2}}}{1+|t|^2}\right)+O\left(\frac{x^{-\frac{5}{2}}}{1+|t|}\right). \tag{8.8}$$

For use in the evaluation of the line integral in (8.8), we expand the expression in (5.27) for $\frac{Z_1'}{Z_1}(z)$, where $\Re\, z = c+\frac{1}{2} = 1+\epsilon_0 > 1, \Im\, z \geq A$ and $A$ is some sufficiently large constant, by keeping the factors $\frac{\chi'}{\chi}(z)$ intact. Assuming RH, we have

$$\frac{Z_1'}{Z_1}(z) = -\sum_{k=0}^{\lfloor\frac{\log T}{\log\log T}\rfloor}\left(\frac{-2}{\frac{\chi'}{\chi}(z)}\right)^k\sum_{m=1}^{\infty}\frac{\Lambda^{(k+1)}(m)}{m^z}+\sum_{k=0}^{\lfloor\frac{\log T}{\log\log T}\rfloor}\left(\frac{-2}{\frac{\chi'}{\chi}(z)}\right)^{k+1}\sum_{m=1}^{\infty}\frac{\lambda^{(k)}(m)}{m^z}$$
$$+O\left(\exp\left(\left(C_{39}(\epsilon_0)-\log\log|z|\right)\frac{\log T}{\log\log T}\right)\right)+O\left(\frac{1}{|z|\log|z|}\right). \tag{8.9}$$

Next, we split the integral into several parts as

$$\frac{1}{2\pi i}\int_{(c)}\frac{Z_1'}{Z_1}(w+\frac{1}{2})K(w,s)x^w\,dw = \tag{8.10}$$

$$\begin{aligned}
&-\sum_{k=0}^{\lfloor\frac{\log T}{\log\log T}\rfloor}(-2)^k\sum_{m=1}^{\infty}\frac{\Lambda^{(k+1)}(m)}{m^{\frac{1}{2}}}\frac{1}{2\pi i}\int_{(c)}\frac{K(w,s)}{\left(\frac{\chi'}{\chi}(w+\frac{1}{2})\right)^k}\left(\frac{x}{m}\right)^w dw\\
&+\sum_{k=0}^{\lfloor\frac{\log T}{\log\log T}\rfloor}(-2)^{k+1}\sum_{m=1}^{\infty}\frac{\lambda^{(k)}(m)}{m^{\frac{1}{2}}}\frac{1}{2\pi i}\int_{(c)}\frac{K(w,s)}{\left(\frac{\chi'}{\chi}(w+\frac{1}{2})\right)^{k+1}}\left(\frac{x}{m}\right)^w dw\\
&+\sum_{k=0}^{\lfloor\frac{\log T}{\log\log T}\rfloor}(-2)^k\sum_{m=1}^{\infty}\frac{\Lambda^{(k+1)}(m)}{m^{\frac{1}{2}}}\frac{1}{2\pi i}\int_{c-iA}^{c+iA}\frac{K(w,s)}{\left(\frac{\chi'}{\chi}(w+\frac{1}{2})\right)^k}\left(\frac{x}{m}\right)^w dw\\
&-\sum_{k=0}^{\lfloor\frac{\log T}{\log\log T}\rfloor}(-2)^{k+1}\sum_{m=1}^{\infty}\frac{\lambda^{(k)}(m)}{m^{\frac{1}{2}}}\frac{1}{2\pi i}\int_{c-iA}^{c+iA}\frac{K(w,s)}{\left(\frac{\chi'}{\chi}(w+\frac{1}{2})\right)^{k+1}}\left(\frac{x}{m}\right)^w dw\\
&+O\left(x^c\int_{-iA}^{iA}\left|\frac{Z_1'}{Z_1}(c+\frac{1}{2}+iy)K(c+iy,s)\right|dy\right)\\
&+O\left(x^c\int_{|y|\geq A}|K(c+iy,s)|\left(\exp\left(\frac{(C_{39}(\epsilon_0)-\log\log|y|)\log T}{\log\log T}\right)+\frac{1}{|y|\log|y|}\right)dy\right)\\
&=I_1+I_2+\cdots+I_6,\ \text{say.}
\end{aligned}$$

We will work with large $|t|\in[2A,T]$; later on $T$ will be a parameter which tends to $\infty$. By using

$$K(c+iy,s)\ll\frac{1}{1+(y-t)^2}, \tag{8.11}$$

it is easily seen that

$$I_5\ll\frac{x^c}{1+t^2},\quad I_6\ll\frac{x^c}{1+|t|}\exp\left(\frac{C_{40}(\epsilon_0)\log T}{\log\log T}\right). \tag{8.12}$$

Now note that since the zero of $\chi'(s)$ is in $(\frac{5}{4},2)$, if we take $\epsilon_0<\frac{1}{5}$, then by (1.16) we have $\left|\frac{\chi'}{\chi}(w+\frac{1}{2})\right|^{-1}\leq C_{41}$, so that

$$\begin{aligned}
I_3&\ll x^c\sum_{k=0}^{\lfloor\frac{\log T}{\log\log T}\rfloor}(2C_{41})^k\sum_{m=1}^{\infty}\frac{\Lambda^{(k+1)}(m)}{m^{c+\frac{1}{2}}}\int_{-A}^{A}|k(c+iy,s)|\,dy \qquad (8.13)\\
&\ll\frac{x^c}{1+t^2}\sum_{k=0}^{\lfloor\frac{\log T}{\log\log T}\rfloor}\left(2C_{41}\frac{\zeta'}{\zeta}(c+\frac{1}{2})\right)^{k+1}\ll\frac{x^c}{1+t^2}C_{42}(\epsilon_0)^{\frac{\log T}{\log\log T}},
\end{aligned}$$

and similarly $I_4$ satisfies the same bound. Thus

$$\frac{1}{2\pi i}\int_{(c)}\frac{Z_1'}{Z_1}(w+\frac{1}{2})K(w,s)x^w\,dw = I_1+I_2+O\left(\frac{x^c}{|t|}\exp\left(\frac{C_{43}(\epsilon_0)\log T}{\log\log T}\right)\right). \quad (8.14)$$

The evaluation of $I_1$ and $I_2$ splits into two cases according as $m \le x$ or not. Consider first the situation when $m \le x$ in which case the line of integration $\Re w = c$ will be pulled to the left. Let $U = 2n+1$. Then, by (1.14), we see that $\frac{\chi'}{\chi}(-U+\frac{1}{2}+it) \gg \log U$. Between the lines $\Re w = c$ and $\Re w = -U$ the integrands have poles coming from the zeros of the $\frac{\chi'}{\chi}$ factor at $w = \kappa_\ell - \frac{1}{2}$, $\ell = 1, 2, \ldots, n$, at $w = \pm i\tau$ by (5.10) and (5.22), and a pole arising from $K(w,s)$ at $w = \frac{1}{2} - \overline{s}$. By the residue theorem we write

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{c-iY}^{c+iY}\frac{K(w,s)}{\left(\frac{\chi'}{\chi}(w+\frac{1}{2})\right)^k}\left(\frac{x}{m}\right)^w dw\\
&= \text{ Residues } + \left(\int_{c-iY}^{-U-iY}+\int_{-U-iY}^{-U+iY}+\int_{-U+iY}^{c+iY}\right)\frac{K(w,s)}{\left(\frac{\chi'}{\chi}(w+\frac{1}{2})\right)^k}\left(\frac{x}{m}\right)^w dw
\end{aligned}$$

with the parameters having the same specifications as in (8.2). The last three integrals all tend to 0 as $n \to \infty$ (i.e. $-U \to -\infty$) and $Y \to \infty$ as in (8.5)-(8.6). Now, applying Cauchy's estimate on a disk of radius $\epsilon_1$ centered at $\kappa_\ell - \frac{1}{2}$, for $k \ge 1$ we have

$$\begin{aligned}
&\underset{w=\kappa_\ell-1/2}{\text{Res}}\left\{\frac{K(w,s)}{\left(\frac{\chi'}{\chi}(w+1/2)\right)^k}\left(\frac{x}{m}\right)^w\right\}\\
&= \frac{1}{(k-1)!}\frac{d^{k-1}}{dw^{k-1}}\left\{K(w,s)\left(\frac{w-(\kappa_\ell-\frac{1}{2})}{\frac{\chi'}{\chi}(w+\frac{1}{2})}\right)^k\left(\frac{x}{m}\right)^w\right\}_{w=\kappa_\ell-\frac{1}{2}}\\
&\ll \frac{C_{44}(\epsilon_1)^k\left(\frac{x}{m}\right)^{\kappa_\ell-\frac{1}{2}+\epsilon_1}}{(\ell+|t|)^2}.
\end{aligned}$$

Summing over all $\ell$, we can include the contribution from all of these residues in

$$\frac{C_{45}(\epsilon_1)^k}{1+|t|}\left(\frac{x}{m}\right)^{\kappa_1-\frac{1}{2}+\epsilon_1}$$

Similarly we see that

$$\underset{w=\pm i\tau}{\text{Res}}\left\{\frac{K(w,s)}{\left(\frac{\chi'}{\chi}(w+1/2)\right)^k}\left(\frac{x}{m}\right)^w\right\} \ll \frac{C_{46}(\epsilon_1)^k\left(\frac{x}{m}\right)^{\epsilon_1}}{1+t^2}, \quad (k \ge 1),$$

and

$$\operatorname*{Res}_{w=\frac{1}{2}-\overline{s}} \left\{ \frac{K(w,s)}{\left(\frac{\chi'}{\chi}(w+\frac{1}{2})\right)^k} \left(\frac{x}{m}\right)^w \right\} = -\left(\frac{\chi'}{\chi}(1-\overline{s})\right)^{-k} \left(\frac{x}{m}\right)^{\frac{1}{2}-\overline{s}}$$

(for $\sigma \in [\frac{3}{2}, \frac{11}{4}]$ this pole is away from the zeros of $\frac{\chi'}{\chi}(1-\overline{s})$). Collecting the foregoing estimates, we obtain (with no error term if $k = 0$)

$$\frac{1}{2\pi i}\int_{(c)} \frac{K(w,s)}{\left(\frac{\chi'}{\chi}(w+\frac{1}{2})\right)^k} \left(\frac{x}{m}\right)^w dw = -\left(\frac{\chi'}{\chi}(1-\overline{s})\right)^{-k} \left(\frac{x}{m}\right)^{\frac{1}{2}-\overline{s}} + O\left(\frac{C_{45}(\epsilon_1)^k}{1+|t|}\left(\frac{x}{m}\right)^{\kappa_1-\frac{1}{2}+\epsilon_1}\right) + O\left(\frac{C_{46}(\epsilon_1)^k}{1+t^2}\left(\frac{x}{m}\right)^{\epsilon_1}\right), \quad (m \le x). \tag{8.15}$$

In case $m > x$, the calculation is done similarly, but now the line of integration is pulled to the right to $\Re w = 2n$, followed by letting the integer $n \longrightarrow \infty$ . The poles encountered are at $w = \frac{1}{2} - \kappa_\ell$, $\ell = 1, 2, \ldots$ from the $\frac{\chi'}{\chi}$ factor, and at $w = s - \frac{1}{2}$ from $K(w, s)$. The result is (with no error term if $k = 0$)

$$\frac{1}{2\pi i}\int_{(c)} \frac{K(w,s)}{\left(\frac{\chi'}{\chi}(w+\frac{1}{2})\right)^k} \left(\frac{x}{m}\right)^w dw = -\left(\frac{\chi'}{\chi}(s)\right)^{-k} \left(\frac{x}{m}\right)^{s-\frac{1}{2}} + O\left(\frac{C_{47}(\epsilon_1)^k}{1+|t|}\left(\frac{x}{m}\right)^{\frac{1}{2}-\kappa_1-\epsilon_1}\right), \quad (m > x). \tag{8.16}$$

Writing, for brevity,

$$\mathfrak{L}(m,s) := \sum_{k=0}^{\lfloor \frac{\log T}{\log\log T} \rfloor} (-2)^k \left(\Lambda^{(k+1)}(m)\left(\frac{\chi'}{\chi}(s)\right)^{-k} + 2\lambda^{(k)}(m)\left(\frac{\chi'}{\chi}(s)\right)^{-k-1}\right), \tag{8.17}$$

we see from (8.15) - (8.17) that

$$
\begin{aligned}
I_1 + I_2 = x^{-\frac{1}{2}} &\left( \sum_{m \le x} \mathfrak{L}(m, 1-\overline{s}) \left(\frac{x}{m}\right)^{1-\overline{s}} + \sum_{m > x} \mathfrak{L}(m,s) \left(\frac{x}{m}\right)^{s} \right) \\
&+ O\left( \frac{x^{\kappa_1 - \frac{1}{2} + \epsilon_1}}{1+|t|} \sum_{k=0}^{\lfloor \frac{\log T}{\log\log T} \rfloor} C_{48}(\epsilon_1)^k \sum_{m \le x} \frac{\Lambda^{(k+1)}(m) + \lambda^{(k)}(m)}{m^{\kappa_1 + \epsilon_1}} \right) \\
&+ O\left( \frac{x^{\epsilon_1}}{1+t^2} \sum_{k=0}^{\lfloor \frac{\log T}{\log\log T} \rfloor} C_{49}(\epsilon_1)^k \sum_{m \le x} \frac{\Lambda^{(k+1)}(m) + \lambda^{(k)}(m)}{m^{\frac{1}{2} + \epsilon_1}} \right) \\
&+ O\left( \frac{x^{\frac{1}{2} - \kappa_1 - \epsilon_1}}{1+|t|} \sum_{k=0}^{\lfloor \frac{\log T}{\log\log T} \rfloor} C_{50}(\epsilon_1)^k \sum_{m > x} \frac{\Lambda^{(k+1)}(m) + \lambda^{(k)}(m)}{m^{1-\kappa_1 - \epsilon_1}} \right).
\end{aligned}
$$

We can put a bound on the error terms by first replacing $m^{-\kappa_1-\epsilon_1}$ with $m^{-1-\epsilon_1}x^{1-\kappa_1}$ in the first series over $m \le x$, and $m^{-\frac{1}{2}-\epsilon_1}$ by $x^{\frac{1}{2}}m^{-1-\epsilon_1}$ in the second series over $m \le x$, and $m^{-1+\kappa_1+\epsilon_1}$ with $m^{-1-\epsilon_1}x^{\kappa_1+2\epsilon_1}$ for $m > x$, and then using (7.13) to bound the resulting Dirichlet series, so that we have

$$
\begin{aligned}
I_1 + I_2 = x^{-\frac{1}{2}} &\left( \sum_{m \le x} \mathfrak{L}(m, 1-\overline{s}) \left(\frac{x}{m}\right)^{1-\overline{s}} + \sum_{m > x} \mathfrak{L}(m,s) \left(\frac{x}{m}\right)^{s} \right) \\
&+ O\left( \frac{x^{\frac{1}{2}+\epsilon_1}}{1+|t|} \exp\left( \frac{C_{51}(\epsilon_1)\log T}{\log\log T} \right) \right). \qquad (8.18)
\end{aligned}
$$

The last step for completing the derivation of the explicit formula for $Z_1(s)$ is replacing the term $-x^{\frac{1}{2}-\overline{s}}\frac{Z_1'}{Z_1}(1-\overline{s})$ in (8.8) with

$$
x^{\frac{1}{2}-\overline{s}}\frac{Z_1'}{Z_1}(\overline{s}) - x^{\frac{1}{2}-\overline{s}}\frac{\chi'}{\chi}(\overline{s}) = x^{\frac{1}{2}-\overline{s}}\log\frac{|t|}{2\pi} + O_\sigma(x^{\frac{1}{2}-\sigma}), \qquad (8.19)
$$

by using (7.16).

Combining (8.8), (8.14), (8.17)-(8.19) we obtain

**Proposition 5** *(Explicit formula; assuming RH) For $s = \sigma + it$ with $\frac{3}{2} \le \sigma \le \frac{11}{4}$, $|t| \in [2A, T]$ with $A$ a sufficiently large constant, and for any arbitrarily small fixed $\epsilon > 0$, we have, as $T \to \infty$,*

$$
\begin{aligned}
\sum_{\varrho} \frac{(2\sigma-1)x^{i\nu}}{(\sigma-\frac{1}{2})^2 + (\nu - t)^2} = -x^{-\frac{1}{2}} &\left( \sum_{m \le x} \mathfrak{L}(m, 1-\overline{s}) \left(\frac{x}{m}\right)^{1-\overline{s}} + \sum_{m > x} \mathfrak{L}(m,s) \left(\frac{x}{m}\right)^{s} \right) \\
&+ x^{\frac{1}{2}-\overline{s}}\log\frac{|t|}{2\pi} + O\left( \frac{x^{\frac{1}{2}+\epsilon}}{|t|} \exp\left( \frac{C_{52}(\epsilon)\log T}{\log\log T} \right) \right) + O_\sigma(x^{\frac{1}{2}-\sigma}), \qquad (8.20)
\end{aligned}
$$

*where $\mathfrak{L}(m,s)$ was defined in (8.17).*

In (8.20) we take $\sigma = \frac{5}{2}$ and sum both sides over $t = \tilde{\upsilon} \in (\frac{T}{2}, T]$, where $\frac{1}{2} + i\tilde{\upsilon}$ runs through the non-trivial zeros of $Z_1(s)$. In view of (5.24) and (1.1) we have

$$\sum_{\upsilon} \sum_{\frac{T}{2} < \tilde{\upsilon} \le T} \frac{4x^{i(\upsilon-\tilde{\upsilon})}}{4+(\upsilon-\tilde{\upsilon})^2} \tag{8.21}$$

$$= -x^{-2} \sum_{m \le x} \sum_{\frac{T}{2} < \tilde{\upsilon} \le T} \mathfrak{L}(m, -\frac{3}{2} + i\tilde{\upsilon}) m^{\frac{3}{2} - i\tilde{\upsilon}} - x^2 \sum_{m > x} \sum_{\frac{T}{2} < \tilde{\upsilon} \le T} \mathfrak{L}(m, \frac{5}{2} + i\tilde{\upsilon}) m^{-\frac{5}{2} - i\tilde{\upsilon}}$$
$$+ \frac{T \log^2 T}{4\pi x^2} \left(1 + O\left(\frac{1}{\log T}\right)\right) + O\left(x^{\frac{1}{2}+\epsilon} \exp\left(\frac{C_{53}(\epsilon) \log T}{\log\log T}\right)\right).$$

We do not hope to get an asymptotic estimate for $x > T$, so just as we did in connection with (7.3), let us assume $x \le T^M$ and first discard the terms with $m > T^M$ for some fixed $M > \frac{4}{3}$. Upon using (1.16) and trivially estimating the sum over $\tilde{\upsilon}$ by (5.24), we have

$$x^2 \sum_{m > T^M} \sum_{\frac{T}{2} < \tilde{\upsilon} \le T} \mathfrak{L}(m, \frac{5}{2} + i\tilde{\upsilon}) m^{-\frac{5}{2} - i\tilde{\upsilon}}$$
$$= x^2 \sum_{k=0}^{\lfloor \frac{\log T}{\log\log T} \rfloor} (-2)^k \sum_{m > T^M} m^{-\frac{5}{2}} \Big[ \sum_{\frac{T}{2} < \tilde{\upsilon} \le T} \Lambda^{(k+1)}(m) \left(\frac{\chi'}{\chi}(\frac{5}{2} + i\tilde{\upsilon})\right)^{-k} m^{-i\tilde{\upsilon}}$$
$$+ \sum_{\frac{T}{2} < \tilde{\upsilon} \le T} 2\lambda^{(k)}(m) \left(\frac{\chi'}{\chi}(\frac{5}{2} + i\tilde{\upsilon})\right)^{-(k+1)} m^{-i\tilde{\upsilon}} \Big]$$
$$\ll x^2 T \log T \Big[ \sum_{k=0}^{\lfloor \frac{\log T}{\log\log T} \rfloor} \left(\frac{2}{\log \frac{T}{2\pi}}\right)^k \sum_{m > T^M} \frac{\Lambda^{(k+1)}(m)}{m^{\frac{5}{2}}}$$
$$+ \sum_{k=0}^{\lfloor \frac{\log T}{\log\log T} \rfloor} \left(\frac{2}{\log \frac{T}{2\pi}}\right)^{k+1} \sum_{m > T^M} \frac{\lambda^{(k)}(m)}{m^{\frac{5}{2}}} \Big].$$

By (6.1) and (6.2) we see

$$\sum_{m > T^M} \frac{\Lambda^{(k+1)}(m)}{m^{\frac{5}{2}}} \ll \frac{(\log T^M)^{k+1}}{T^{\frac{3}{2}M}}, \quad \sum_{m > T^M} \frac{\lambda^{(k)}(m)}{m^{\frac{5}{2}}} \ll \frac{(\log T^M)^{k+2}}{T^{\frac{3}{2}M}},$$

so that

$$x^2 \sum_{m > T^M} \sum_{\frac{T}{2} < \tilde{\upsilon} \le T} \mathfrak{L}(m, \frac{5}{2} + i\tilde{\upsilon}) m^{-\frac{5}{2} - i\tilde{\upsilon}} \ll x^2 T^{1 - \frac{3M}{2}} \exp\left(\frac{C_{54} \log T}{\log\log T}\right). \tag{8.22}$$

For the $m \le T^M$ terms we use Proposition 4. We first note that the contribution of the error term of (7.26) to (8.21) is

$$x \exp\left(\frac{C_{55} \log T \log\log\log T}{\log\log T}\right), \tag{8.23}$$

by a calculation similar to that leading to (8.22). Putting these together we have

$$\begin{aligned}
&\sum_{\upsilon} \sum_{\frac{T}{2} < \tilde{\upsilon} \le T} \frac{4x^{i(\upsilon - \tilde{\upsilon})}}{4 + (\upsilon - \tilde{\upsilon})^2} \qquad (8.24)\\
&= \frac{U(0,0;x) - 2U(0,1;x) + U(1,1;x)}{2\pi x^2} + \frac{x^2}{2\pi}\left(\tilde{U}(0,0;x) - 2\tilde{U}(0,1;x) + \tilde{U}(1,1;x)\right)\\
&\quad + \frac{T \log^2 T}{4\pi x^2}\left(1 + O\left(\frac{1}{\log T}\right)\right) + O\left(x \exp\left(\frac{C_{56} \log T \log\log\log T}{\log\log T}\right)\right),
\end{aligned}$$

where, for $\iota_1$ and $\iota_2$ taking on the values 0 or 1 and $\Delta(k,\iota;n)$ defined in (6.13)-(6.14),

$$\begin{aligned}
U(\iota_1,\iota_2;x) := &\sum_{k_1,k_2=0}^{\lfloor \frac{\log T}{\log\log T} \rfloor} \int_{\frac{T}{2}}^{T} \left(\frac{2}{\log \frac{t}{2\pi}}\right)^{k_1+k_2+\iota_1+\iota_2} dt \qquad (8.25)\\
&\times \sum_{m \le x} \Delta(k_1 + [\iota_1 = 0], \iota_1; m)\Delta(k_2 + [\iota_2 = 0], \iota_2; m) m,\\
\tilde{U}(\iota_1,\iota_2;x) := &\sum_{k_1,k_2=0}^{\lfloor \frac{\log T}{\log\log T} \rfloor} \int_{\frac{T}{2}}^{T} \left(\frac{2}{\log \frac{t}{2\pi}}\right)^{k_1+k_2+\iota_1+\iota_2} dt \qquad (8.26)\\
&\times \sum_{x < m \le T^M} \frac{\Delta(k_1 + [\iota_1 = 0], \iota_1; m)\Delta(k_2 + [\iota_2 = 0], \iota_2; m)}{m^3}.
\end{aligned}$$

Formulas (6.58)-(6.59) in Proposition 3 are for the evaluation of the sums over $m$ in (8.25)-(8.26), but for this application we need to extend the range of the last sum of (8.26) to all $m > x$. We have, using (6.1) and (6.2),

$$\begin{aligned}
&\sum_{m > T^M} \frac{\Delta(k_1 + [\iota_1 = 0], \iota_1; m)\Delta(k_2 + [\iota_2 = 0], \iota_2; m)}{m^3} \le \sum_{m > T^M} \frac{(\log m)^{k_1+k_2+\iota_1+\iota_2+2}}{m^3}\\
&\ll \frac{(\log T^M)^{k_1+k_2+\iota_1+\iota_2+1}}{T^{2M}}. \qquad (8.27)
\end{aligned}$$

When plugged in (8.26) this brings an error

$$\begin{aligned}
&\ll T^{1-2M} \log T \sum_{k_1,k_2=0}^{\lfloor \frac{\log T}{\log\log T} \rfloor} \left(\frac{2M \log T}{\log \frac{T}{4\pi}}\right)^{k_1+k_2+\iota_1+\iota_2} \ll T^{1-2M} \log T \sum_{k_1,k_2=0}^{\lfloor \frac{\log T}{\log\log T} \rfloor} (3M)^{k_1+k_2}\\
&\ll T^{1-2M} (\log T)(3M)^{\frac{3\log T}{\log\log T}} \ll T^{1-\frac{3}{2}M}, \qquad (8.28)
\end{aligned}$$

which is very small for $M > \frac{4}{3}$. In applying (6.58)-(6.59), the error terms of these formulas give rise to error terms of the form

$$\ll T \sum_{k_1,k_2=0}^{\lfloor \frac{\log T}{\log\log T} \rfloor} \frac{1}{\max(k_1,k_2)!} \left( \frac{2C_{18}\log x}{\log \frac{T}{4\pi}} \right)^{k_1+k_2+\ell}, \quad (\ell = 0,1,2),$$

and recalling that $x \le T^M$, we see that these error terms can be bounded as

$$\begin{aligned} T \sum_{k_1,k_2=0}^{\lfloor \frac{\log T}{\log\log T} \rfloor} \frac{C_{57}^{k_1+k_2}}{\max(k_1,k_2)!} &\ll T \sum_{k_2=0}^{\lfloor \frac{\log T}{\log\log T} \rfloor} \frac{C_{57}^{k_2}}{k_2!} \sum_{k_1 \le k_2} C_{57}^{k_1} \\ &\ll T \sum_{k_2=0}^{\lfloor \frac{\log T}{\log\log T} \rfloor} \frac{C_{58}^{k_2}}{k_2!} \le e^{C_{58}} T = C_{59} T. \end{aligned} \tag{8.29}$$

Now, going back to (8.24) we have

$$\begin{aligned} &\sum_{\upsilon} \sum_{\frac{T}{2} < \tilde{\upsilon} \le T} \frac{4x^{i(\upsilon-\tilde{\upsilon})}}{4+(\upsilon-\tilde{\upsilon})^2} \qquad (8.30) \\ &= \frac{1}{2\pi} \sum_{k_1,k_2=0}^{\lfloor \frac{\log T}{\log\log T} \rfloor} \Big[ \frac{P(k_1+1,k_2+1,0,0)(\log x)^{k_1+k_2+1}}{(k_1+k_2+1)!} \int_{\frac{T}{2}}^{T} \left( \frac{2}{\log \frac{t}{2\pi}} \right)^{k_1+k_2} dt \\ &\qquad - 2\frac{P(k_1+1,k_2,0,1)(\log x)^{k_1+k_2+2}}{(k_1+k_2+2)!} \int_{\frac{T}{2}}^{T} \left( \frac{2}{\log \frac{t}{2\pi}} \right)^{k_1+k_2+1} dt \\ &\qquad + \frac{P(k_1,k_2,1,1)(\log x)^{k_1+k_2+3}}{(k_1+k_2+3)!} \int_{\frac{T}{2}}^{T} \left( \frac{2}{\log \frac{t}{2\pi}} \right)^{k_1+k_2+2} dt \Big] \\ &\quad + \frac{T\log^2 T}{4\pi x^2} \left( 1 + O\left( \frac{1}{\log T} \right) \right) + O\left( x \exp\left( \frac{C_{56}\log T \log\log\log T}{\log\log T} \right) \right) + O(T). \end{aligned}$$

Here we plug in the values of $P(k_1,k_2,\iota_1,\iota_2)$ given in (6.36) and we use

$$\int_{\frac{T}{2}}^{T} \left( \frac{2}{\log \frac{t}{2\pi}} \right)^{\ell} dt = \frac{T}{2} \left( \frac{2}{\log \frac{T}{2\pi}} \right)^{\ell} \left( 1 + O\big(\frac{\ell}{\log T}\big) \right), \ (\ell \in \mathbb{N}, \ell \ll \frac{\log T}{\log\log T}), \tag{8.31}$$

and upon organizing the terms we obtain

$$\sum_{\upsilon}\sum_{\frac{T}{2}<\tilde{\upsilon}\le T}\frac{4x^{i(\upsilon-\tilde{\upsilon})}}{4+(\upsilon-\tilde{\upsilon})^2} \tag{8.32}$$
$$= \frac{T\log x}{4\pi}\Big[1-4\frac{\log x}{\log\frac{T}{2\pi}}+\sum_{k=0}^{\lfloor\frac{\log T}{\log\log T}\rfloor}\frac{2\cdot k!}{(2k+2)!}\left(\frac{2\log x}{\log\frac{T}{2\pi}}\right)^{2k+2}\Big]\cdot\left(1+O\left(\frac{\log\log T}{\log T}\right)\right)$$
$$+\frac{T\log^2 T}{4\pi x^2}\left(1+O\left(\frac{1}{\log T}\right)\right)+O\left(x\exp\left(\frac{C_{56}\log T\log\log\log T}{\log\log T}\right)\right)+O(T).$$

We would like to extend the sum over $k$ to an infinite sum. Observe that for $x=T$ we no longer have an asymptotic estimate, so we take $1\le x\le T^{1-\epsilon}$. Then $0\le\frac{2\log x}{\log\frac{T}{2\pi}}<2$, and with $K=\lceil\frac{\log T}{\log\log T}\rceil$ we have

$$\sum_{k=K}^{\infty}\frac{k!}{(2k+2)!}2^{2k+2}\le\sum_{k=K}^{\infty}\left(\frac{1}{k+1}\right)^{k+2}4^{k+1}\le\frac{1}{K}\sum_{k=K}^{\infty}\left(\frac{4}{k+1}\right)^{k+1}$$
$$\le\frac{1}{K}\left(\frac{4}{K+1}\right)^{K+1}\left(1+\frac{4}{K+1}+\left(\frac{4}{K+1}\right)^2+\dots\right)=\frac{1}{K}\left(\frac{4}{K+1}\right)^{K+1}\frac{1}{1-\frac{4}{K+1}}$$
$$\ll\frac{\log\log T}{\log T}\left(\frac{4\log\log T}{\log T}\right)^{\frac{\log T}{\log\log T}}\ll T^{-1+\epsilon}. \tag{8.33}$$

Hence (8.32) can be re-written as

$$\sum_{\upsilon}\sum_{\frac{T}{2}<\tilde{\upsilon}\le T}\frac{4x^{i(\upsilon-\tilde{\upsilon})}}{4+(\upsilon-\tilde{\upsilon})^2} \tag{8.34}$$
$$= \frac{T\log x}{4\pi}\Big[1-4\frac{\log x}{\log\frac{T}{2\pi}}+\sum_{k=0}^{\infty}\frac{2\cdot k!}{(2k+2)!}\left(\frac{2\log x}{\log\frac{T}{2\pi}}\right)^{2k+2}\Big]\cdot\left(1+O\left(\frac{\log\log T}{\log T}\right)\right)$$
$$+\frac{T\log^2 T}{4\pi x^2}\left(1+O\left(\frac{1}{\log T}\right)\right)+O(T),\qquad (1\le x\le T^{1-\epsilon}).$$

Now we do the same process as after (7.32), and obtain

**Theorem 2** *Assume RH. Let* $\frac{1}{2}+i\upsilon$ *and* $\frac{1}{2}+i\tilde{\upsilon}$ *run through the critical zeros of* $Z_1(s)$. *We have, as* $T\to\infty$, *uniformly for* $1\le x\le T^{1-\epsilon}$,

$$F_{Z_1,Z_1}(x,T) := \sum_{0<\upsilon,\tilde{\upsilon}\le T} \frac{4x^{i(\upsilon-\tilde{\upsilon})}}{4+(\upsilon-\tilde{\upsilon})^2} \tag{8.35}$$
$$= \frac{T\log x}{2\pi}\Big[1-4\frac{\log x}{\log\frac{T}{2\pi}}+\sum_{k=0}^{\infty}\frac{2\cdot k!}{(2k+2)!}\left(\frac{2\log x}{\log\frac{T}{2\pi}}\right)^{2k+2}\Big]\cdot\left(1+O\left(\frac{\log\log T}{\log T}\right)\right)$$
$$+\frac{T\log^2 T}{2\pi x^2}\left(1+O\left(\frac{1}{\log T}\right)\right)+O(T).$$

*The formula (1.3) also holds for $F_{Z_1,Z_1}(x,T)$.*

*Remark:* The asymptotic value in (8.35) is the same as the value found by Farmer, Gonek and Lee [5] for the pair correlation of the zeros of the derivative of the Riemann $\xi$-function. In [5] an explanation of why the leading order terms of the pair correlation functions for $\xi'$ and $Z'$ should be the same was given. Since the leading order of the the correlations are the same, the same conclusions can be drawn from Theorem 3 as Farmer, Gonek and Lee's results. Thus we can say that, assuming RH, a positive proportion of the gaps between consecutive zeros of $Z_1(s)$ are smaller than 0.89661 times the average spacing ($\sim \frac{2\pi}{\log T}$), and more than 85.838% of the zeros of $Z_1(s)$ are simple. For the proofs of these results and references to earlier related results we refer the reader to [5].

We now define, suppressing the dependence on $T$ in the notation,

$$\mathcal{F}(\alpha) := \left(\frac{T}{\pi}\log T\right)^{-1}\left[F_{\zeta,\zeta}\big((\frac{T}{2\pi})^{\alpha},T\big)+2F_{\zeta,Z_1}\big((\frac{T}{2\pi})^{\alpha},T\big)+F_{Z_1,Z_1}\big((\frac{T}{2\pi})^{\alpha},T\big)\right], \tag{8.36}$$

so that

$$\mathcal{F}(\alpha) = \left(\frac{T}{\pi}\log T\right)^{-1}\sum_{\substack{0<t_1,t_2\le T\\ \zeta\cdot Z_1(\frac{1}{2}+it_j)=0,\,(j=1,2)}}\frac{4}{4+(t_1-t_2)^2}\,(\frac{T}{2\pi})^{i\alpha(t_1-t_2)}. \tag{8.37}$$

Note that we have taken $x=\frac{T}{2\pi}$ because of the powers in the infinite series in (8.35). This won't make any difference in the asymptotics for $F_{\zeta,\zeta}$ and $F_{\zeta,Z_1}$. Then we know from (1.4) and Theorems 1 and 2 that, for $0\le|\alpha|\le 1-\epsilon$ with any arbitrarily small but fixed $\epsilon$,

$$\mathcal{F}(\alpha)\sim 2|\alpha|-4|\alpha|^2+\frac{1}{2}\sum_{m=1}^{\infty}\frac{(m-1)!}{(2m)!}(2|\alpha|)^{2m+1}+2T^{-2|\alpha|}\log T, \tag{8.38}$$

as $T\to\infty$.

In order to show the existence of small gaps between zeros of $\zeta\cdot Z_1$ along with a value for the size of such gaps, we follow the method developed by Montgomery [17] and later on also used in [5]. So we will take $h(u)$ to be a minorant of the characteristic function of the interval $[-1,1]$ such that the domain of its Fourier transform $\hat{h}(\alpha)$ includes $[-1,1]$ and is non-negative everywhere. Furthermore, to minimize losses we will require $h(0)=1$. Montgomery [17] used the pair $h(u)=\max(1-|u|,0)$, $\hat{h}(\alpha)=\left(\frac{\sin\pi\alpha}{\pi\alpha}\right)^2$; in [5], $h(u)=\left(\frac{\sin\pi u}{\pi u}\right)^2\frac{1}{1-u^2}$ which is the Selberg minorant of the characteristic function of the interval $[-1,1]$ with Fourier transform $\hat{h}(\alpha)=\max(1-|\alpha|+\frac{\sin 2\pi|\alpha|}{2\pi},0)$ was taken (the use of the latter Fourier pair in such a context goes back to [7]). Let $m_{t_1}$ denote the multiplicity of a zero $\frac{1}{2}+t_1$ of $\zeta\cdot Z_1(s)$, and $\lambda\in(0,1)$. Then we see that

$$\sum_{0<t_1\leq T} m_{t_1} + 2\sum_{\substack{0<t_1,t_2\leq T\\ 0<t_1-t_2\leq\frac{\pi\lambda}{\log T}}} 1 \geq \sum_{0<t_1,t_2\leq T}\frac{4}{4+(t_1-t_2)^2}\, h\left((t_1-t_2)\frac{\log T}{\pi\lambda}\right), \tag{8.39}$$

and since

$$h(\frac{2u}{\lambda}) = \int_{-\infty}^{\infty}\hat{h}(\alpha)e^{2\pi i\alpha\frac{2u}{\lambda}}\,d\alpha = \frac{\lambda}{2}\int_{-\infty}^{\infty}\hat{h}(\frac{\lambda\beta}{2})e^{2\pi i\beta u}\,d\beta,$$

we have

$$\begin{aligned}\sum_{0<t_1\leq T} m_{t_1} + 2\sum_{\substack{0<t_1,t_2\leq T\\ 0<t_1-t_2\leq\frac{\pi\lambda}{\log T}}} 1 &\gtrsim \left(\frac{T}{\pi}\log T\right)\frac{\lambda}{2}\int_{-\infty}^{\infty}\mathcal{F}(\beta)\hat{h}(\frac{\lambda\beta}{2})\,d\beta\\ &\gtrsim \left(\frac{T}{\pi}\log T\right)\frac{\lambda}{2}\int_{-1+\epsilon}^{1-\epsilon}\mathcal{F}(\beta)\hat{h}(\frac{\lambda\beta}{2})\,d\beta.\end{aligned}$$

In the last step we have used the non-negativity of $\hat{h}$ and $\mathcal{F}$, the latter seen in the same way as (1.3). We may assume that

$$\sum_{0<t_1\leq T} m_{t_1} \sim \frac{T}{\pi}\log T,$$

for otherwise there would be infinitely many multiple zeros of $\zeta\cdot Z_1$ and there wouldn't be any need to prove a result for existence of small gaps between zeros. Then

$$\sum_{\substack{0<t_1,t_2\leq T\\ 0<t_1-t_2\leq\frac{\pi\lambda}{\log T}}} 1 \geq (\frac{1}{2}-\epsilon)\left(\frac{T}{\pi}\log T\right)\left(-1+\frac{\lambda}{2}\int_{-1+\epsilon}^{1-\epsilon}\mathcal{F}(\beta)\hat{h}(\frac{\lambda\beta}{2})\,d\beta\right). \tag{8.40}$$

Since this holds for any fixed $\epsilon>0$, we want to find an as small as possible value of $\lambda$ for which

$$-1+\frac{\lambda}{2}\int_{-1+\epsilon}^{1-\epsilon}\mathcal{F}(\beta)\hat{h}(\frac{\lambda\beta}{2})\,d\beta > 0. \tag{8.41}$$

Choosing the Fourier pair $h$, $\hat{h}$ as in [5], by (8.38) the condition (8.41) is cast into

$$\lambda \int_0^1 \Big[2\beta - 4\beta^2 + \frac{1}{2}\sum_{m=1}^{\infty} \frac{(m-1)!}{(2m)!}(2\beta)^{2m+1} + 2T^{-2\beta}\log T\Big]\Big(1 - \frac{\lambda\beta}{2} + \frac{\sin\pi\lambda\beta}{2\pi}\Big)\,d\beta$$
$$> 1. \tag{8.42}$$

Some computing reveals that (8.42) is satisfied when we take $\lambda = 0.78842$, and we can state

**Corollary 2** *Assume RH. There exist infinitely many gaps of size* $< 0.78842$ *of the average gap (*$\sim \dfrac{\pi}{\log T}$*) in the sequence composed of the zeros and maxima points on the critical line of* $\zeta(s)$*.*

*Remark:* Since the critical zeros of $\zeta$ and $Z_1$ are interlaced these small gaps are between a zero of $\zeta$ and a zero of $Z_1$. The result of Corollary 2 doesn't seem not very strong, but it is better than the trivial bound 1, and it is the first result of its kind. Applying (8.38) for the proportion of simple zeros of $\zeta \cdot Z_1$ doesn't give a bright result. By some elementary considerations and Hall's result given in (5.24) we may take that a non-simple zero $\frac{1}{2} + it_1$ of $\zeta \cdot Z_1$ has multiplicity at least 3. Then, Montgomery's argument for the proportion of simple zeros, together with $\sum_{\substack{0<t_1\le T\\ t_1:\text{simple}}} 1 \ge \sum_{0<t_1\le T} \frac{3-m_{t_1}}{2}$ and a numerical fact from the proof of Corollary 1.3 of [5], gives the rather dismal estimate $\sum_{\substack{0<t_1\le T\\ t_1:\text{simple}}} 1 \gtrsim 0.38\left(\frac{T}{\pi}\log T\right)$. That these results aren't as strong as those obtained from $F_{\zeta,\zeta}(x,T)$ by Montgomery is related to the fact that $F_{\zeta,Z_1}(x,T)$ changes sign at $x \approx \sqrt{T}$.

## 9 Correlation of the zeros of two Dirichlet $L$-functions

In this section, we apply the ideas of §3 and §4 to correlating the zeros of two Dirichlet $L$-functions associated with primitive Dirichlet characters. Let $\chi$ be a primitive character to the moduli $q_\chi$ (so that $q_\chi \ge 3$). In this section the Dirichlet characters will always be taken to be primitive characters to moduli which are fixed (i.e. not a function of $T$). The non-trivial zeros of $L(s,\chi)$ have real parts in the strip $\sigma \in (0,1)$, and the number of these with ordinates in $[0,T]$ is

$$N_\chi(T) = \frac{T}{2\pi}\log\frac{q_\chi T}{2\pi} - \frac{T}{2\pi} + O(\log q_\chi T), \quad (T \ge 4) \tag{9.1}$$

(see, e.g. Corollary 14.7 of [19]). We shall assume the Generalized Riemann Hypothesis (GRH) for Dirichlet $L$-functions that all of the non-trivial zeros of $L(s,\chi)$ are on the critical line. Let $\rho_\chi = \frac{1}{2} + i\gamma_\chi$ run through these zeros. Quoting from [23], the analogue of Montgomery's explicit formula (2.2) is

$$\sum_{\gamma_\chi} \frac{(2\sigma-1)x^{i\gamma_\chi}}{(\sigma-\frac{1}{2})^2+(t-\gamma_\chi)^2} = -x^{-\frac{1}{2}}\Big(\sum_{n\le x}\Lambda(n)\chi(n)(\frac{x}{n})^{1-\sigma+it} + \sum_{n>x}\Lambda(n)\chi(n)(\frac{x}{n})^{\sigma+it}\Big)$$
$$-\frac{L'}{L}(1-\sigma+it,\chi)x^{\frac{1}{2}-\sigma+it} + x^{-\frac{1}{2}}\sum_{r=1}^{\infty}\frac{(2\sigma-1)x^{-2r-\mathfrak{a}_\chi}}{(\sigma-1-it-2r-\mathfrak{a}_\chi)(\sigma+it+2r+\mathfrak{a}_\chi)}, \quad (9.2)$$

valid under GRH for $\sigma > 1$, and all $x \ge 1$. Here

$$\mathfrak{a}_\chi = \begin{cases} 0 & \text{if} \quad \chi(-1) = 1, \\ 1 & \text{if} \quad \chi(-1) = -1. \end{cases}$$

We employ (9.2) with $\sigma = \frac{5}{2}$, then while going through a procedure paralleling that of §3 we substitute

$$\frac{L'}{L}(1-\sigma+it,\chi) = -\log\left(\frac{q_\chi|t|}{2\pi}\right) - \frac{L'}{L}(\sigma-it,\overline{\chi}) + O\left(\frac{1}{|t|}\right), (-2<\sigma<3, |t|\ge 1) \quad (9.3)$$

which follows from logarithmic differentiation of the functional equation of $L(s,\chi)$, and we obtain the analogue of (3.3)

$$\sum_{\gamma_\chi}\frac{4x^{i(\gamma_\chi-t)}}{4+(\gamma_\chi-t)^2} = -x^{-2}\sum_{n\le x}\Lambda(n)\chi(n)n^{\frac{3}{2}-it} - x^2\sum_{n>x}\Lambda(n)\chi(n)n^{-\frac{5}{2}-it}$$
$$+x^{-2}\big(\log q_\chi|t| + O(1)\big) \quad (9.4)$$

for $|t|\ge 1$ and $x \ge 1$.

Now let $\psi$ be a primitive character to the modulus $q_\psi$, and let $t$ run through the ordinates $\gamma_\psi \in (1,T]$ of the zeros of $L(s,\psi)$. Summing (9.4) over this set of $t$ values, we have

$$\sum_{\gamma_\chi}\sum_{1<\gamma_\psi\le T}\frac{4x^{i(\gamma_\chi-\gamma_\psi)}}{4+(\gamma_\chi-\gamma_\psi)^2} = \frac{T\log^2 T}{2\pi x^2}\left(1+O_{q_\chi,q_\psi}\left(\frac{1}{\log T}\right)\right) \quad (9.5)$$
$$-x^{-2}\sum_{n\le x}\Lambda(n)\chi(n)n^2\sum_{1<\gamma_\psi\le T}n^{-\rho_\psi} - x^2\sum_{n>x}\Lambda(n)\chi(n)n^{-2}\sum_{1<\gamma_\psi\le T}n^{-\rho_\psi},$$

where we have used

$$\sum_{1<\gamma_\psi\le T}\big(\log q_\chi\gamma_\psi + O(1)\big) = \frac{T}{2\pi}\log^2 T\left(1+O_{q_\chi,q_\psi}\left(\frac{1}{\log T}\right)\right) \quad (9.6)$$

which follows from (9.1) by Stieltjes integration.

At this point we need the analogue of the Landau-Gonek formula (4.3) for $L(s,\psi)$ to be used in (9.5). A meticulous GRH dependent version was given by Fujii [6].

**Proposition 6** *For $y, T > 1$ we have*

$$\sum_{1<\gamma_\psi\le T} y^{\rho_\psi} = -\frac{T}{2\pi}\psi(y)\Lambda(y) + O(y\log(yq_\psi T)\log\log 3y) + O\left(\log y \min\left(T, \frac{y}{\langle y\rangle}\right)\right) + O\left(\log(q_\psi T)\min(T, \frac{1}{\log y})\right) + O\left(\frac{y\log^2 q_\psi}{\log 2y}\right). \tag{9.7}$$

The proof of this proposition is similar to Gonek's proof of (4.3) in [13], so we omit it. The only slight difference is in estimating the contribution of the segment along $t = 1$ (in case there is no $\gamma_\psi = 1$; if there is a $\gamma_\psi = 1$, then this segment can be shifted a little bit), we use $\frac{L'}{L}(\sigma + i) \ll \log^2 q_\psi$ for $-1 < \sigma < 2$, giving rise to the last error term in (9.7). If GRH is assumed, as we have already done, and if $y = n \ge 2$, then the argument gives

$$\sum_{1<\gamma_\psi\le T} n^{-\rho_\psi} = -\frac{T}{2\pi n}\overline{\psi(n)}\Lambda(n) + O(\log n\log\log n) + O(\log q_\psi T) + O\left(\frac{\log^2 q_\psi}{\log n}\right). \tag{9.8}$$

Plugging (9.8) into (9.5) we have

$$\sum_{\gamma_\chi}\sum_{1<\gamma_\psi\le T}\frac{4x^{i(\gamma_\chi-\gamma_\psi)}}{4+(\gamma_\psi-\gamma_\chi)^2} = \frac{T\log^2 T}{2\pi x^2}\left(1 + O_{q_\chi,q_\psi}\left(\frac{1}{\log T}\right)\right) + \frac{T}{2\pi x^2}\sum_{n\le x} n\Lambda^2(n)(\overline{\psi}\chi)(n) + \frac{Tx^2}{2\pi}\sum_{n>x}\frac{\Lambda^2(n)(\overline{\psi}\chi)(n)}{n^3} + O(x\log 2x\log\log 3x) + O(x\log q_\psi T) + O\left(\frac{x\log^2 q_\psi}{\log 2x}\right). \tag{9.9}$$

The summation conditions for the double sum on the left-hand side can be changed into $0 \le \gamma_\chi, \gamma_\psi \le T$ within an error of $O(\log(q_\chi T)\log(q_\psi T)\log T)$ as can be seen by a standard estimation.

If $\chi \ne \psi$, then from

$$\sum_{n\le x}\Lambda(n)(\overline{\psi}\chi)(n) \ll x^{\frac{1}{2}}\log^2 x, \quad (q_\chi q_\psi \le x), \tag{9.10}$$

on GRH ([2], Chapter 20) , we have

$$\frac{T}{2\pi x^2}\sum_{n\le x} n\Lambda^2(n)(\overline{\psi}\chi)(n) + \frac{Tx^2}{2\pi}\sum_{n>x}\frac{\Lambda^2(n)(\overline{\psi}\chi)(n)}{n^3} \ll Tx^{-\frac{1}{2}}\log^3 2x, \tag{9.11}$$

so that there is no main term of size $T\log T$ for $x > T^\epsilon$.

If $\chi = \psi$, then with $q_\chi = q_\psi \le x$, the terms in (9.11) are the same as in (4.4) for the pair correlation of zeros of $\zeta(s)$ except for the summands with $(n, q_\chi) > 1$. The effect of these latter terms are

$$\ll \frac{T}{x^2} \sum_{\substack{n \le x \\ (n, q_\chi) > 1}} n\Lambda^2(n) \ll \frac{T \log x \log q_\chi}{x}. \tag{9.12}$$

This completes our calculation in this section, and we state our findings as

**Theorem 3** *Assume GRH. Let $\frac{1}{2} + i\gamma_\chi$ and $\frac{1}{2} + i\gamma_\psi$ run through the critical zeros of $L(s,\chi)$ and $L(s,\psi)$ respectively. Assume that the moduli $q_\chi$, $q_\psi$ are fixed. Then, as $T \to \infty$,*

$$\begin{aligned} F_{\chi,\psi}(x,T) &:= \sum_{0<\gamma_\chi \le T} \sum_{0<\gamma_\psi \le T} \frac{4x^{i(\gamma_\chi - \gamma_\psi)}}{4 + (\gamma_\chi - \gamma_\psi)^2} \qquad (9.13) \\ &= [\chi = \psi] \cdot \left( \frac{T \log x}{2\pi} + O\big(\frac{T \log x \log q_\chi}{x}\big) \right) + \frac{T \log^2 T}{2\pi x^2} \left( 1 + O_{q_\chi, q_\psi}\left( \frac{1}{\log T} \right) \right) \\ &\quad + O(T x^{-\frac{1}{2}} \log^3 2x) + O(x \log 2x \log\log 3x) + O(x \log T). \end{aligned}$$

*Remark:* We note that if $\psi \equiv 1$ is taken then the preceding calculation is the same, and so we obtain the correlation of critical zeros of $\zeta(s)$ with those of $L(s,\chi)$.
*Remark:* If $\chi$ is a non-real character and $\psi = \overline{\chi}$, then the zeros of $L(s,\psi)$ occurring in $F_{\chi,\psi}(x,T)$ are the reflections with respect to the real axis of the zeros of $L(s,\chi)$ which are below the real axis, and for $x \ge T^\epsilon$ the main term is again 0. Thus the zeros of $L(s,\chi)$ which are above the real axis can be said to be not correlated to those which are below the real axis.
*Remark:* In [23] a certain average over correlations of zeros of two Dirichlet $L$-functions of the same modulus were obtained by following Montgomery's method. Our theorem here is more particular.
*Remark:* Gallagher's citation of oral communication in [7] seems to indicate that a result of the same kind as Theorem 3 was observed by A. Fujii.

We now work on the product $L(s,\chi) \cdot L(s,\psi)$, where $\chi \ne \psi$ are primitive characters. Similar to (8.36), let

$$\mathfrak{F}(\alpha) := \left( \frac{T}{\pi} \log T \right)^{-1} \left[ F_{\chi,\chi}(T^\alpha, T) + 2F_{\chi,\psi}(T^\alpha, T) + F_{\psi,\psi}(T^\alpha, T) \right], \tag{9.14}$$

so that

$$\mathfrak{F}(\alpha) = \left( \frac{T}{\pi} \log T \right)^{-1} \sum_{\substack{0 < t_1, t_2 \le T \\ L((\frac{1}{2} + it_j, \chi) \cdot L((\frac{1}{2} + it_j, \psi) = 0,\ (j=1,2)}} \frac{4}{4 + (t_1 - t_2)^2} T^{i\alpha(t_1 - t_2)}. \tag{9.15}$$

Then we know from Theorem 3 that, for $0 \le |\alpha| \le 1 - \epsilon$ with any arbitrarily small but fixed $\epsilon$,

$$\mathfrak{F}(\alpha) \sim |\alpha| + 2T^{-2|\alpha|}\log T, \quad \text{as } T \to \infty. \tag{9.16}$$

For small gaps between zeros of $L(s,\chi)\cdot L(s,\psi)$, proceeding as in (8.39)-(8.42), we reach the condition

$$\lambda \int_0^1 \left[\beta + 2T^{-2\beta}\log T\right]\left(1 - \frac{\lambda\beta}{2} + \frac{\sin \pi\lambda\beta}{2\pi}\right) d\beta > 1. \tag{9.17}$$

Some computing reveals that (9.17) is satisfied with $\lambda = 0.68742$, and we have

**Corollary 3** *Assume RH. There exist infinitely many gaps of size* $< 0.68742$ *of the average gap (*$\sim \frac{\pi}{\log T}$*) in the sequence composed of the zeros of* $L(s,\chi)\cdot L(s,\psi)$*, where* $\chi \neq \psi$ *are primitive characters.*

Note that the small gaps detected here may be coming from a single $L$-function.

## 10 Further problems to study

The method presented in this article may be applied to correlating the ordinates of zeros of the Riemann zeta-function or Dirichlet $L$-functions to various sequences of real numbers (and possibly generalizable to other functions from the Selberg class). Moreover as in §8 one may try to work out the pair correlation of zeros of other related functions such as $Z_1(s)$. Furthermore, our method is applicable to products of not just two Dirichlet $L$-functions, but in fact to products of $k$ Dirichlet $L$-functions. In any such problem it is desirable to obtain results which have implications for the distribution of zeros on the critical line. Another problem of interest is to form conjectures about the behaviour of the correlations of Theorems 1 and 2 for $x \geq T$ as Montgomery [17] did for the pair correlation of zeta zeros. Montgomery drew upon the Hardy-Littlewood conjecture about sums of the kind $\sum_{n \leq x} \Lambda(n)\Lambda(n+h)$ for formulating the conjecture given in (1.8). The conjecture that all zeta zeros are simple also provided guidance in the sense that his conjecture should agree with it. In the case of the correlation sums of Theorems 1 and 2, extension to $x \geq T$ will involve formulating Hardy-Littlewood type of conjectures for the arithmetic functions $\Lambda^k(n)$ and $\lambda^k(n)$, or else one may rely upon the random matrix model to first formulate the analogues of the pair correlation conjecture (1.10) and then pass to the analogues of (1.8) therefrom.

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