Garaev, M. Z. and C. Y. Yıldırım. (2007) "On Small Distances Between Ordinates of Zeros of $\zeta(s)$ and $\zeta'(s)$," International Mathematics Research Notices, Vol. 2007, Article ID rnm091, 14 pages.
doi:10.1093/imrn/rnm091

On Small Distances Between Ordinates of Zeros of $\zeta(s)$ and $\zeta'(s)$

M. Z. Garaev[1] **and C. Y. Yıldırım**[2]

[1]Instituto de Matemáticas, Universidad Nacional Autónoma de México, Campus Morelia, Apartado Postal 61-3 (Xangari), C.P. 58089, Morelia, Michoacán, México, and [2]Department of Mathematics, Boğaziçi University, Bebek, Istanbul 34342, & Feza Gürsey Enstitüsü, Çengelköy, Istanbul, P.K. 6, 81220, Turkey

Correspondence to be sent to: M. Z. Garaev, Instituto de Matemáticas, Universidad Nacional Autónoma de México, Campus Morelia, Apartado Postal 61-3 (Xangari), C.P. 58089, Morelia, Michoacán, México. e-mail: garaev@matmor.unam.mx

We prove that for any zero $\beta' + i\gamma'$ of $\zeta'(s)$ there exists a zero $\beta + i\gamma$ of $\zeta(s)$ such that $|\gamma - \gamma'| \ll \sqrt{|\beta' - \frac{1}{2}|}$, and we provide some other related results.

1 Introduction

In this paper $s = \sigma + it$ will denote a complex variable, where σ and t are real, and T will denote a large parameter.

The relations between the zeros of a function and the zeros of its derivatives have been the object of much study. The case of the Riemann zeta-function $\zeta(s)$ presents many puzzles beginning with the Riemann hypothesis (RH). Speiser [11] showed that RH is equivalent to $\zeta'(s)$ having no zeros in $0 < \sigma < \frac{1}{2}$. From Riemann's original work (proofs for some parts of which were provided later by other mathematicians), it is well-known that the nontrivial zeros of $\zeta(s)$, to be denoted by $\rho = \beta + i\gamma$, are to be found only in the

Received October 1, 2006; Revised March 20, 2007; Accepted August 20, 2007
Communicated by Dennis Hejhal

See http://www.oxfordjournals.org/our_journals/imrn/ for proper citation instructions.

critical strip, i.e. $0 \le \beta \le 1$, and the number of nontrivial zeros with $\gamma \in [0, T]$ is

$$N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi e} + \frac{7}{8} + S(T) + O\left(\frac{1}{T}\right)$$

as $T \to \infty$. Here for t not the ordinate of a zero, $S(t) := \frac{1}{\pi} \arg \zeta(\frac{1}{2} + it)$ obtained by continuous variation along the line segments joining 2, $2 + it$, $\frac{1}{2} + it$, starting with the value 0; if t is the ordinate of a zeta zero, $S(t) := S(t+0)$. It is also well-known that $S(T) = O(\log T)$. Titchmarsh [12, Theorem 11.5 (C)] established the existence of a constant E, between 2 and 3, such that $\zeta'(s)$ does not vanish in the half-plane $\sigma > E$, while $\zeta'(s)$ has infinitely many zeros in any strip between $\sigma = 1$ and $\sigma = E$. Berndt [1] showed that the number of nonreal zeros of $\zeta'(s)$, which are to be denoted by $\rho' = \beta' + i\gamma'$, with $\gamma' \in [0, T]$ is

$$N'(T) = \frac{T}{2\pi} \log \frac{T}{4\pi e} + O(\log T).$$

Levinson and Montgomery [8] in addition to proving a quantified version of Speiser's theorem and that the only zeros of $\zeta'(s)$ in $\sigma \le 0$ are its 'trivial zeros' on the negative real axis which occur between the trivial zeros of $\zeta(s)$, obtained results revealing that the zeros of $\zeta'(s)$ are mostly clustered around $\sigma = \frac{1}{2}$, and most of the nonreal zeros of $\zeta'(s)$ lie to the right of $\sigma = \frac{1}{2} - \frac{w(t)}{\log t}$, where $w(t) \to \infty$ as $t \to \infty$. From the fact that $\Re \frac{\zeta'(s)}{\zeta(s)} < 0$ on $\sigma = \frac{1}{2}$, except at zeros of $\zeta(s)$, they observed that $\zeta'(\frac{1}{2} + i\gamma') = 0$ can occur only if $\frac{1}{2} + i\gamma'$ is a multiple zero of $\zeta(s)$. Levinson and Montgomery also proved

$$\sum_{0 < \gamma' \le T} (\beta' - \tfrac{1}{2}) \sim \frac{T}{2\pi} \log \log T,$$

which has the immediate interpretation that $\beta' - \frac{1}{2}$ is often much larger than the average gap between the consecutive zeros of $\zeta(s)$. In [2] Conrey and Ghosh showed that for any fixed $\nu > 0$, a positive proportion of zeros of $\zeta'(s)$ are in the region $\sigma \ge \frac{1}{2} + \frac{\nu}{\log t}$. We note that the works cited above (except for Titchmarsh's book) deal more generally with $\zeta^{(k)}(s)$ and contain other results which we have not mentioned here.

Soundararajan [10] addressed these matters expressing his belief that the magnitude of $\beta' - \frac{1}{2}$ is usually of order $\frac{1}{\log \gamma'}$, and the average is high because of few zeros which are abnormally distant from $\sigma = \frac{1}{2}$. He also wrote to the effect that, the more distant ρ' is from the critical line the larger the gap between the two zeros of $\zeta(s)$ which straddle ρ'. Soundararajan announced two conjectures:

Conjecture 1.1. *For $\nu \in \mathbb{R}$, let*

$$m^-(\nu) = \liminf_{T\to\infty} \frac{1}{N'(T)} \# \left\{ \rho' : \beta' \le \frac{1}{2} + \frac{\nu}{\log T}, \quad 0 \le \gamma' \le T \right\}$$

and $m^+(\nu)$ by replacing lim inf *by* lim sup *in the above. Then for all ν we have $m^-(\nu) = m^+(\nu) =: m(\nu)$. Further, $m(\nu)$ is a nonnegative, nondecreasing, continuous function with the properties: $m(\nu) = 0$ for $\nu \le 0$, $0 < m(\nu) < 1$ for $\nu > 0$, and $m(\nu) \to 1$ as $\nu \to \infty$.* □

Conjecture 1.2. *Assume RH. The following two statements are equivalent:*

(i) $\liminf_{\gamma'\to\infty} (\beta' - \frac{1}{2}) \log \gamma' = 0$;

(ii) $\liminf_{\gamma\to\infty} (\gamma^+ - \gamma) \log \gamma = 0$, *where γ^+ is the least ordinate of a zero of $\zeta(s)$ with $\gamma^+ > \gamma$.* □

Towards these conjectures he showed that there exists a constant C such that $m^-(C) > 0$ unless RH is 'badly violated', and assuming RH he obtained $m^-(\nu) > 0$ for $\nu \ge 2.6$. Zhang [13] made considerable progress for Conjecture 1.1 by proving unconditionally that $m^-(\nu) > 0$ for sufficiently large ν. Assuming RH and Montgomery's [9] pair correlation conjecture in the weak form

$$\liminf_{T\to\infty} \frac{1}{N(T)} \sum_{\substack{\gamma_n \le T \\ \gamma_{n+1}-\gamma_n \le \frac{\alpha}{\log T}}} 1 > 0$$

for any fixed $\alpha > 0$, Zhang also showed that $m^-(\nu) > 0$ for any $\nu > 0$, and Feng [5] was able to dispense with the assumption of RH in obtaining this result. Here and in what follows we use the notation that the nontrivial zeros $\rho_n = \beta_n + i\gamma_n$ of $\zeta(s)$ in the upper half-plane are indexed as $0 < \gamma_1 \le \gamma_2 \le \ldots$, with the understanding that the ordinate of a zero of multiplicity m appears m times consecutively in this sequence. Moreover, Zhang [13] showed under RH that when α_1 and α_2 are positive constants satisfying $\alpha_1 < 2\pi$ and $\alpha_2 > \alpha_1 \left(1 - \sqrt{\frac{\alpha_1}{2\pi}}\right)^{-1}$, if it happens that $(\gamma^+ - \gamma) \log \gamma < \alpha_1$ for ρ with sufficiently large γ, then there exists ρ' such that $|\rho' - \rho| < \alpha_2 (\log \gamma)^{-1}$, thereby proving that "(ii) implies (i)". The other half of Conjecture 1.2, namely "(i) implies (ii)", remains open.

2 Statement of the results

For a $\rho' = \beta' + i\gamma'$ let of all ordinates of zeros of $\zeta(s)$, γ_c be the one for which $|\gamma_c - \gamma'|$ is smallest (if there are more than one such zero of $\zeta(s)$, take γ_c to be the imaginary part of any one of them).

The following lemma is an immediate consequence of Lemmas 2 and 3 of [13].

Lemma 2.1. Assume RH. Let $\rho = \frac{1}{2} + i\gamma$ be a simple zero of $\zeta(s)$ with $\gamma > 0$. Then

$$\sum_{\beta' > \frac{1}{2}} \frac{\beta' - \frac{1}{2}}{(\beta' - \frac{1}{2})^2 + (\gamma - \gamma')^2} = \frac{1}{2}\log\gamma + O(1). \qquad \square$$

Assuming RH and that $\frac{1}{2} + i\gamma_c$ is a simple zero, we have

$$\frac{\beta' - \frac{1}{2}}{(\beta' - \frac{1}{2})^2 + (\gamma_c - \gamma')^2} \leq \frac{1}{2}\log\gamma_c + O(1).$$

Hence, if $(\beta' - \frac{1}{2})\log\gamma'$ is small (which may happen, since (i) is believed to be true), then $|\gamma_c - \gamma'| \gg \sqrt{\frac{\beta' - \frac{1}{2}}{\log\gamma'}}$. Our Theorem 2.2 may cause one to believe that $|\gamma_c - \gamma'| \ll \sqrt{\frac{|\beta' - \frac{1}{2}|}{\log\gamma'}}$ for all sufficiently large γ'. This may in turn suggest

$$|\gamma_c - \gamma'| \asymp \sqrt{\frac{|\beta' - \frac{1}{2}|}{\log\gamma'}},$$

although one might also suspect that the right-hand side is off by a factor of size a power of $\log\log\gamma'$, where the power may vary depending on the size of $\beta' - \frac{1}{2}$ (the power may become as high as $\frac{1}{2}$ for $\beta' - \frac{1}{2} \gg 1$) in view of the conjecture made by Farmer, Gonek and Hughes [4] based upon arguments from random matrix theory that $\limsup_{t\to\infty} \frac{|S(t)|}{\sqrt{\log t \log\log t}} = \frac{1}{\pi\sqrt{2}}$.

Theorem 2.2. For any zero $\beta' + i\gamma'$ of $\zeta'(s)$ with a large γ', there exists γ_n such that $\gamma' - 1 \leq \gamma_n \leq \gamma_{n+2} \leq \gamma' + 1$ and

$$\min\{|\gamma_c - \gamma'|\log\gamma',\ |\gamma_{n+2} - \gamma_n|\log\gamma_n\} \ll \left(\left|\beta' - \tfrac{1}{2}\right|\log\gamma'\right)^{\frac{1}{2}}. \qquad \square$$

Note that we haven't formulated the result in Theorem 2.2 in terms of $\gamma_{n+1} - \gamma_n$ because if there are infinitely many zeta zeros off the critical line, then since these zeros occur symmetrically with respect to the critical line this difference will be trivially 0 infinitely often. In fact the statement of Theorem 2.2 holds more generally with γ_{n+n_0} in place of γ_{n+2}, where n_0 is any fixed integer.

We also obtain unconditionally the following upper-bound.

Theorem 2.3. For any zero $\beta' + i\gamma'$ of $\zeta'(s)$ we have

$$|\gamma_c - \gamma'| \ll \left|\beta' - \tfrac{1}{2}\right|^{\frac{1}{2}}. \qquad \square$$

Besides the two statements in Conjecture 1.2, let us pose the following statement:

(iii) $\liminf_{\gamma'\to\infty} |\gamma_c - \gamma'| \log \gamma' = 0.$

In particular, from Theorem 2.2 we immediately see that if (i) holds, then either (iii) is true or $\liminf_{n\to\infty} (\gamma_{n+2} - \gamma_n) \log \gamma_n = 0$.

Combining Theorem 2.2 with Zhang's result which was mentioned at the end of §1, we derive in §5

Corollary 2.4. Assume that RH and (i) hold. Then (iii) is true. □

Conjecture 1.2 claims that, under RH, (i) implies (ii). We establish the following weaker result.

Theorem 2.5. Assume RH (so that $\beta' \geq 1/2$ for every ρ') and

$$\liminf_{\gamma'\to\infty} \left(\beta' - \frac{1}{2}\right) (\log \gamma')(\log\log \gamma')^2 = 0.$$

Then

$$\liminf_{n\to\infty} (\gamma_{n+1} - \gamma_n)(\log \gamma_n) = 0.$$ □

We briefly recount some known conditional results related to Theorems 2.2 and 2.3. Guo [6] (see also [13] for a generalization) has proved, under RH, if for ρ' with $T \leq \gamma' \leq 2T$ and $\frac{1}{2} < \beta' < \frac{1}{2} + g(T)$ (where $g(T) \to 0$ as $T \to \infty$) there exists a zero $\rho_1' = \beta_1' + i\gamma_1'$ of $\zeta'(s)$ such that $|\rho_1' - \rho'| \ll \beta' - \frac{1}{2}$, then $|\gamma_c - \gamma'| \ll \beta' - \frac{1}{2}$. In the light of the foregoing discussion, in Guo's result the condition of the existence of such a zero ρ_1' is crucial and probably can not be removed. Zhang's paper contains the following result implicitly (see (3.5)–(3.6) of [13]). Assume RH and $\gamma_{n+1} - \gamma_n > \frac{2\pi\lambda}{\log T}$ with $\gamma_n > \frac{T}{\log T}$, where $\lambda > 1$ is such that the condition $\#\left\{n : n < N(T), \gamma_{n+1} - \gamma_n > \frac{2\pi\lambda}{\log T}\right\} > c_0 T \log T$ is satisfied with a constant $c_0 > 0$ (from [3] this condition is known to hold with $\lambda = 1.33$). Then, there exists ρ' such that $|\rho' - \rho_n| < \frac{\nu}{\log T}$, where ν is such that $(\frac{\nu}{\nu + 2\pi\lambda})^2 > \frac{\lambda+1}{2\lambda}$.

3 Preliminaries

We shall use some well-known properties of $\zeta(s)$ which can be found in [7] or [12]. We recall the functional equation

$$\pi^{-\frac{s}{2}}\Gamma(\tfrac{s}{2})\zeta(s) = \pi^{-\frac{1-s}{2}}\Gamma(\tfrac{1-s}{2})\zeta(1-s), \tag{3.1}$$

and the partial fraction representation

$$\frac{\zeta'(s)}{\zeta(s)} = b - \frac{1}{s-1} - \frac{1}{2}\frac{\Gamma'\left(\frac{s}{2}+1\right)}{\Gamma\left(\frac{s}{2}+1\right)} + \sum_{\rho}\left(\frac{1}{s-\rho} + \frac{1}{\rho}\right),$$

where $b = -\frac{\gamma}{2} - 1 + \log 2\pi = -\sum_{n=1}^{\infty}\left(\frac{1}{\rho_n} + \frac{1}{\overline{\rho_n}}\right) + \frac{\log \pi}{2}$. Using

$$\frac{\Gamma'(s)}{\Gamma(s)} = \log|t| + O(1), \qquad (0 \le \sigma \le 2, |t| > 2), \tag{3.2}$$

we see that in the region $0 \le \sigma \le 1, |t| > 2$ the Riemann zeta-function satisfies

$$\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty}\left(\frac{1}{s-\rho_n} + \frac{1}{s-\overline{\rho_n}}\right) - \frac{1}{2}\log|t| + O(1).$$

Taking real parts and observing that in the region $0 \le \sigma \le 1,\ t > 2$ the bound

$$\sum_{n=1}^{\infty}\Re\frac{1}{s-\overline{\rho}_n} = \sum_{n=1}^{\infty}\frac{\sigma-\beta_n}{(\sigma-\beta_n)^2 + (t+\gamma_n)^2} = O(1)$$

is valid, because $\sum_{n=1}^{\infty}\gamma_n^{-2}$ is convergent and the $|\sigma - \beta_n|$ are bounded, we have

$$\Re\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty}\frac{\sigma-\beta_n}{(\sigma-\beta_n)^2 + (t-\gamma_n)^2} - \frac{1}{2}\log t + O(1), \qquad (0 \le \sigma \le 1, t > 2). \tag{3.3}$$

From the simple properties of the nontrivial zeros of $\zeta(s)$, we know that

$$\sum_{n=1}^{\infty}\frac{1}{1+(\gamma_n - T)^2} \ll \log T.$$

for any real number $T \ge 2$. In particular, we know

$$\sum_{|\gamma_n - T| \le 1} 1 \ll \log T, \qquad \sum_{|\gamma_n - T| \ge 1}\frac{1}{(\gamma_n - T)^2} \ll \log T. \tag{3.4}$$

It is also useful to remember that for every large $T > T_0 > 0$, $\zeta(s)$ has a zero $\beta + i\gamma$ which satisfies

$$|\gamma - T| \ll \frac{1}{\log\log\log T} \tag{3.5}$$

([12], Theorem 9.12) and that for any fixed positive h, however small,

$$\sum_{T\le\gamma_n\le T+h} 1 > K\log T, \qquad (K = K(h) > 0) \tag{3.6}$$

([12], Theorem 9.14). The following lemma will play a role in the proof of Theorems 2.2 and 2.3.

Lemma 3.1. For any real numbers $a > 0$, x_1 and x_2, we have

$$\left|\frac{x_1}{x_1^2+a} - \frac{x_2}{x_2^2+a}\right| \le \frac{|x_1-x_2|}{a}.$$

□

Proof. If $f(x) = \frac{x}{x^2+a}$, then

$$|f'(x)| = \left|\frac{(x^2+a)-2x^2}{(x^2+a)^2}\right| \le \frac{x^2+a}{(x^2+a)^2} \le \frac{1}{a},$$

whence the result follows by the mean-value theorem. ■

4 Proof of Theorems 2.2 and 2.3

We can assume that $\left|\beta' - \frac{1}{2}\right|$ is small, otherwise the statements are trivial in view of (3.5) and (3.6). We also can assume that $\beta' \ne \frac{1}{2}$, since otherwise, $\beta' + i\gamma'$ is a multiple zero of $\zeta(s)$ and again the results become trivial. We also assume that γ' is a large positive number and $\gamma' \ne \gamma_n$ for any n.

Let $s = \sigma + it$ be in the region $0 \le \sigma \le 1$, $|t| > 2$. Taking logarithmic derivatives in the functional equation (3.1), and using (3.2), we have

$$\frac{\zeta'(s)}{\zeta(s)} + \frac{\zeta'(1-s)}{\zeta(1-s)} = -\log|t| + O(1).$$

Since $\zeta'(\beta'+i\gamma') = \zeta'(\beta'-i\gamma') = 0$, setting $s = \beta' - i\gamma'$ we obtain

$$\Re\frac{\zeta'(\beta'+i\gamma')}{\zeta(\beta'+i\gamma')} - \Re\frac{\zeta'(1-\beta'+i\gamma')}{\zeta(1-\beta'+i\gamma')} = \log\gamma' + O(1). \tag{4.1}$$

Calculating the left-hand side of (4.1) via (3.3) with $s = \beta' + i\gamma'$ and $s = 1-\beta'+i\gamma'$ gives

$$\sum_{n=1}^{\infty}\left(\frac{\beta'-\beta_n}{(\beta'-\beta_n)^2+(\gamma'-\gamma_n)^2} - \frac{1-\beta'-\beta_n}{(1-\beta'-\beta_n)^2+(\gamma'-\gamma_n)^2}\right) = \log\gamma' + O(1),$$

so that

$$\sum_{n=1}^{\infty}\left|\frac{\beta'-\beta_n}{(\beta'-\beta_n)^2+(\gamma'-\gamma_n)^2}-\frac{1-\beta'-\beta_n}{(1-\beta'-\beta_n)^2+(\gamma'-\gamma_n)^2}\right|\geq \log\gamma'+O(1).$$

Using Lemma 3.1 with

$$x_1=\beta'-\beta_n,\quad x_2=1-\beta'-\beta_n,\quad a=(\gamma'-\gamma_n)^2>0,$$

we have

$$\sum_{n=1}^{\infty}\left|\frac{\beta'-\beta_n}{(\beta'-\beta_n)^2+(\gamma'-\gamma_n)^2}-\frac{1-\beta'-\beta_n}{(1-\beta'-\beta_n)^2+(\gamma'-\gamma_n)^2}\right|\leq 2|\beta'-\tfrac{1}{2}|\sum_{n=1}^{\infty}\frac{1}{(\gamma'-\gamma_n)^2}.$$

Thus, we obtain

$$2\left|\beta'-\tfrac{1}{2}\right|\sum_{n=1}^{\infty}\frac{1}{(\gamma'-\gamma_n)^2}\geq \log\gamma'+O(1), \tag{4.2}$$

an estimate which basically depends upon just the Hadamard factorization theorem and the functional equation of $\zeta(s)$.

First we prove Theorem 2.2. Without loss of generality, we can assume that

$$2\left|\beta'-\tfrac{1}{2}\right|\sum_{n=1}^{\infty}\frac{1}{(\gamma'-\gamma_{2n})^2}\geq 2\left|\beta'-\tfrac{1}{2}\right|\sum_{n=1}^{\infty}\frac{1}{(\gamma'-\gamma_{2n-1})^2}$$

which implies the key estimate

$$4\left|\beta'-\tfrac{1}{2}\right|\sum_{n=1}^{\infty}\frac{1}{(\gamma'-\gamma_{2n})^2}\geq \log\gamma'+O(1). \tag{4.3}$$

We recall that γ' is a large number and $|\beta'-\frac{1}{2}|$ is small. Then, from (4.3) we have

$$\left|\beta'-\tfrac{1}{2}\right|\sum_{n=1}^{\infty}\frac{1}{(\gamma'-\gamma_{2n})^2}\geq \frac{\log\gamma'}{8}.$$

According to (3.4), for a small $\left|\beta'-\frac{1}{2}\right|$, we have

$$\left|\beta'-\tfrac{1}{2}\right|\sum_{|\gamma_n-\gamma'|\geq 1}\frac{1}{(\gamma'-\gamma_n)^2}\leq \frac{\log\gamma'}{16},$$

which implies

$$\left|\beta' - \tfrac{1}{2}\right| \sum_{|\gamma_{2n}-\gamma'|\le 1} \frac{1}{(\gamma'-\gamma_{2n})^2} \ge \frac{\log\gamma'}{16}. \tag{4.4}$$

Denote

$$\delta = C\sqrt{\frac{|\beta'-\frac{1}{2}|}{\log\gamma'}},$$

where $C > 0$ will be chosen to be sufficiently large. Divide the interval $[\gamma'-1, \gamma'+1]$ into small subintervals of the type

$$I_k = [\gamma' + k\delta, \gamma' + (k+1)\delta] \cap [\gamma'-1, \gamma'+1],$$

where k runs through integers and $-\frac{1}{\delta} - 1 \le k \le \frac{1}{\delta} + 1$. If there exists n such that $\gamma_{2n} \in I_{-1} \cup I_0$, then

$$|\gamma_{2n} - \gamma'| \log\gamma' \le \delta\log\gamma' = C\sqrt{|\beta'-\tfrac{1}{2}|\log\gamma'}$$

and we are done in this case. Otherwise, we can rewrite (4.4) in the form

$$\left|\beta'-\tfrac{1}{2}\right| \sum_{-\frac{1}{\delta}-1\le k\le -2} \sum_{\gamma_{2n}\in I_k} \frac{1}{(\gamma'-\gamma_{2n})^2} + \left|\beta'-\tfrac{1}{2}\right| \sum_{1\le k\le \frac{1}{\delta}+1} \sum_{\gamma_{2n}\in I_k} \frac{1}{(\gamma'-\gamma_{2n})^2} \ge \frac{\log\gamma'}{16}.$$

If for some k we have two numbers n_1, n_2 with $\gamma_{2n_1} \in I_k$, and $\gamma_{2n_2} \in I_k$, then

$$|\gamma_{2n_1} - \gamma_{2n_2}| \log\gamma' \le \delta\log\gamma' = C\sqrt{|\beta'-\tfrac{1}{2}|\log\gamma'}$$

and we are done in this case too. For this reason we can assume that for a given k we have at most one n with $\gamma_{2n} \in I_k$, in which case we have

$$|\gamma_{2n} - \gamma'| \ge |k+1|\delta, \quad \text{when} \quad -\frac{1}{\delta} - 1 \le k \le -2,$$

and

$$|\gamma_{2n} - \gamma'| \ge k\delta, \quad \text{when} \quad 1 \le k \le \frac{1}{\delta} + 1.$$

Hence we have

$$\frac{\log\gamma'}{16} \le \left|\beta' - \tfrac{1}{2}\right| \sum_{|k|>0} \frac{1}{k^2\delta^2} \le 4C^{-2}\log\gamma'.$$

However, this can not hold if $C > 8$. The proof of Theorem 2.2 is finished.

Now we proceed to prove Theorem 2.3. Let $d = \min_n |\gamma' - \gamma_n|$. If we prove that $d \ll \left|\beta' - \frac{1}{2}\right|^{\frac{1}{2}}$, then we are done. From (3.4) we know

$$2\left|\beta' - \tfrac{1}{2}\right| \sum_{d \le |\gamma' - \gamma_n| < 1} \frac{1}{(\gamma' - \gamma_n)^2} \ll \frac{\left|\beta' - \frac{1}{2}\right| \log\gamma'}{d^2},$$

and

$$2\left|\beta' - \tfrac{1}{2}\right| \sum_{|\gamma' - \gamma_n| \ge 1} \frac{1}{(\gamma' - \gamma_n)^2} \ll \left|\beta' - \tfrac{1}{2}\right| \log\gamma'.$$

Plugging these estimates into (4.2) and recalling the fact that γ' is large, we obtain

$$\frac{\left|\beta' - \frac{1}{2}\right| \log\gamma'}{d^2} + \left|\beta' - \tfrac{1}{2}\right| \log\gamma' \gg \log\gamma'.$$

Since $d = o(1)$ by (3.5), this reduces to

$$\frac{\left|\beta' - \frac{1}{2}\right| \log\gamma'}{d^2} \gg \log\gamma',$$

completing the proof of Theorem 2.3.

5 Proof of Corollary 2.4

In order to prove Corollary 2.4, assume that (iii) is not true, i.e.

$$\liminf_{\gamma' \to \infty} |\gamma_c - \gamma'| \log\gamma' \ge c_1 > 0. \tag{5.1}$$

Then, there exists a constant T_0 such that

$$|\gamma - \gamma'| \log\gamma' > \frac{c_1}{2}$$

for $\gamma > T_0$, $\gamma' > T_0$, and there can be at most finitely many multiple zeros of $\zeta(s)$. Hence, assuming RH, Theorem 2.2 implies

$$\liminf_{\gamma\to\infty}(\gamma^+ - \gamma)\log\gamma = 0.$$

Therefore, in Zhang's result which was mentioned at the end of §1, we can take α_1 to be small (and therefore, we can take α_2 to be small too) and deduce that for any ρ with a large γ there exists a zero ρ' of $\zeta'(s)$ such that

$$|\gamma - \gamma'|\log\gamma \le |\rho' - \rho|\log\gamma < \frac{c_1}{2}.$$

This contradicts (5.1) and proves Corollary 2.4.

6 Proof of Theorem 2.5

The proof presented here stems from an idea of Haseo Ki. We now work under the assumptions that the RH is true, and

$$\liminf_{\gamma'\to\infty}(\beta' - \tfrac{1}{2})(\log\gamma')(\log\log\gamma')^2 = 0. \tag{6.1}$$

For our purpose we may also assume that all but finitely many of the zeta zeros are simple, because otherwise $\liminf(\gamma_{n+1} - \gamma_n)\log\gamma_n = 0$ holds trivially.

For a $\rho' = \beta' + i\gamma'$, member of a sequence with the property (6.1), there are two possibilities:
Either

$$|\gamma_c - \gamma'| \le |\gamma_{n+1} - \gamma_n|, \quad \forall\gamma_n \in [\gamma' - 1, \gamma' + 1],$$

or

$$\exists\gamma_n \in [\gamma' - 1, \gamma' + 1], \quad |\gamma_{n+1} - \gamma_n| < |\gamma_c - \gamma'|.$$

If there is a subsequence of ρ' satisfying the second possibility, we have by Theorem 2.2 for the corresponding γ_n,

$$|\gamma_{n+1} - \gamma_n|(\log\gamma_n) \ll (|\beta' - \tfrac{1}{2}|\log\gamma')^{\frac{1}{2}}.$$

Thus, in this case we don't even need the full strength of the condition (6.1) to conclude that $\liminf_{n\to\infty}(\gamma_{n+1}-\gamma_n)\log\gamma_n = 0$.

From now on we may take that after a point on all ρ' from a sequence with the property (6.1) satisfy the first possibility. Suppose

$$\liminf_{n\to\infty}(\gamma_{n+1}-\gamma_n)\log\gamma_n > 0,$$

so that there exists a fixed $\delta > 0$ such that

$$\gamma_{n+1}-\gamma_n > \frac{\delta}{\log\gamma_n}$$

for all sufficiently large n.

We apply the formula [12, Theorem 9.6 (A)]

$$\frac{\zeta'}{\zeta}(s) = \sum_{|\gamma-t|\le 1}\frac{1}{s-\rho} + O(\log t)$$

at $s=\rho'$, where ρ' is a member of a sequence obeying (6.1). So we can write

$$0 = \frac{1}{\rho'-\rho_c} + \sum_{\substack{\rho\ne\rho_c\\ |\gamma-\gamma'|\le 1}}\frac{1}{\rho'-\rho} + O(\log\gamma'). \tag{6.2}$$

We now examine the sum occurring in this formula. Clearly,

$$\left|\sum_{\substack{\rho\ne\rho_c\\ |\gamma-\gamma'|\le 1}}\frac{1}{\rho'-\rho}\right| \le \sum_{\substack{\gamma\ne\gamma_c\\ |\gamma-\gamma'|\le 1}}\frac{1}{|\gamma-\gamma'|}.$$

By our (case 1) assumption we have, for all positive integers j,

$$|\gamma_{c\pm j}-\gamma'| \ge \frac{j\delta}{3\log\gamma'}$$

(here $\gamma_{c\pm j}=\gamma_{n_0\pm j}$ when $\gamma_c=\gamma_{n_0}$). Since the sum is over the zeros with γ in an interval of radius 1 around γ', we see that j can be at most as large as $\frac{\kappa\log\gamma'}{\delta}$ with some absolute constant κ. Therefore

$$\sum_{\substack{\gamma\ne\gamma_c\\ |\gamma-\gamma'|\le 1}}\frac{1}{|\gamma-\gamma'|} \ll \frac{(\log\gamma')(\log\log\gamma')}{\delta}.$$

Hence we can rewrite (6.2) as

$$0 = \frac{1}{\rho' - \rho_c} + O\left(\frac{(\log\gamma')(\log\log\gamma')}{\delta}\right),$$

from which we see that

$$\frac{1}{\sqrt{(\beta' - \frac{1}{2})^2 + (\gamma' - \gamma_c)^2}} \leq \frac{\kappa_1(\log\gamma')(\log\log\gamma')}{\delta} \tag{6.3}$$

for some absolute constant κ_1. Now recall that ρ' satisfies the first possibility, so that by Theorem 2.2 we have

$$|\gamma' - \gamma_c| \leq \kappa_2\left(\frac{\beta' - \frac{1}{2}}{\log\gamma'}\right)^{\frac{1}{2}}$$

for some absolute constant κ_2. Using this in (6.3) we get

$$\frac{1}{(\beta' - \frac{1}{2})^2 + \kappa_2^2\left(\frac{\beta' - \frac{1}{2}}{\log\gamma'}\right)} \leq \frac{\kappa_1^2(\log\gamma')^2(\log\log\gamma')^2}{\delta^2}.$$

Now the quadratic formula yields

$$\beta' - \tfrac{1}{2} \geq \frac{\delta^2}{2(\kappa_1\kappa_2)^2(\log\gamma')(\log\log\gamma')^2}$$

for sufficiently large γ', which contradicts the assumption (6.1).

This completes the proof of Theorem 2.5.

Acknowledgments

This work was supported by Project PAPIIT-IN105605 from the UNAM. Yıldırım is grateful to Instituto de Matemáticas, UNAM, Campus Morelia, México for its hospitality. The authors are grateful to Professor Haseo Ki for sharing his thoughts which led to the present form of Theorem 2.5, improving upon the former version greatly.

References

[1] Berndt, B. C. "The Number of Zeros for $\zeta^{(k)}(s)$." *Journal of the London Mathematical Society-Second Series (2)* 2 (1970): 577–580.

[2] Conrey, J. B., and A. Ghosh. "Zeros of Derivatives of the Riemann Zeta-Function Near the Critical Line." *In Analytic Number Theory (Allerton Park, Ill. 1989), Progress in Mathematics, 85*. Boston, MA: Birkhäuser, (1990): 95–110.

[3] Conrey, J. B., A. Ghosh, D. Goldston, S. M. Gonek, and D. R. Heath-Brown. "On the Distribution of Gaps Between Zeros of the Zeta-Function." *Quarterly Journal of Mathematics Oxford (2)* 36 (1985): 43–51.

[4] Farmer, D. W., S. M. Gonek, and C. P. Hughes. "The Maximum Size of L-functions." arXiv:math. NT/0506218.

[5] Feng, S. "A Note on the Zeros of the Derivative of the Riemann Zeta Function Near the Critical Line." *Acta Arithmetica* 120.1 (2005): 59–68

[6] Guo, C. R. "On the Zeros of $\zeta(s)$ and $\zeta'(s)$." *Journal of Number Theory* 54 (1995): 206–210.

[7] Karatsuba, A. A., and S. M. Voronin. *The Riemann Zeta-Function*. Berlin, New York: Walter de Gruyter, 1992.

[8] Levinson, N., and H. L. Montgomery. "Zeros of the Derivatives of the Riemann Zeta-Function." *Acta Mathematica* 133 (1974): 49–65.

[9] Montgomery, H. L. "The Pair Correlation of Zeros of the Riemann Zeta-Function" *in Analytic Number Theory (St. Louis, Mo., 1972), Proceedings of Symposia in Pure Mathematics, 24*. Providence, RI: American Mathematical Society, (1973): 181–193.

[10] Soundararajan, K. "The Horizontal Distribution of Zeros of $\zeta'(s)$." *Duke Mathematical Journal* 91, no. 1 (1998): 33–59.

[11] Speiser, A. "Geometrisches zur Riemannschen Zetafunktion." *Mathematische Annalen* 110 (1934): 514–521.

[12] Titchmarsh, E. C. *The Theory of the Riemann Zeta-Function*, 2nd edn, edited by D. R. Heath-Brown. Oxford: Clarendon Press, 1986.

[13] Zhang, Y. "On the Zeros of $\zeta'(s)$ Near the Critical line." *Duke Mathematical Journal* 110, no. 3 (2001): 555–572.