ON THE ZEROS OF THE SECTIONS OF THE EXPONENTIAL FUNCTION

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Let ν_j denote a zero of the k-th partial sum of the Maclaurin series for e^z. We find a sharper bound than previously known for $k+1-\sum_{j=1}^{k} e^{-\nu_j}$ by making use of a sequence of functions introduced by Dieudonné.

§1. Introduction

The Riemann zeta-function has the functional equation

$$\zeta(1-s) \;=\; \chi(1-s)\zeta(s)\;; \qquad \chi(1-s) \;=\; \pi^{\frac{1}{2}-s}\,\frac{\Gamma(\frac{s}{2})}{\Gamma(\frac{1-s}{2})}\;.$$

In investigating the zeros ρ_k of $\zeta^{(k)}(s)$ Conrey and Ghosh [3] showed, assuming the Riemann Hypothesis, that as $T \to \infty$

$$\sum_{0<Im\rho_k\le T} \chi(\rho_k) \;\sim\; \alpha_k \frac{T}{2\pi}$$

where

$$\alpha_k \;=\; k+1-\sum_{j=1}^{k} e^{-\nu_j}$$

and ν_j is a root of

$$P_k(z) \;:=\; \sum_{r=0}^{k} \frac{z^r}{r!}$$

(the k-th polynomial-section in the Maclaurin expansion of e^z).

The Eneström-Kakeya theorem (the absolute value of any zero of $a_0+a_1x+\cdots+a_kx^k$, $a_i \in \mathbb{R}^+$ is at most $\max(\frac{a_0}{a_1}, \frac{a_1}{a_2}, \ldots, \frac{a_{k-1}}{a_k})$) implies $|\nu_j| \le k$. It was shown by Szegö [8] that the numbers $\frac{\nu_j}{k}$ cluster around the simple closed curve $\Gamma = \{z : |ze^{1-z}| = 1, |z| \le 1\}$ as $k \to \infty$ and conversely each point of the Szegö curve is a limit point of the normalized zeros. Moreover Szegö [8] and also Dieudonné [4] found that as $k \to \infty$ the proportion of the normalized zeros which cluster along a given arc of Γ is asymptotic to $\frac{1}{2\pi}\Delta\, arg\, ze^{1-z}$ as z moves on the arc. Buckholtz [1] added that $\frac{\nu_j}{k}$ always lies in the exterior of Γ within a distance of $\frac{2e}{\sqrt{k}}$ from Γ.

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Conrey and Ghosh [2] proved that if $0 < \beta < 1 - \log 2$ then

$$|\alpha_k| \ \le \ e^{-\beta k}$$

for all sufficiently large k. They also gave the values of α_k for $k \le 15$ from which it is seen that $\frac{\log|\alpha_k|}{k}$ is mostly between -1.45 and -1.50 (there seems to be deviations though, e.g. -1.93 when $k = 6$). It is indeed remarkable that α_k is exponentially small whereas most of the terms $e^{-\nu_j}$ are exponentially large functions of k.

Here we show that a stronger estimate holds.

Theorem. *For any fixed* $m > 0$

$$\alpha_k \ \ll_m \ k^{-m} e^{-k} \ .$$

§2. Reformulation of the problem

It will be convenient to write $n = k + 1$. We recall the following definition due to Dieudonné [4] .

$$\begin{aligned} f_n(z) \ &:= \ \frac{n!}{(nz)^n}\Big(e^{nz} - \sum_{p=0}^{n-1} \frac{(nz)^p}{p!}\Big) \\ &= \ 1 + \frac{z}{(1+\frac{1}{n})} + \frac{z^2}{(1+\frac{1}{n})(1+\frac{2}{n})} + \cdots + \frac{z^p}{(1+\frac{1}{n})(1+\frac{2}{n})\cdots(1+\frac{p}{n})} + \cdots \end{aligned} \tag{1}$$

Then $P_k(\nu_j) \ = \ 0$ may be expressed as

$$e^{-\nu_j} \ = \ \frac{n!}{\nu_j^n f_n(\frac{\nu_j}{n})} \ . \tag{2}$$

Let

$$\frac{1}{f_n(z)} \ = \ 1 + \sum_{p=1}^{\infty} d_p z^p \ , \tag{3}$$

valid in a certain neighbourhood of $z = 0$ (see §3 (vi)), and put

$$s_r \ (= \ s_r(k)) \ = \ \sum_{j=1}^{k} \nu_j^r \ . \tag{4}$$

It follows that

$$\sum_{j=1}^{k} e^{-\nu_j} \ = \ (k+1)!\Big(s_{-(k+1)} + \sum_{p=1}^{\infty} \frac{d_p s_{p-(k+1)}}{(k+1)^p}\Big) \ .$$

The relevant negative-power sums of the zeros of $P_k(z)$ are easily calculated in terms of its coefficients via the Newton-Girard formulae and found to be

$$s_{-1} = -1, \qquad s_{-2} \;=\; s_{-3} = \cdots = s_{-k} \;=\; 0\;, \qquad s_{-(k+1)} = \frac{1}{k!}.$$

Using these values we have

$$\sum_{j=1}^{k} e^{-\nu_j} \;=\; k+1+\frac{(k+1)!}{(k+1)^{k+1}}\Big(-(k+1)d_k+kd_{k+1}+\sum_{p=1}^{\infty}\frac{d_{p+k+1}s_p}{(k+1)^p}\Big)\;. \qquad (5)$$

§3. The tail of the exponential series

Our main objective is to find upper bounds for the coefficients $d_p = d_p(n)$ (defined by (3)) then the Theorem will be deduced immediately. For this purpose below we give the relevant properties of the functions $f_n(z)$ that were derived in [9].

(i) For each $n \in \mathbb{Z}^+$ and for all $z \in \mathbb{C}$

$$f_n(z)(1-z) \;=\; 1-\frac{zf_n'(z)}{n}\;.$$

(ii) Dieudonné [4] showed that as $n \to \infty$ the function $f_n(z)$ tends uniformly to $\frac{1}{1-z}$ in the domain $\{z : |z| < 1, |1-z| > r\}$ for any small fixed $r > 0$. Similarly $f_n'(z)$ tends uniformly to $\frac{1}{(1-z)^2}$ in the same domain.

(iii) For $\frac{r_0}{\sqrt{n}} \le |1-z| \le \frac{1}{C}$ and $|z| \le 1$ (where r_0 and C are arbitrarily large fixed positive numbers) the asymptotic estimates of (ii) continue to hold .

(iv) When $0 \le |1-z| \le \frac{r_0}{\sqrt{n}}$

$$\frac{f_n(z)}{\sqrt{n}} \longrightarrow \sqrt{\frac{\pi}{2}}\; w\Big(\sqrt{\frac{n}{2}}(1-z)i\Big)$$

uniformly as $n \to \infty$ where

$$w(z) \;=\; e^{-z^2}\Big(1+\frac{2i}{\sqrt{\pi}}\int_0^z e^{t^2}dt\Big)\;.$$

The values of $w(z)$ have been tabulated by Faddeeva and Terent'ev [5].

(v) By a crude version of an assertion of Ramanujan [7] $f_n(1) \sim \sqrt{\frac{\pi}{2}n}$ as $n \to \infty$. It is readily seen from (1) that $f_n'(1) = n$. Then employing (i) we find for the l-th derivative (l fixed)

$$f_n^{(l)}(1) \sim \begin{cases} ((l-1)(l-3)\cdots 4\cdot 2)n^{\frac{l+1}{2}}, & \text{odd} \quad l \geq 3, \\ ((l-1)(l-3)\cdots 3\cdot 1)\sqrt{\frac{\pi}{2}}n^{\frac{l+1}{2}}, & \text{even} \quad l \geq 2. \end{cases}$$

(vi) Buckholtz's study [1] reveals that the zeros of $f_n(z)$ satisfy $|ze^{1-z}| > 1$ and $|z| > 1$, located within a distance of $\frac{2e}{\sqrt{n}}$ from the curve $\Gamma_1 = \{z : \ |ze^{1-z}| = 1, |z| \geq 1\}$. So the inversion of the series for $f_n(z)$ for $|z| \leq 1$ in (5) is justified. In fact from the work of Fettis-Caslin-Cramer [6] on the error function it follows that the zeros of $f_n(z)$ which are closest to $z = 1$ are at $1 + \frac{\sqrt{2}(1.35 \pm i1.99) + o(1)}{\sqrt{n}}$.

(vii) Let $g_n(z) = \frac{f_n'}{f_n}(z)$. As a consequence of the previous paragraphs, when $|z| \leq 1$,

$$|g_n^{(m)}(z)| \ll_m \ n^{\frac{m+1}{2}}, \quad \text{fixed } m = 0, 1, 2, \ldots \ .$$

(viii) By virtue of (i) instead of the coefficients of $\frac{1}{f_n(z)}$ we consider those of $g_n(z) = \frac{n}{n+1} + \sum_{p=1}^{\infty} \delta_p z^p$ with $\delta_p = nd_{p+1}$. Applying the Cauchy formula on the unit disk we have, if $p \geq m$,

$$\begin{aligned} p(p-1)\cdots(p-m+1)\delta_p &= \frac{(g_n^{(m)})^{(p-m)}(0)}{(p-m)!} \\ &\leq \frac{1}{2\pi}\int_0^{2\pi} |g_n^{(m)}(e^{i\theta})| d\theta \\ &\ll_m n^{\frac{m+1}{2}} \end{aligned}$$

so that

$$d_p \ll_m k^{\frac{m-1}{2}} p^{-m} \ .$$

Substituting the result of (viii) and the trivial bound

$$|s_p| \leq k^{p+1} \quad (p \geq 1)$$

in (5) the proof of the Theorem is completed.

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