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PRIMES IN TUPLES III: On the difference $p_{n+\nu} - p_n$

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Dedicated to Prof. Eduard Wirsing
on the occasion of his 75th birthday

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1. Introduction

As an approximation to the twin prime problem, Hardy and Littlewood initiated the investigation of

$$\Delta_\nu := \liminf_{n\to\infty} \frac{p_{n+\nu} - p_n}{\log p_n}, \tag{1.1}$$

where p_n denotes the nth prime. They considered only the case $\nu = 1$, and proved using the circle method, under the assumption that all Dirichlet-$\mathcal{L}$-functions $L(s,\chi)$ have no zeros in the half-plane $\mathrm{Re}(s) > \Theta$, that

$$\Delta_1 \leqslant \frac{1+2\Theta}{3}. \tag{1.2}$$

Thus, under the Generalized Riemann Hypothesis where $\Theta = \frac{1}{2}$, they proved

$$\Delta_1 \leqslant \frac{2}{3}. \tag{1.3}$$

The estimate $\Delta_1 \leqslant 1$ (or, in general, $\Delta_\nu \leqslant \nu$) is a trivial consequence of the prime number theorem, and the first non-trivial unconditional result that

$$\Delta_1 < 1, \tag{1.4}$$

was obtained by Erdős [3] in 1940 using Brun's sieve.

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The method of Hardy–Littlewood was improved and made unconditional by Bombieri and Davenport [1] in 1965 where they proved

$$\Delta_\nu \leqslant \nu - \frac{1}{2}. \tag{1.5}$$

The removal of the Generalized Riemann Hypothesis here was made possible by the newly available and now celebrated Bombieri–Vinogradov theorem. In the case of $\nu = 1$ Bombieri and Davenport combined their method with that of Erdős and obtained

$$\Delta_1 \leqslant \frac{2+\sqrt{3}}{8} = 0.46650\ldots\ . \tag{1.6}$$

Later Huxley [11, 12] refined this method and obtained

$$\Delta_1 \leqslant 0.44254\ldots, \text{ and later } \quad \Delta_1 \leqslant 0.43494\ldots, \tag{1.7}$$

and for general $\nu \geqslant 2$

$$\Delta_\nu \leqslant \nu - \frac{5}{8} + O\left(\frac{1}{\nu}\right). \tag{1.8}$$

Huxley also generalized the estimate (1.8) for primes in arithmetic progressions $a(\bmod q)$ with a fixed q and any a with $(a,q)=1$. Denoting by $p'_n = p'_n(q,a)$ the nth prime in this progression, the prime number theorem for arithmetic progressions implies the average distance between primes in this progression is $\phi(q)\log p'_n$, and Huxley proved in [10] the generalization of (1.8)

$$\liminf_{n\to\infty} \frac{p'_{n+\nu} - p'_n}{\varphi(q)\log p'_n} \leqslant \nu - \frac{5}{8} + O\left(\frac{1}{\nu}\right). \tag{1.9}$$

Finally, about 20 years ago, Maier [13] found the surprising result that there exist short intervals with a higher than expected proportion of primes, and also intervals with a smaller than expected proportion. This result when applied to intervals at the average spacing yields trivially

$$\Delta_\nu \leqslant e^{-\gamma}\nu, \tag{1.10}$$

and in particular $\Delta_1 \leqslant e^{-\gamma} = 0.56145\ldots$. While the estimate (1.10) is substantially better than (1.8) it does not improve on the results of Bombieri-Davenport and Huxley when $\nu = 1$. A few years after his initial work Maier [14] addressed this by combining his matrix method with Huxley's work, thus using all three of the significantly different methods to obtain

$$\Delta_\nu \leqslant e^{-\gamma}\left(\nu - \frac{5}{8} + O\left(\frac{1}{\nu}\right)\right) \tag{1.11}$$

and for $\nu = 1$

$$\Delta_1 \leqslant e^{-\gamma}\cdot 0.44254 = 0.2486\ldots. \tag{1.12}$$

Recently, the first and the third authors [6] used higher correlations of short divisor sums which approximate the von Mangoldt function to obtain without any of the earlier methods

$$\Delta_\nu \leqslant \left(\sqrt{\nu} - \frac{1}{2}\right)^2, \text{ and in particular, } \Delta_1 \leqslant \frac{1}{4}. \tag{1.13}$$

Finally, in 2005, the present authors [7] found an alternative, improved form of this method which can be interpreted within the framework of Selberg's λ^2-sieve. They obtained by this method

$$\Delta_\nu \leqslant \left(\sqrt{\nu} - 1\right)^2, \text{ and in particular, } \Delta_1 = 0. \tag{1.14}$$

At the same time, J. Sivak [16] combined Maier's method and an earlier version of the method [6] leading to (1.13) (see [5]) to prove

$$\Delta_\nu \leqslant e^{-\gamma}\left(\nu - \frac{\sqrt{\nu}}{2}\right). \tag{1.15}$$

The aim of the present work is to show that Maier's matrix method and the approach in [7] can be combined successfully to yield

$$\Delta_\nu \leqslant e^{-\gamma}\left(\sqrt{\nu} - 1\right)^2, \tag{1.16}$$

even in the extended form for small differences between primes in arithmetic progressions modulo q. In contrast to Huxley's work we can allow q to tend to infinity with the size N of the primes as long as

$$q \ll_A (\log\log N)^A, \tag{1.17}$$

where A is an arbitrary constant.

Our present work will be heavily based on [7], and on the basic idea of Maier [13] but we will install some refinements of our work and the works of Maier [13, 14]. These features are

(i) our present results will be effective, in contrast to our work [7], which used the original Bombieri–Vinogradov theorem;

(ii) our present result makes possible an effective localization of the dense blocks of primes $[p_n, \ldots, p_{n+\nu}]$ with small differences $p_{n+\nu} - p_n$ within intervals of type

$$[N/3, N]; \tag{1.18}$$

whereas Maier's original approach yields a much weaker localization of the type $[\log^c N, N]$ for the relevant dense blocks [14] (or dense intervals, in [13]). We emphasize however, that we still only obtain these dense blocks rarely.

Finally we remark the work [6] contains a somewhat more detailed description of the three earlier mentioned methods ([9]–[1], [3], [13, 14]) besides a full proof of (1.13).

A further interesting feature of our present proof is that we do not need the important theorem of Gallagher [4] about the average of the singular series which played a decisive role in our work [7] in the proof of (1.14).

The exact result we prove is formulated in the following theorem.

Theorem 1. *Let ν be an arbitrary fixed positive integer. Let ε and A be arbitrary fixed positive numbers. Let q and N be arbitrary, sufficiently large integers, satisfying*

$$q_0(A,\varepsilon,\nu) < q < (\log\log N)^A, \quad N > N_0(A,\varepsilon,\nu), \tag{1.19}$$

and let a be arbitrary with $(a,q)=1$. Let $p_1', p_2', \dots$ denote the consecutive primes $\equiv a(\bmod q)$. Then there exists a block of $\nu+1$ primes $p_n', \dots, p_{n+\nu}'$ such that

$$\frac{p_{n+\nu}' - p_n'}{\varphi(q)\log p_n'} < e^{-\gamma}\left(\sqrt{\nu}-1\right)^2 + \varepsilon, \quad p_n' \in [N/3, N]. \tag{1.20}$$

Consequently (cf. (1.16)),

$$\Delta_\nu(q,a) := \liminf_{n\to\infty} \frac{p_{n+\nu}' - p_n'}{\varphi(q)\log p_n'} \leqslant e^{-\gamma}\left(\sqrt{\nu}-1\right)^2, \tag{1.21}$$

and in particular

$$\Delta_1(q,a) = 0. \tag{1.22}$$

Remark. The constant $N_0(A,\varepsilon,\nu)$ is ineffective for $A>2$ by Siegel's theorem; however, it can be given explicitly if $A \leqslant 2$.

2. Preparation for the proof of the Theorem

We will define, similarly to Maier [14]

$$z = \frac{\log N}{\log_2^2 N}, \quad \widetilde{P} = \prod_{p\leqslant z} p, \quad R = N^{\frac{1}{4}-\frac{\varepsilon}{10}}, \tag{2.1}$$

$$h = e^{-\gamma}\left(\left(\sqrt{\nu}-1\right)^2 + \varepsilon\right)\varphi(q)\log(3N), \quad Y = e^{3z} = \exp\left(\frac{3\log N}{\log_2^2 N}\right), \tag{2.2}$$

where $\log_n x$ denotes the n-fold iterated logarithm function.

The Siegel zeros cause serious irregularities in the distribution of primes in arithmetic progressions and through this, in Bombieri–Vinogradov (type) theorem(s). In order to deal with this situation we quote the well-known

Lemma 1 (Landau–Page Theorem). *There exists a c_0 such that for any $Y > C(c_0)$ there exists at most one modulus q_1 and at most one real primitive character $\chi_1 \pmod{q_1}$ such that*

$$L(1-\delta,\chi_1,q_1) = 0, \quad \delta \leqslant \frac{c_0}{\log Y}, \quad q_1 \leqslant Y. \tag{2.3}$$

If q_1 exists, then $q_1 > \log^2 Y$.

A proof of this (with an unspecified, but explicitly calculable c_0) is contained in Chapter 14 of [2], with $c_0 = 1/2 + o(1)$ in [15], so we are entitled to choose $c_0 = 1/3$, $C(1/3) = C_0$.

Using our value $Y = Y(N)$ from (2.2) we can define a quantity q_1' with $c_0 = 1/3$ and the above chosen modulus $q_1 = q_1(Y) = q_1(N)$, if it exists, by $q_1' = q_1$. If q_1 does not exist for our given N, we set $q_1' = 1$ and define

$$G\left(\frac{q_1'}{(q_1', q)}\right) = p_1, \quad P = \frac{q\widetilde{P}}{(\widetilde{P}, p_1)}, \quad P_1 = [P, p_1], \quad M = \left\lceil \frac{N}{3P} \right\rceil, \tag{2.4}$$

where $G(n)$ denotes the greatest prime factor of n, $G(1) = 1$, $(a, b) = \gcd(a, b)$, $[a, b] = \mathrm{lcm}[a, b]$.

Our strategy will be to look for blocks of close primes among numbers of the form

$$mP + i, \quad m \in (M, 2M], \quad i \in \mathcal{P}^* = \mathcal{P}_{q,a} \cap (z, h], \tag{2.5}$$

where $\mathcal{P}_{q,a}$ denotes the set of primes $\equiv a (\mathrm{mod}\, q)$,

$$\mathcal{P}_{q,a} = \{p_j'\}_{j=1}^{\infty}. \tag{2.6}$$

This is, even for $q = 1$, somewhat different from the method of Maier, since he uses arithmetic progressions of type $m\widetilde{P} + i$, and restricts z (equivalently N) to a rare set, to avoid Siegel zeros for the modulus P.

All real primitive characters are the products of Legendre symbols with different odd primes, and possibly either the unique real character mod 4 or one of the two primitive real characters mod 8 (see for instance [2], Chapter 5, equation (6)). Thus q_1 is the product of odd primes with exponent 1 and the prime 2 at most with exponent 3, and $q_1 > \log N > z > 8q$. Hence we have by (2.4)

$$p_1 \nmid q \ \text{ and } \ p_1 \nmid P \ \text{ if } \ p_1 > 1 \Longleftrightarrow q_1 \ \text{ exists.} \tag{2.7}$$

Consequently, in both cases $p_1 = 1$ or $p_1 > 1$ we have

$$q \mid P, \quad mP + i \equiv a (\mathrm{mod}\, q) \ \text{ if } i \in \mathcal{P}^*. \tag{2.8}$$

As indicated in (2.5) we will consider only k-tuples $\mathcal{H}$ of the form

$$\mathcal{H} = \{h_j\}_{j=1}^{k} \subset \mathcal{P}^*, \quad h_j \text{ distinct} \tag{2.9}$$

and set, as in [7],

$$\mathcal{P}_{\mathcal{H}}(n) = (n + h_1) \dots (n + h_k), \tag{2.10}$$

where the value of n will be always restricted to multiples of P. A further change compared with [7] will be that if q_1 exists, then we exclude from the sieving process those λ_d (see (3.1)), where $p_1 \mid d$. In this way we have by (2.7) that

$$p_1 \nmid [d_1, d_2, P] \Longrightarrow q_1 \nmid [d_1, d_2, P] \tag{2.11}$$

if $p_1 > 1 \Leftrightarrow q_1$ exists. This will assure that primes will be regularly distributed in arithmetic progressions with a difference $[d_1, d_2, P]$ due to the absence of

Siegel-zeros. This fact has to be used in the proof of Proposition 2. We will use the notation

$$\theta(n) = \begin{cases} \log n & \text{if } n \text{ is prime.} \\ 0, & \text{otherwise.} \end{cases} \tag{2.12}$$

According to our choice $n = mP$ and (2.11) we will need a Bombieri–Vinogradov-type theorem for multiples m of P, which are, however, not multiples of the exceptional modulus q_1 if q_1 exists, that is, we will have $q_1 \nmid mP$.

Our Lemma 2 follows from the proof of our Theorem 6 in [8], since none of the moduli dP appearing in (2.17), is a multiple of the possibly existing exceptional modulus q_1 in Lemma 1.

Lemma 2. *Let P, Y, N be integers with*

$$P^2 \leqslant Y \leqslant N, \quad Y \geqslant \exp\bigl(2\sqrt{\log N}\bigr), \tag{2.13}$$

and suppose that with the exceptional modulus q_1 defined by Lemma 1 we have for a given prime factor p_1 of q_1

$$p_1 \nmid P \quad \text{if } q_1 \text{ exists.} \tag{2.14}$$

Let

$$D^* = N^{1/2} P^{-3} \exp\bigl(-\sqrt{\log N}\bigr), \tag{2.15}$$

$$E^*(N, q) = \max_{X \leqslant N} \max_{(a,q)=1} \left| \sum_{\substack{p \equiv a (\operatorname{mod} q) \\ p \leqslant X}} \log p - \frac{X}{\varphi(q)} \right|. \tag{2.16}$$

Then we have, with explicitly calculable constants C_1 and c_2

$$\sideset{}{^*}\sum_{\substack{d \leqslant D^* \\ (d,P)=1}} E^*(N, dP) \leqslant C_1 \frac{N}{P} \exp\left(-\frac{c_2 \log N}{\log Y} \right), \tag{2.17}$$

where $\sum^*$ means that the summation refers only for d's with $p_1 \nmid d$ if q_1 exists.

We note that with the choice of P, Y, N, R in (2.1)–(2.4) the conditions (2.13)–(2.14) and $D^* > R^2$ are satisfied and we obtain an error term of size

$$\frac{N}{P} (\log N)^{-\frac{c_2}{3} \log_2 N}. \tag{2.18}$$

3. Two basic propositions

We will define the weights similarly as in Section 2 of [7], but we will exclude divisors d which are multiples of p_1 and consider only sets $\mathcal{H} \subset \mathcal{P}^*$. So, let

$$\Lambda_R(n; \mathcal{H}, \ell) := \frac{1}{(k+\ell)!} \sideset{}{^*}\sum_{\substack{d \mid \mathcal{P}_{\mathcal{H}}(n) \\ d \leqslant R}} \lambda_d := \frac{1}{(k+\ell)!} \sideset{}{^*}\sum_{\substack{d \mid \mathcal{P}_{\mathcal{H}}(n) \\ d \leqslant R}} \mu(d) \left(\log \frac{R}{d} \right)^{k+\ell}, \tag{3.1}$$

where $\sum^*$ means that the summation is extended only for d with $p_1 \nmid d$ if $p_1 > 1 \Leftrightarrow q_1$ exists. Here k will be chosen as a sufficiently large constant, $\ell = \left[\sqrt{k}\right]$. Let further

$$\Psi''_R(k, \ell, n, h) := \sum_{\substack{\mathcal{H}; |\mathcal{H}|=k \\ \mathcal{H} \subset \mathcal{P}^*}} \Lambda_R(n; \mathcal{H}, \ell). \tag{3.2}$$

With these definitions we will consider the weighted sum:

$$S''_{\mathcal{R}}(M, k, \ell, P, \nu) \tag{3.3}$$
$$:= \frac{1}{M(he^\gamma/\varphi(q))^{2k+1}} \sum_{m=M+1}^{2M} \left(\sum_{i \in \mathcal{P}^*} \theta(mP + i) - \nu \log N \right) \Psi''_{\mathcal{R}}(k, \ell, mP, h)^2.$$

The following two propositions supply the basis for our proof, similarly to Propositions 1 and 2 of [7].

In the propositions below we suppose that

$$|\mathcal{H}_1| = |\mathcal{H}_2| = k, \quad |\mathcal{H}_1 \cap \mathcal{H}_2| = r, \quad \mathcal{H} = \mathcal{H}_1 \cup \mathcal{H}_2 \subset \mathcal{P}^*. \tag{3.4}$$

Proposition 1. *If* $R \leqslant M^{1/2} (\log N)^{-(3k+\ell)}$ *then*

$$\sum_{m=M+1}^{2M} \Lambda_R(mP; \mathcal{H}_1, \ell) \Lambda_R(mP; \mathcal{H}_2, \ell) \sim M \binom{2\ell}{\ell} \frac{(\log R)^{r+2\ell}}{(r+2\ell)!} \left(\frac{P_1}{\varphi(P_1)} \right)^{|\mathcal{H}|}. \tag{3.5}$$

Proposition 2. *Let* $h_0 \in \mathcal{P}^*$, $\mathcal{H}^0 = \mathcal{H} \cup \{h_0\}$, $R \leqslant M^{1/4} P^{-3/2} \exp\left(-\sqrt{\log N}\right)$. *Then we have*

$$\sum_{m=M+1}^{2M} \Lambda_R(mP; \mathcal{H}_1, \ell) \Lambda_R(mP; \mathcal{H}_2, \ell) \theta(mP + h_0) \tag{3.6}$$
$$\sim \begin{cases} \binom{2\ell}{\ell} \frac{(\log R)^{r+2\ell}}{(r+2\ell)!} M \left(\frac{P_1}{\varphi(P_1)} \right)^{|\mathcal{H}^0|}, & \text{if } h_0 \notin \mathcal{H} \\ \binom{2\ell+1}{\ell+1} \frac{(\log R)^{r+2\ell+1}}{(r+2\ell+1)!} M \left(\frac{P_1}{\varphi(P_1)} \right)^{|\mathcal{H}^0|}, & \text{if } h_0 \in \mathcal{H},\ h_0 \notin \mathcal{H}_1 \cap \mathcal{H}_2 \\ \binom{2\ell+2}{\ell+1} \frac{(\log R)^{r+2\ell+1}}{(r+2\ell+1)!} M \left(\frac{P_1}{\varphi(P_1)} \right)^{|\mathcal{H}^0|}, & \text{if } h_0 \in \mathcal{H}_1 \cap \mathcal{H}_2. \end{cases}$$

4. Proofs of Propositions 1 and 2

The proofs will follow closely the proofs of the relevant Propositions 1 and 2 in Sections 7–9 of [7], although the results are different. For example, the singular series $\mathfrak{S}(\mathcal{H})$ does not appear at all on the right-hand side of (3.5) and (3.6), contrary to Section 7 of [7]. This is due to the fact that primes below z have no effect on our problem, due to the condition $\mathcal{H} \subset \mathcal{P}^*$, as explained below.

In the proof of Proposition 1 we proceed similarly to Section 7. However, any prime $p \mid P_1$ will be absent in the Euler-product representation of $F(s)$, since by $\mathcal{H} \subset \mathcal{P}^*$ the condition

$$p_1 \nmid d \mid \mathcal{P}_{\mathcal{H}}(mP) \tag{4.1}$$

trivially implies

$$(d, P_1) = 1. \tag{4.2}$$

If we consider any prime $p \nmid P_1$, then

$$p \mid \mathcal{P}_{\mathcal{H}}(mP) \Longleftrightarrow \exists i : p \mid mP + h_i \tag{4.3}$$

allows for $m \in (M, 2M]$ exactly $\nu_p(\mathcal{H})$ residue classes $\bmod p$, and thereby for any d with (4.1)

$$\nu_d(\mathcal{H}) \left(\frac{M}{d} + O(1) \right) \tag{4.4}$$

values for m, exactly as in Section 6 of [7] (here with M, instead of N there).

Accordingly, the change is just in the evaluation of $G(0,0)$. This is different from [7], since the factors for $p \mid P_1$ are missing in the representation of $F(s_1, s_2)$. So we obtain:

$$G(0,0) = \prod_{p \mid P_1} \left(1 - \frac{1}{p}\right)^{-|\mathcal{H}|} \overline{\mathfrak{S}}_z(\mathcal{H}) = \left(1 + O\left(\frac{\log_2 z}{\log z}\right)\right) \left(\frac{P_1}{\varphi(P_1)}\right)^{|\mathcal{H}|}, \tag{4.5}$$

where $\overline{\mathfrak{S}}_z(\mathcal{H})$ is the tail of the singular series,

$$\overline{\mathfrak{S}}_z(\mathcal{H}) := \prod_{p > z, p \neq p_1} \left(1 - \frac{\nu_p(\mathcal{H})}{p}\right) \left(1 - \frac{1}{p}\right)^{-|\mathcal{H}|} = 1 + O\left(\frac{\log_2 z}{\log z}\right). \tag{4.6}$$

In order to see (4.6) we note that for $p > h$ we have $\nu_p(\mathcal{H}) = |\mathcal{H}|$ (while $\nu_p(\mathcal{H}) \leqslant |\mathcal{H}|$ trivially for any p), and thus

$$\begin{aligned} |\log \overline{\mathfrak{S}}_z(\mathcal{H})| &\ll |\mathcal{H}| \sum_{p=z+1}^{h} \frac{1}{p} + |\mathcal{H}|^2 \sum_{p > h} \frac{1}{p^2} \\ &\ll |\mathcal{H}|^2 \log \frac{\log h}{\log z} \ll |\mathcal{H}|^2 \log\left(1 + \frac{\log_2 z}{\log z}\right). \end{aligned} \tag{4.7}$$

In case of Proposition 2 the proof is again analogous to that of Proposition 2 in Section 9 of [7]. The analysis of the error terms in the evaluation of the integral $\mathcal{T}$ is again the same; the differences in the remaining parts are the following.

If $mP + h_0$ is prime, $h_0 \in \mathcal{P}^*$, then for $z < p < R$

$$p \mid P_{\mathcal{H}}(mP) \Longleftrightarrow \exists i : p \mid mP + h_i, \quad h_i \not\equiv h_0 (\bmod p), \tag{4.8}$$

which allows for $m \in (M, 2M]$ exactly $\nu_p^*(\mathcal{H}) = \nu_p(\mathcal{H}^0) - 1$ residue classes $\bmod\, p$ and we have thereby for any d by (4.1)–(4.2), similarly to Section 9 of [7], $\nu_d^*(\mathcal{H})$ suitable residue classes for m modulo d. Thus

$$\sum_{\substack{M<m\leqslant 2M\\ d\mid P_{\mathcal{H}}(mP)}} \theta(mP+h_0) = \nu_d^*(\mathcal{H})\left(\frac{M}{\varphi(d)}\cdot\frac{P}{\varphi(P)} + O\big(\max_{X\leqslant N}\max_{(a,dP)=1}|E(X,dP,a)|\big)\right). \quad (4.9)$$

Now, applying Lemma 2 in place of the usual Bombieri–Vinogradov theorem, the error will be admissible if k and ℓ are bounded. The main term of (4.9) is apart from the factor $\frac{P}{\varphi(P)}$ the same as in [7], and thus the error analysis of [7] is again valid for the evaluation of the crucial integral $\mathcal{T}$. Thus, the only change is in the evaluation of $G(0,0)$ in Cases I–III, as in (4.50–(4.7) above. In the product representation of $F(s)$ the primes $p \mid P_1$ are again absent (cf. Section 9 of [7]):

$$F(s_1,s_2) = \prod_{p\nmid P_1}\left(1 - \frac{\nu_p^*(\mathcal{H}_1^0)}{(p-1)p^{s_1}} - \frac{\nu_p^*(\mathcal{H}_2^0)}{(p-1)p^{s_2}} + \frac{\overline{\nu}_p^*((\mathcal{H}_1\overline{\cap}\mathcal{H}_2)^0)}{(p-1)p^{s_1+s_2}}\right) = \quad (4.10)$$
$$= G(s_1,s_2)\frac{\zeta(1+s_1+s_2)^{|(\mathcal{H}_1\cap\mathcal{H}_2)^0|-1}}{\zeta(1+s_1)^{|\mathcal{H}_1^0|-1}\zeta(1+s_2)^{|\mathcal{H}_2^0|-1}}.$$

On the other hand, for $p \nmid P_1$ we have exactly the same factor in our present function F as in [7], and therefore we have in all Cases I–III, similarly to (4.5)–(4.7) above and Section 9 of [7]

$$G(0,0) = \prod_{p\mid P_1}\left(1-\frac{1}{p}\right)^{-\left[|\mathcal{H}_1^0|+|\mathcal{H}_2^0|-|(\mathcal{H}_1\cap\mathcal{H}_2)^0|-1\right]}\overline{\mathfrak{S}}_z(\mathcal{H}^0) \quad (4.11)$$
$$= \left(1+O\left(\frac{\log_2 z}{\log z}\right)\right)\left(\frac{P_1}{\varphi(P_1)}\right)^{|\mathcal{H}^0|-1}.$$

Multiplying with the extra factor $P/\varphi(P) \sim P_1/\varphi(P_1)$ in (4.9), the new exponent will be $|\mathcal{H}^0|$.

5. Proof of the Theorem

Propositions 1 and 2 immediately imply a weaker form of (1.21), namely

$$\Delta_\nu(q,a) \leqslant e^{-\gamma}(\nu-1), \quad (5.1)$$

which already implies $\Delta_1(q,a) = 0$, for example.

In order to prove this it is enough to consider an arbitrary fixed k-tuple of primes $\mathcal{H}_0 = \{h_i\}_{i=1}^k \subset \mathcal{P}^*$ where according to (5.1) we choose now, differently from (2.2),

$$h := e^{-\gamma}(\nu - 1 + \varepsilon)\varphi(q)\log N, \quad (5.2)$$

We consider further, differently from (3.3) the simpler sum

$$S_R^*(M,\mathcal{H}_0,\ell,P,\nu) := \sum_{m=M+1}^{2M}\Bigg(\sum_{i\in\mathcal{P}^*}\theta(mP+i)-\nu\log N\Bigg)\Lambda_R^2(mP;\mathcal{H}_0,\ell). \quad (5.3)$$

From Propositions 1 and 2 we obtain, similarly to Section 3 of [7],

$$S_R^* \sim \binom{2\ell}{\ell}\frac{M(\log R)^{k+2\ell}}{(k+2\ell)!}\left(\frac{P_1}{\varphi(P_1)}\right)^k \log N \quad (5.4)$$
$$\times\left\{\frac{P_1}{\varphi(P_1)}\frac{\pi(h,q,a)-\pi(z,q,a)-k}{\log N}+\frac{2k(2\ell+1)}{(k+2\ell+1)(\ell+1)}\frac{\log R}{\log N}-\nu\right\}.$$

Choosing $k>k_0(\varepsilon)$, $\ell=\big[\sqrt{k}\big]$, and taking into account (see [2, Ch. 19])

$$\frac{P_1}{\varphi(P_1)}(\pi(h,q,a)-\pi(z,q,a)-k)\sim e^{\gamma}\log z\frac{h}{\varphi(q)\log h}\sim\frac{e^{\gamma}h}{\varphi(q)}, \quad (5.5)$$

we obtain a positive lower bound for S_R^* in (5.4), which proves the existence of at least $\nu+1$ primes among the numbers $mP+i$ $(i\in\mathcal{P}^*)$ for some value $m\in(M,2M]$.

Now we turn to the proof of the stronger relation (1.20), following the method of Section 10 in [7], naturally with appropriate changes, since our weighted sum (3.3) is different from that in Section 10 of [7].

Our task will now be actually easier than in case of [7], since, similarly to the proof of (5.1), we do not need the theorem of Gallagher for the average of the singular series. Instead of it we need just the trivial consequence of the prime number theorem for arithmetic progressions [2, Ch. 19] that for any fixed j we have for $z\to\infty$

$$\sum_{\substack{\mathcal{H};\mathcal{H}\subset\mathcal{P}^*\\|\mathcal{H}|=j}}1=\binom{\pi(h,q,a)-\pi(z,q,a)}{j}\sim\frac{1}{j!}\left(\frac{h}{\varphi(q)\log h}\right)^j\sim\frac{1}{j!}\left(\frac{h}{\varphi(q)\log z}\right)^j. \quad (5.6)$$

Now we can follow the arguments of Section 10 mutatis mutandis, with the following changes:

(i) $\mathfrak{S}(\mathcal{H})$ and $\mathfrak{S}(\mathcal{H}^0)$ are replaced by 1; N by M,

(ii) the factor h^j will be replaced by

$$\left(\frac{P_1}{\varphi(P_1)}\right)^j\left(\frac{h}{\varphi(q)\log z}\right)^j=\left(\frac{(1+o(1))he^{\gamma}}{\varphi(q)}\right)^j \quad (5.7)$$

by $\log z\sim\log h$, and accordingly, the definition of x,

(iii) $x=\dfrac{\log R}{h}$ will be replaced by $x=\dfrac{\varphi(q)\log R}{he^{\gamma}}$.

In this way we obtain now exactly the same bound $(\sqrt{\nu}-1)^2\log N$ for $he^{\gamma}/\varphi(q)$ as in [7] for h, and this proves our theorem.

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