A sum over the zeros of partial sums of e^x

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1. Introduction

In this paper we prove the following.

Theorem. *Let $\nu_j (j = 1, 2, \ldots, k)$ be a zero of the k-th partial sum of e^x:*

$$P_k(x) = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^k}{k!} \qquad \text{and} \qquad P_k(\nu_j) = 0 .$$

We have

$$\sum_{j=1}^{k} \frac{e^{-2\nu_j}}{\nu_j^2} = \begin{cases} 2k + 4 + \frac{2}{k} + O(k^{-\frac{3}{2}}), & \text{if } k \text{ is odd} , \\ 2k + 4 - \frac{6}{k} + O(k^{-\frac{3}{2}}), & \text{if } k \text{ is even.} \end{cases}$$

The motivation for studying this particular sum arises from its appearance in the expression obtained for certain mean values of the Riemann zeta-function ([12]).

In 1988 Conrey and Ghosh [2] showed that

$$\sum_{j=1}^{k} e^{-\nu_j} = k + 1 + O(e^{-\beta k}) \tag{1}$$

where β is any positive number less than $1 - \log 2$. Conrey and Ghosh remarked that the error term in (1) is an exponentially small sum of terms most of which are exponentially large as functions of k. Our method provides an improvement on (1):

$$\sum_{j=1}^{k} e^{-\nu_j} = k + 1 + O(k^2 e^{-k}). \tag{2}$$

By the Eneström-Kakeya theorem all zeros of a polynomial $a_0 + a_1 x + \cdots + a_k x^k$ with real and positive coefficients are of absolute value at most $\max(\frac{a_0}{a_1}, \frac{a_1}{a_2}, \ldots, \frac{a_{k-1}}{a_k})$. Hence $|\nu_j| \leq k$. In fact Szegö [8], who started the study of zeros of partial sums, proved that the numbers $\frac{\nu_j}{k+1}$ cluster around the simple closed curve $\Gamma = \{z : |ze^{1-z}| = 1, |z| \leq 1\}$ as $k \to \infty$ (see also Dieudonné [3]).

For brevity we put $n = k+1$. Another way of expressing $P_k(x) = 0$ is, writing $x = nz$,

$$\frac{n!e^{nz}}{(nz)^n} = 1 + \frac{z}{1+\frac{1}{n}} + \frac{z^2}{(1+\frac{1}{n})(1+\frac{2}{n})} + \cdots + \frac{z^p}{(1+\frac{1}{n})(1+\frac{2}{n})\cdots(1+\frac{p}{n})} + \cdots =: f_n(z)\,. \quad (3)$$

Let

$$a_p = a_p(n) = \frac{1}{(1+\frac{1}{n})(1+\frac{2}{n})\cdots(1+\frac{p}{n})} \quad (p = 1, 2, \ldots), \quad a_0 = 1 \quad (4)$$

and note that in (3) we have defined the entire function $f_n(z)$ by its power series representation

$$f_n(z) = \sum_{p=0}^{\infty} a_p z^p\,. \quad (5)$$

2. Properties of the functions $f_n(z)$

In this section we present some facts about $f_n(z)$. We refer the reader to the original sources for those results existing in the literature.

As a direct consequence of the definition of $f_n(z)$ we have

Lemma 1. *For each positive integer n and for all $z \in \mathbb{C}$*

$$f_n(z)(1-z) = 1 - \frac{zf_n'(z)}{n}\,.$$

Buckholtz [1] showed that the zeros of $P_{n-1}(nz)$, $\frac{\nu_j}{n}$, lie strictly outside the Szegö curve Γ and are within a distance of $\frac{2e}{\sqrt{n}}$ from Γ. The next lemma is obtained following the remarks in [1].

Lemma 2. *The zeros of $f_n(z)$ satisfy $|z| > 1$ and $|ze^{1-z}| > 1$. Given $\epsilon > 0$, arbitrarily small but fixed. For every sufficiently large $n \geq N(\epsilon)$ the zeros of $f_n(z)$ are within a distance $\frac{\sqrt{8\pi}+\epsilon}{\sqrt{n}}$ of the curve*
$\Gamma_1 = \{z : |ze^{1-z}| = 1, |z| \geq 1\}$.

The point $z = 1$ is of special interest in studying $f_n(z)$ (cf. [1]).

Lemma 3. *For each positive integer n*

$$f_n(1) = \frac{n!}{2}\left(\frac{e}{n}\right)^n + \frac{1}{3} + \frac{4}{135n} + O\left(\frac{1}{n^2}\right)$$

$$f_n'(1) \;=\; n$$

The value of $f_n(1)$ was asserted in a slightly different and stronger form by Ramanujan [7] and the result presented here is contained in the works of Szegö [9] and Watson [10]. Note that by Stirling's formula,

$$n!(\frac{e}{n})^n \frac{1}{\sqrt{2\pi n}} \;=\; 1 + \frac{1}{12n} + \frac{1}{288n^2} - \frac{139}{51840n^3} + \cdots , \tag{6}$$

we see that

$$f_n(1) \;=\; \sqrt{\frac{\pi}{2}n} + O(1) .$$

Let $r > 0$ be an arbitrarily small fixed number and D_r' be the set obtained by removing from $\overline{D}(0,1)$ (the closed unit disk) the points of distance less than r to 1.

Lemma 4. *For $z \in D_r'$*

$$\begin{aligned} f_n(z) &= \frac{1}{1-z}(1+\lambda_n(z)) \\ f_n'(z) &= \frac{1}{(1-z)^2}(1+\mu_n(z)) \end{aligned}$$

where $\lambda_n(z)$ and $\mu_n(z)$ tend uniformly to 0 in D_r' as $n \to \infty$.

Dieudonné [3] proved the first equality of Lemma 4 by showing that $|\frac{z}{n}f_n'(z)| \to 0$ as $n \to \infty$ uniformly for $z \in D_r'$. The second part can be proved similarly ([11]).

We find that the following way of expressing $f_n(z)$ is more suitable for determining the value of $f_n(z)$. The approach we take is contained in Newman and Rivlin [6].

Lemma 5. *Let $z \neq 0$ and write $z = 1 + \frac{s}{\sqrt{n}}$. We have*

$$f_n(z) \;=\; 1 + \frac{e^{s\sqrt{n}}}{(1+\frac{s}{\sqrt{n}})^n}\Big[\Big(\sqrt{\frac{\pi n}{2}} - \frac{2}{3} + \frac{1}{12}\sqrt{\frac{\pi}{2n}} + \frac{4}{135n}\cdots\Big) + \sqrt{n}\int_0^s (1+\frac{\zeta}{\sqrt{n}})^n e^{-\zeta\sqrt{n}} d\zeta\Big] .$$

Proof. From (3) for $z \neq 0$ we have

$$\begin{aligned} f_n(z) &= \frac{n!}{(nz)^n}\Big(e^{nz} - \sum_{p=0}^{n-1}\frac{(nz)^p}{p!}\Big) \\ &= 1 + \frac{n!}{(nz)^n}\Big(e^{nz} - \frac{1}{n!}\int_0^\infty e^{-t}(t+nz)^n dt\Big) . \end{aligned}$$

Putting $t = \sqrt{n}(\zeta - s)$ we obtain

$$f_n(z) \; = \; 1 + \frac{e^{s\sqrt{n}}}{(1+\frac{s}{\sqrt{n}})^n}\Big(\frac{n!e^n}{n^n} - \sqrt{n}\int_s^\infty (1+\frac{\zeta}{\sqrt{n}})^n e^{-\zeta\sqrt{n}} d\zeta\Big) \tag{7}$$

where the path of integration is the horizontal line from s to the right to ∞. In (7) we take $z = 1$. Using Lemma 3 and Stirling's formula, (6), we have

$$\begin{aligned}\int_0^\infty (1+\frac{\zeta}{\sqrt{n}})^n e^{-\zeta\sqrt{n}} d\zeta \; &= \; \frac{1}{\sqrt{n}}\Big(\frac{n!}{2}(\frac{e}{n})^n + \frac{2}{3} - \frac{4}{135n}\cdots\Big) \\ &= \; \sqrt{\frac{\pi}{2}} + \frac{2}{3\sqrt{n}} + \sqrt{\frac{\pi}{2}}\frac{1}{12n} - \frac{4}{135n^{\frac{3}{2}}}\cdots \end{aligned} \tag{8}$$

and to complete the proof (8) is substituted into (7).

From Lemma 4 we deduce that for any fixed $z \in \overline{D}(0,1)\backslash\{1\}$ $\lim_{n\to\infty} \frac{f_n'}{f_n^2}(z) = 1$ while by Lemma 3 we know that $\lim_{n\to\infty} \frac{f_n'}{f_n^2}(1) = \frac{2}{\pi}$. The discontinuity of $\lim_{n\to\infty} \frac{f_n'}{f_n^2}$ is due to the fact that $f_n(z)$ has (at least) a pair of zeros (complex conjugates of each other) which tend to 1 as $n \to \infty$.

We examine the behaviour of $f_n(z)$ when $s = \sqrt{n}(z-1)$ is in a fixed compact set S. By taking logarithms it is seen that

$$\frac{e^{s\sqrt{n}}}{(1+\frac{s}{\sqrt{n}})^n} \; = \; e^{\frac{s^2}{2} - \frac{s^3}{3n^{\frac{1}{2}}} + \frac{s^4}{4n} + \frac{s^5}{5n^{\frac{3}{2}}} + \cdots} \; . \tag{9}$$

Hence as $n \to \infty$ $\frac{e^{s\sqrt{n}}}{(1+\frac{s}{\sqrt{n}})^n}$ converges uniformly to $e^{\frac{s^2}{2}}$ in S. Also by the dominated convergence theorem $\int_s^\infty (1+\frac{\zeta}{\sqrt{n}})^n e^{-\zeta\sqrt{n}} d\zeta$ converges uniformly to $\int_s^\infty e^{\frac{-\zeta^2}{2}} d\zeta$ in S (see [6]). Thus, as $n \to \infty$, from (7) we have

$$\frac{f_n(z)}{\sqrt{n}} \longrightarrow \sqrt{\frac{\pi}{2}}\, w(\frac{s}{\sqrt{2}\,i}) \tag{10}$$

uniformly for all s in a fixed compact set where

$$w(z) \; = \; e^{-z^2}(1 + \frac{2i}{\sqrt{\pi}} \int_0^z e^{t^2} dt) \; . \tag{11}$$

If $f_n(1+\frac{s_n}{\sqrt{n}}) = 0$ then as $n \to \infty$ (s_n) tends to a limit s_L which satisfies

$$1 - \frac{1}{\sqrt{2\pi}} \int_{s_L}^\infty e^{\frac{-\zeta^2}{2}} d\zeta \; = \; 0 \; .$$

It has been determined in Fettis, Caslin and Cramer [5] that the zero of $erfc(z) := \frac{2}{\sqrt{\pi}}\int_z^\infty e^{-t^2}\,dt$ which is in the upper-half plane and closest to the origin is approximately at $-1.35 \pm i1.99$. Thus $f_n(z)$ has zeros at

$$1 + \frac{\sqrt{2}(1.35 \pm i1.99) + o(1)}{\sqrt{n}}$$

and for s in a fixed compact set S and $1 + \frac{s}{\sqrt{n}} \in \overline{D}(0,1)$ the function $w(z)$ defined by (11) is nonvanishing. Hence for sufficiently large n

$$\left|\frac{f_n'}{f_n^2}(z)\right| \;<\; M_S \tag{12}$$

where M_S is some constant which may depend only on the compact set S. The values of $f_n(z)$ and $\frac{f_n'}{f_n^2}(z)$ may be computed from the tabulated values of $w(z)$ (see [4]).

Lemma 6. *There is a constant M such that for all $n \geq 1$ and $|z| \leq 1$*

$$\left|\frac{f_n'}{f_n^2}(z)\right| \;<\; M\,.$$

Proof. We take C to be a large fixed positive number. By Lemma 4 given $\epsilon > 0$ there exists $N = N(\epsilon, C)$ such that for all $n \geq N$ and $z \in D'_{\frac{1}{C}}$

$$1 - \epsilon \;<\; \left|\frac{f_n'}{f_n^2}(z)\right| \;<\; 1 + \epsilon\,. \tag{13}$$

We put

$$M_1 \;=\; \max_{\substack{1 \leq n < N \\ |z| \leq 1}} \left|\frac{f_n'}{f_n^2}(z)\right| \tag{14}$$

and from now on we shall suppose n is large and $|z-1| < \frac{1}{C}$. By the maximum modulus theorem we may restrict our attention to those z that are on the unit circle. Since (12) gives a bound M_0 when $0 \leq r \leq r_0$, where $r := |s|$ and r_0 is fixed but may be arbitrarily large, from now on we suppose that $r_0 \leq r \leq \frac{\sqrt{n}}{C}$. For z on the upper unit semicircle we have

$$s \;=\; -\frac{r^2}{2\sqrt{n}} + ir\sqrt{1 - \frac{r^2}{4n}} \;=\; -\frac{r^2}{2\sqrt{n}} + irq(r)\,, \qquad \text{say.} \tag{15}$$

Using this in (9) we see that for $|z| = 1$

$$\begin{aligned}
\frac{e^{s\sqrt{n}}}{(1+\frac{s}{\sqrt{n}})^n} &= e^{-\frac{r^2}{2}-iq(r)\left(\frac{r^3}{6n^{\frac{1}{2}}}+\frac{r^5}{30n^{\frac{3}{2}}}+\frac{r^7}{140n^{\frac{5}{2}}}+\frac{r^9}{630n^{\frac{7}{2}}}+\cdots\right)} \\
&= e^{-\frac{r^2}{2}-i\left(\frac{r^3}{6n^{\frac{1}{2}}}+\frac{r^5}{80n^{\frac{3}{2}}}+\frac{3r^7}{896n^{\frac{5}{2}}}+\cdots\right)} \\
&= e^{-\frac{r^2}{2}-i\psi_n(r)}\ \text{, say.} \qquad (16)
\end{aligned}$$

In Lemma 5 it will be understood that the integral $\int_{\zeta=0}^{\zeta=s}$ is taken on the path such that $1+\frac{\zeta}{\sqrt{n}}$ goes from 1 to z following the unit circle. Call $|\zeta| = \rho$. Differentiating (15) yields

$$\begin{aligned}
d\zeta &= \left(i\sqrt{1-\frac{\rho^2}{4n}}-\frac{\rho}{\sqrt{n}}-\frac{i\rho^2}{4n\sqrt{1-\frac{\rho^2}{4n}}}\right)d\rho \\
&= \left(i-\frac{\rho}{\sqrt{n}}-\frac{3i\rho^2}{8n}-\frac{5i\rho^4}{32n^2}\cdots\right)d\rho \\
&= \left(i-\frac{\rho}{\sqrt{n}}-i\sum_{m=1}^{\infty} h_m\left(\frac{\rho^2}{n}\right)^m\right)d\rho\ , \qquad \text{say}\ ,
\end{aligned}$$

and by Lemma 5 we have

$$\frac{f_n(z)-1}{\sqrt{n}} = \int_0^r e^{\frac{\rho^2-r^2}{2}+i(\psi_n(\rho)-\psi_n(r))}\left(i-\frac{\rho}{\sqrt{n}}-i\sum_{m=1}^{\infty} h_m\left(\frac{\rho^2}{n}\right)^m\right)d\rho + O(e^{-\frac{r^2}{2}})\ . \qquad (17)$$

It suffices to consider the main term on the right-hand side of (17) which is

$$i\int_0^r e^{\frac{\rho^2-r^2}{2}+i(\psi_n(\rho)-\psi_n(r))}d\rho\ . \qquad (18)$$

In (18) the contribution of ρ far from r is small. Let $\alpha > 3$ be fixed. We have

$$\int_0^{r-\frac{\alpha\log r}{r}} e^{\frac{\rho^2-r^2}{2}}d\rho \ll r^{1-\alpha}\ .$$

For $\int_{r-\frac{\alpha\log r}{r}}^{r}$ we make the substitution

$$v = r(r-\rho)$$

which transforms (18) into

$$\frac{i}{r}\int_0^{\alpha\log r} e^{-v+\frac{v^2}{2r^2}-ig(v,r)}dv$$

where $g(v,r) = \psi_n(r - \frac{v}{r}) - \psi_n(r)$. Integrating by parts we get

$$\frac{i}{r} + O(\frac{1}{\sqrt{n}}) + O(\frac{1}{r^2})$$

The contribution from the terms other than the main term on the right-hand side of (17) may be absorbed into the last error terms. As C can be chosen arbitrarily large for $r_0 \leq r \leq \frac{\sqrt{n}}{C}$ we have

$$f_n(z) \sim \frac{i\sqrt{n}}{r} . \tag{19}$$

Differentiating both sides of (17) with respect to r and proceeding as for $f_n(z)$ we find that

$$f_n'(z) \sim -\frac{n}{r^2} . \tag{20}$$

Thus there exists an absolute constant M_2 such that

$$\left|\frac{f_n'}{f_n^2}(z)\right| < M_2 \qquad (r_0 \leq r \leq \frac{\sqrt{n}}{C}, \ n > N) . \tag{21}$$

To complete the proof of Lemma 6 we take

$$M = \max(1+\varepsilon, M_0, M_1, M_2)$$

(cf. Equations (12-14, 21).)

Lemma 7. There is an absolute constant K such that for all $n \geq 1$ and $|z| \leq 1$

$$\left|\frac{1}{(f_n(z))^2} - (1-z)^2\right| < \frac{K}{n}.$$

Proof. From Lemma 1 we have

$$\frac{1}{(f_n(z))^2} - (1-z)^2 = \frac{z}{n}\frac{f_n'}{f_n^2}(z)(2 - \frac{z}{n}f_n'(z))$$

for $|z| \leq 1$. The result follows readily by Lemma 6.

3. Proof of the theorem

Taking squares in (3) gives

$$\frac{e^{-2\nu_j}}{\nu_j^2} = \frac{(n!)^2}{\nu_j^{2n+2}(f_n(\frac{\nu_j}{n}))^2} \tag{22}$$

where $\frac{\nu_j}{n} \in D(0;1)$ (the open unit disk). By Lemma 2 the power series (5) can be inverted in $|z| \le 1$ and we write

$$\frac{1}{(f_n(z))^2} = \sum_{p=0}^{\infty} b_p(n) z^p \, . \tag{23}$$

The straightforward computation of the coefficients $b_p = b_p(n)$ doesn't seem to obey an easy pattern.

Let the sum of the r-th power of the roots of $P_k(x)$ be denoted as

$$s_r = s_r(k) = \sum_{j=1}^{k} \nu_j^r \, . \tag{24}$$

Inserting (23) in (22) we have

$$\sum_{j=1}^{k} \frac{e^{-2\nu_j}}{\nu_j^2} = ((k+1)!)^2 \Big(s_{-(2k+4)} + \sum_{p=1}^{\infty} \frac{b_p s_{p-(2k+4)}}{(k+1)^p} \Big) \, . \tag{25}$$

The quantities s_r, can be calculated in terms of the coefficients of $P_k(x)$. The relevant s_r have the following values.

$$\begin{aligned}
s_{-1} &= -1, \qquad s_{-2} = s_{-3} = \cdots = s_{-k} = 0 \, , \\
s_{-(k+1)} &= \frac{1}{k!}, \quad s_{-(k+2)} = -\frac{1}{k!}, \quad s_{-(k+3)} = \frac{1}{2!k!}, \quad s_{-(k+4)} = \frac{-1}{3!k!}, \cdots \\
s_{-(2k-1)} &= \frac{(-1)^{k-2}}{(k-2)!k!}, \quad s_{-2k} = \frac{(-1)^{k-1}}{(k-1)!k!}, \quad s_{-(2k+1)} = \frac{(-1)^k}{(k!)^2}, \\
s_{-(2k+2)} &= \frac{(-1)^{k+1}+1}{k!(k+1)!}, \quad s_{-(2k+3)} = \frac{(-1)^{k+2}+1}{k!(k+2)!} + \frac{-2}{k!(k+1)!} \, , \\
s_{-(2k+4)} &= \frac{(-1)^{k+3}+1}{k!(k+3)!} + \frac{-2}{k!(k+2)!} + \frac{2}{k!(k+1)!} \, .
\end{aligned} \tag{26}$$

Using (26) in (25) we obtain

$$\sum_{j=1}^{k} \frac{e^{-2\nu_j}}{\nu_j^2} = \begin{cases} 2k+4+\frac{2}{k}+\sum_{p=3}^{\infty} b_p c_p + O(\frac{1}{k^2}), & \text{if } k \text{ is odd,} \\ 2k+4-\frac{6}{k}+\sum_{p=3}^{\infty} b_p c_p + O(\frac{1}{k^2}), & \text{if } k \text{ is even,} \end{cases} \tag{27}$$

where

$$c_p = \frac{((k+1)!)^2 s_{p-(2k+4)}}{(k+1)^p} \qquad (p = 3, 4, \ldots) \, .$$

From (26)

$$c_p = \begin{cases} \frac{(-1)^{k+3-p}(k+1)!}{(k+1)^{p-1}(k+3-p)!} & \text{for} \quad 3 \le p \le k+3 \ , \\ 0 & \text{for} \quad k+4 \le p \le 2k+2 \end{cases} \tag{28}$$

and trivially

$$|c_p| \le \frac{((k+1)!)^2 k^{p-(2k+3)}}{(k+1)^p} \qquad \text{for} \qquad p \ge 2k+3 \ . \tag{29}$$

Now by (23) and Lemma 7

$$\Big|\sum_{p=3}^{\infty} b_p z^p\Big| = \Big| \frac{1}{(f_n(z))^2} - (1-z)^2 + O(\frac{1}{n})\Big| < \frac{A}{n} \tag{30}$$

for $|z| \le 1$ where A is an absolute constant. Hence by Cauchy's estimates on the unit disk we deduce that

$$|b_p| = |b_p(n)| < \frac{A}{n} \qquad (p \ge 3) \ . \tag{31}$$

So from (28), (29) and (31) we obtain

$$\begin{aligned} \sum_{p=3}^{\infty} b_p c_p &\ll \frac{1}{n}\sum_{p=3}^{k+3} |c_p| + \frac{1}{n}\sum_{p=2k+3}^{\infty} \frac{((k+1)!)^2 k^{p-(2k+3)}}{(k+1)^p} \\ &= \frac{n!}{n^{n+2}}\sum_{r=0}^{n-1} \frac{n^r}{r!} + O(e^{-2n}) \\ &\ll n^{-\frac{3}{2}}. \end{aligned} \tag{32}$$

Substituting (32) in (27) the proof of the theorem is completed.

Acknowledgements

I thank Professor J.B. Friedlander, my thesis advisor, for his encouragement and many helpful suggestions while I worked on this problem at the University of Toronto. I also thank the referee for correcting an error in the analysis of Eq. (32) thereby leading to a change in the final result.

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