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THE MEAN VALUE OF $|\zeta(\frac{1}{2}+it)|^2$ AT THE ZEROS OF $Z^{(k)}(t)$

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Introduction. The functional equation of the Riemann zeta-function may be expressed in the asymmetric form

$$(1) \qquad \zeta(1-s) \ = \ \chi(1-s)\zeta(s) \ ; \qquad \chi(1-s) \ = \ \pi^{\frac{1}{2}-s}\,\frac{\Gamma(\frac{s}{2})}{\Gamma(\frac{1-s}{2})}, \qquad s \in \mathrm{C} \ .$$

The real function $Z(t)$ is defined as

$$(2) \qquad Z(t) \ = \ \{\chi(\frac{1}{2}+it)\}^{-\frac{1}{2}}\zeta(\frac{1}{2}+it)$$

and we put

$$(3) \qquad Z_k(s) \ = \ \big(\chi(s)\big)^{\frac{1}{2}}\frac{d^k}{ds^k}\ \big((\chi(s))^{-\frac{1}{2}}\zeta(s)\big)$$

so that $|Z_k(\frac{1}{2}+it)| = |Z^{(k)}(t)|$.

In [1] Conrey and Ghosh proved on the Riemann Hypothesis (RH) that $\sum_{0<\gamma_1\le T}|\zeta(\frac{1}{2}+i\gamma_1)|^2 \sim \frac{e^2-5}{2}\frac{TL^2}{2\pi}$. Here $\frac{1}{2}+i\gamma_1$ runs through the zeros of $Z_1(s)$ ($|\zeta(\frac{1}{2}+i\gamma_1)|$ is a maximum on $\mathrm{Re}\, s=\frac{1}{2}$) and $L=\log\frac{T}{2\pi}$. They also stated on RH $\sum_{0<\gamma_1\le T}|\zeta(\frac{1}{2}+i\gamma_1+i\frac{2\pi\alpha}{L})|^2 \sim C(\alpha)\frac{TL^2}{2\pi}$ giving $C(\alpha)$ explicitly and deduced that the gaps between the maxima of $\zeta(s)$ can be 1.4 times the expected average. In this paper we prove the following extension of this result.

Theorem. *Assume RH and let k be a fixed natural number. Let γ_k run through the zeros of $Z^{(k)}(t)$ (i.e. $Z_k(\frac{1}{2}+i\gamma_k)=0$). Then as $T\to\infty$:*

$$\sum_{0<\gamma_k<T}\big|\zeta(\frac{1}{2}+i\gamma_k+i\frac{2\pi\alpha}{L})\big|^2 = \frac{TL^2}{2\pi}\{1+2(k+1)\frac{\cos 2\pi\alpha-1}{(2\pi\alpha)^2}$$

$$+\,2\,\mathrm{Re}\sum_{j=1}^{k}\frac{e^{u_j}-u_j-1}{u_j^2}\}+O_k(TL)$$

$$\sim \frac{TL^2}{2\pi}c_k(\alpha)\ , \quad \textit{say},$$

where $u_j = -2\nu_j + 2\pi\alpha i$ *and* ν_j *is a zero of the* k*-th partial sum for the exponential function, i.e.:* $1+\nu_j+\frac{\nu_j^2}{2!}+\cdots+\frac{\nu_j^k}{k!}=0 \qquad (j=1,2,\ldots,k)$.

Assuming RH, $Z(t)$ has asymptotically $\frac{T}{2\pi}\log T$ zeros in $[0,T]$ as $T\to\infty$. By repeated application of Rolle's theorem the same is true of $Z^{(k)}(t)$. Comparison with the classical result $\int_0^T |\zeta(\frac12+it)|^2 dt \sim T\log T$ (see [4], Ch. 7) makes it clear that the average of $|\zeta(\frac12+i\gamma_k)|^2$ over the zeros, $\gamma_k\in[0,T]$, of $Z^k(t)$ is $c_k(0)$ times the average of $|\zeta(\frac12+it)|^2$ over all $t\in[0,T]$. The sum over the zeros of partial sums of e^x has been evaluated in [6] as

$$\text{(4)}\qquad \sum_{j=1}^{k}\frac{e^{-2\nu_j}+2\nu_j-1}{\nu_j{}^2}=\begin{cases}2k+2+\frac{2}{k}+O\big(\frac{\log k}{k^2}\big) & (k \text{ odd and } k>1)\\ 2k+2-\frac{6}{k}+O\big(\frac{\log k}{k^2}\big) & (k \text{ even})\,.\end{cases}$$

Using (4) in the Theorem we have

Corollary. $$\sum_{0<\gamma_k<T}|\,\zeta(\tfrac12+i\gamma_k)\,|^2\sim\begin{cases}\frac{TL^2}{2\pi}\big(1+\frac{1}{k}+O\big(\frac{\log k}{k^2}\big)\big) & (k \text{ odd and } k>1)\\ \frac{TL^2}{2\pi}\big(1-\frac{3}{k}+O\big(\frac{\log k}{k^2}\big)\big) & (k \text{ even})\end{cases}$$

From the Theorem we may compute values for β_k such that for all sufficiently large T there exists γ_k and γ_k^+, consecutive zeros of $Z^{(k)}(t)$ with $T<\gamma_k<\gamma_k^+\le 2T$ and $\gamma_k^+-\gamma_k>\beta_k\frac{2\pi}{L}$. The computation is based on determining β_k such that $\int_{-\beta_k/2}^{\beta_k/2} c_k(\alpha)d\alpha=1$.

A table of the values of $c_k(0)$ and β_k for $1\le k\le 28$ has been given in [5]. As k increases β_k decreases monotonically. Some sample values are $\beta_1=1.4$, $\beta_2=1.295$, $\beta_3=1.224$, $\beta_{28}=1.035$.

Preliminaries. In this section we present the lemmas that will be used in the proof. The constants implied by $\ll$ or O symbols may, in general, depend on k but we suppress this in our notation as k is a fixed integer and $T\to\infty$.

Lemma 1. *The function* $\chi(s)$ *defined in (1) satisfies in* $|\mathrm{Im}\ s|=|t|\ge 1$
$\frac{\chi'}{\chi}(s)=-\log\frac{|t|}{2\pi}+O\Big(\frac{1}{1+|t|}\Big)$ *and* $\Big(\frac{\chi'}{\chi}(s)\Big)^{(k)}\ll\frac{1}{1+|t|} \quad (k\ge 1)$.

For the proof see [3].

By differentiating (1) k times the functional equation of $Z_k(s)$ is obtained:

Lemma 2. $Z_k(s) = (-1)^k \chi(s) Z_k(1-s)$ *for all* $s \in \mathrm{C}$.

Lemma 3. *Assuming RH,* $Z_k(s)$ *has at most* $O(\log T)$ *zeros with ordinates in [0,T] off the critical line.*

Lemma 3 is shown in the same way as in the case $k=1$ (see [1] or [5]).

Lemma 4. *Assuming R.H., the zeros of* $Z_k(s)$ *which are not on* $\sigma = \frac{1}{2}$ *are within a distance* $\frac{1}{9}$ *from* $\sigma = \frac{1}{2}$.

Proof. On R.H. $\frac{\zeta^{(j)}}{\zeta}(s) \ll (\log t)^{j+1-2\sigma}$ uniformly for $\frac{1}{2} \le \sigma_0 < \sigma \le \sigma_1 < 1$ and $t \ge 2$. The case $j=1$ is Theorem 14.5 of Titchmarsh [4], and applying Cauchy's theorem in a disk of radius $(\log \frac{t}{2\pi})^{-1}$ around s the bound for $\frac{\zeta^{(j)}}{\zeta}(s)$ for $j>1$ is obtained. Thus $\frac{Z_k}{\zeta}(s) \ll (\log t)^{k+1-2\sigma} + (\log t)^k$ (cf. Eq. (3)) and the assertion follows immediately.

Lemma 5. *Let* $\mathcal{Z}_k(s,T) = \left(\frac{L}{2} + \frac{d}{ds}\right)^k \zeta(s)$, *where* $L = \log \frac{T}{2\pi}$. *Assuming R.H.,*

$$\frac{Z_k'}{Z_k}(s) - \frac{\mathcal{Z}_k'}{\mathcal{Z}_k}(s,T) \ll \frac{U}{T} .$$

for $\sigma \ge \frac{5}{8}$ *and* $T \le t \le T+U \le 2T$.

Proof. In [1] it was proved upon R.H. that $\frac{Z_1'}{Z_1}(s) - \frac{\mathcal{Z}_1'}{\mathcal{Z}_1}(s,T) \ll \frac{U}{T}$ and the result can be proved by induction on k (see [5]).

Lemma 6. *Assume R.H. At* $s=1$ $\mathcal{Z}_k(s,T)$ *has a pole of order* $k+1$. *There are* k *zeros of* $\mathcal{Z}_k(s,T)$ *located at* $z_j = 1 - \frac{2}{L}\nu_j + O_k\left(\frac{1}{L^2}\right)$ $(j=1,\ldots,k)$, *where* ν_j *'s are the roots of* $\sum_{r=0}^{k} \frac{\nu^r}{r!} = 0$. *There are no other zeros or poles of* $\mathcal{Z}_k(s,T)$ *with* $\frac{5}{8} \le \sigma \le 2$. *Thus we have*

$$\frac{\mathcal{Z}_k'}{\mathcal{Z}_k}(s-i\delta) = \frac{-(k+1)}{s-i\delta-1} + \sum_{j=1}^{k} \frac{1}{s-i\delta-z_j} + W(s,T)$$

where $W(s,T)$ *is regular for* $\frac{5}{8} \le \sigma \le \frac{9}{8}$.

Proof. The statement about the poles follows from the facts that $\zeta^{(k)}(s)$ has a pole of order $k+1$ at $s=1$ and $\zeta(s)$ and its derivatives don't have any other poles in $\sigma \geq \frac{5}{8}$. Since we assume R.H. we may consider $W_k(s) =: \left(\frac{2}{L}\right)^k \frac{\mathcal{Z}_k(s,T)}{\zeta(s)} = \sum_{j=0}^{k} \binom{k}{j} \frac{\zeta^{(j)}}{\zeta}(s) \left(\frac{2}{L}\right)^j$.

The change in the argument of $W_k(s)$ along the rectangle with vertices $2 \pm iT$ and $\frac{5}{8} \pm iT$ is 0 because, for $1 \leq j \leq k$, $\frac{1}{L^j} \frac{\zeta^{(j)}}{\zeta}(s) = o(L)$ on this rectangle. By the argument principle the number of zeros of $W_k(s)$ is equal to the number of its poles counted according to multiplicity and the latter is k.

Let now $\eta = 1 - \frac{2}{L}\mu$. Since $\frac{\zeta^{(j)}}{\zeta}(\eta) = \frac{(-1)^j j!}{(\eta-1)^j} + \sum_{n=-(j-1)}^{\infty} C_{jn}(\eta-1)^n$, where C_{jn} are constants, we have

$$W_k(\eta) = \sum_{j=0}^{k} \binom{k}{j} \mu^{-j} j! + \sum_{j=0}^{k} \binom{k}{j} \left(\frac{2}{L}\right)^j \sum_{n=-(j-1)}^{\infty} C_{jn} \left(\frac{-2}{L}\mu\right)^n$$

so that if $W_k(\eta) = 0$ then $\mu = \nu + O(\frac{1}{L})$ where $\sum_{r=0}^{k} \frac{\nu^r}{r!} = 0$. Hence we conclude that $\mathcal{Z}_k(s,T)$ has k zeros located at $z_j = 1 - \frac{2}{L}\nu_j + O\left(\frac{1}{L^2}\right)$; $j = 1, \ldots, k$.

Lemma 7. *For $\sigma \geq \frac{9}{8}$, there is an absolutely convergent Dirichlet series such that*

$$\frac{\mathcal{Z}_k'}{\mathcal{Z}_k}(s,T) = \sum_{m=1}^{\infty} \frac{a_k(m)}{m^s} + O(T^{-1})$$

where, as $T \to \infty$, for any $\epsilon > 0$ $a_k(m) = a_k(m,L) \ll_\epsilon T^\epsilon$ for $m \ll T$.

This result has been proved in [2].

We quote the last lemma from Gonek [3]:

Lemma 8. *Let $\{b_n\}_{n=1}^{\infty}$ be a sequence of complex numbers such that for any $\varepsilon > 0$, $b_n \ll n^\varepsilon$. Let $\sigma > 1$ and let j be a non-negative integer. Then for sufficiently large T*

$$\frac{1}{2\pi} \int_1^T \Big(\sum_{m=1}^{\infty} b_n n^{-\sigma-it}\Big) \chi(1-\sigma-it) \big(\log \frac{t}{2\pi}\big)^j dt = \sum_{1 \leq n \leq \frac{T}{2\pi}} b_n (\log n)^j + O\big(T^{\sigma-\frac{1}{2}} (\log T)^j\big).$$

Proof of the Theorem. As the real part of the zeros of $Z_k(s)$ are in $(\frac{1}{2}-\frac{1}{9},\frac{1}{2}+\frac{1}{9})$ (Lemma 4) the residue theorem allows us to write

$$\sum_{\substack{\rho_k=\beta_k+i\gamma_k\\ T<\gamma_k<T+U}} \zeta(\rho_k+i\delta)\zeta(1-\rho_k-i\delta) \;=\; \frac{1}{2\pi i}\int_R \frac{Z_k'}{Z_k}(s)\zeta(s+i\delta)\zeta(1-s-i\delta)ds\ ,$$

where we take R to be the rectangle with vertices $\frac{5}{8}+iT$, $\frac{5}{8}+i(T+U)$, $\frac{3}{8}+i(T+U)$, $\frac{3}{8}+iT$, described in the positive sense.

On R.H., by Lemmas 3 and 4, only $O(L)$ of the zeros of $Z_k(s)$ are off the critical line and for such zeros ρ_k to the right of $\sigma=\frac{1}{2}$ $|\zeta(\rho_k)|^2 \ll T^{\frac{1}{4}}$. Thus

$$(5)\qquad \sum_{\substack{\rho_k=\frac{1}{2}+i\gamma_k\\ T<\gamma_k<T+U}} \left|\zeta(\frac{1}{2}+i\gamma_k+i\delta)\right|^2 \;=\; \frac{1}{2\pi i}\int_R \frac{Z_k'}{Z_k}(s)\zeta(s+i\delta)\zeta(1-s-i\delta)ds + O(T^{\frac{1}{4}}L)\ .$$

We can assume without loss of generality that the contour R is at a distance $\gg L^{-1}$ from the zeros of $Z_k(s)$. Then the integral along the horizontal sides is $\ll T^{\frac{1}{7}}$ and it can be absorbed into the last error term. Using the functional equations of $\zeta(s)$ and $\frac{Z_k'}{Z_k}(s)$, Lemma 1 and the well-known result $\int\limits_T^{T+U} \left|\zeta(\frac{1}{2}+it)\right|^2 dt = UL+O(U)+O(T^{\frac{1}{2}+\epsilon})$ (see Theorem 7.4 [4]), with $U=T^{\frac{3}{4}}$, (5) is reduced to

$$(6)\qquad \sum_{T<\gamma_k<T+U} |\zeta\big(\frac{1}{2}+i\gamma_k+i\delta\big)|^2 \;=\; \frac{1}{2\pi i}\int\limits_{\frac{5}{8}+iT}^{\frac{5}{8}+i(T+U)} \frac{Z_k'}{Z_k}(s)\big(\zeta(s+i\delta)\big)^2\chi(1-s-i\delta)ds$$

$$\;=\; 2\,\mathrm{Re}\, I + \frac{UL^2}{2\pi} + O(UL)\ , \qquad \text{say}\ .$$

In the integrand of I changing $\frac{Z_k'}{Z_k}(s)$ to $\frac{\mathcal{Z}_k'}{\mathcal{Z}_k}(s,T)$ (cf. Lemma 5), produces an error of $O\Big(U^2T^{\frac{-7}{8}+\epsilon}\Big)$. Next, in order to use the Dirichlet series approximation to $\frac{\mathcal{Z}_k'}{\mathcal{Z}_k}(s-i\delta,T)$ we move the integral to $\sigma=\frac{9}{8}$ where the approximating series is absolutely convergent. Then by Lemma 7, estimating the integral on the horizontal sides trivially, we have

$$I \;=\; \frac{1}{2\pi i}\int\limits_{\frac{9}{8}+iT}^{\frac{9}{8}+i(T+U)} \sum_{m=1}^{\infty} \frac{a_k(m)m^{i\delta}}{m^s}\,\chi(1-s)\big(\zeta(s)\big)^2 ds + O(T^{\frac{5}{8}}) + O(U^2T^{-\frac{7}{8}+\varepsilon})\ .$$

Applying Lemma 8 we get

$$I \;=\; \sum_{\frac{T}{2\pi}\le mn\le \frac{T+U}{2\pi}} a_k(m)\, m^{i\delta} d(n) + O_\varepsilon(T^{\frac{5}{8}+\varepsilon})\ .$$

This sum is converted back to another integral via Perron's inversion formula so that

$$I \;=\; \Big[\frac{1}{2\pi i}\int_{\frac{9}{8}-iT}^{\frac{9}{8}+iT} \frac{Z_k'}{Z_k}(s-i\delta,T)\big(\zeta(s)\big)^2 ds \frac{x^s}{s} ds + O_\varepsilon\big(x^{\frac{9}{8}}T^{-1}+x^\varepsilon\big)\Big]\Big|_{x=\frac{T}{2\pi}}^{x=\frac{T+U}{2\pi}} + O_\varepsilon(T^{\frac{5}{8}+\varepsilon})\ .$$

The last integral can be evaluated using the residue theorem by carrying the line of integration to $\sigma = \frac{5}{8}$. The residues are calculated from the information in Lemma 6 and returning to (6) we obtain

$$\sum_{\substack{\rho_k=\frac{1}{2}+i\gamma_k\\ T<\gamma_k<T+U}} |\zeta(\frac{1}{2}+i\gamma_k+i\delta)|^2 \;=\; \frac{UL^2}{2\pi}\Big[1+2\mathrm{Re}\,\Big\{\sum_{j=1}^{k}\frac{e^{-2\nu_j+2\pi\alpha i}-(-2\nu_j+2\pi\alpha i)-1}{(-2\nu_j+2\pi\alpha i)^2}\Big\}$$
$$+\,2(k+1)\,\frac{\cos(2\pi\alpha)-1}{(2\pi\alpha)^2}\Big] + O(UL)$$

where $\delta = \frac{2\pi\alpha}{L}$ with $|\alpha|\le 2$, say. This is the desired result with $U = T^{\frac{3}{4}}$ from which the theorem is readily deduced.

References

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