# Mean values of the functional equation factors at the zeros of derivatives of the Riemann Zeta Function and Dirichlet $L$-functions

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*This paper is dedicated to Krishna Alladi on the occasion of his $60^{th}$ birthday.*

**Abstract** In this work average values of the functional equation factors of the Riemann zeta-function and Dirichlet $L$-functions at the zeros of derivatives of these functions are given with the intention of shedding a little light on the interaction between two such functions.

**Key words:** The Riemann zeta-function, Dirichlet $L$-functions, derivatives, zeros

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## 1 Introduction

The Riemann zeta-function is defined as

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad (\sigma > 1),$$

where $s = \sigma + it$ and $\sigma$ and $t$ are real numbers, and then it can be continued analytically to the whole complex plane (with a simple pole at $s = 1$ with residue 1), satisfying the functional equation

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$$\zeta(s) = \chi_\zeta(s)\zeta(1-s),$$

where

$$\chi_\zeta(s) := \pi^{s-\frac{1}{2}} \frac{\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)}.$$

Apart from the trivial zeros at $s = -2, -4, \ldots$, $\zeta(s)$ has non-real (non-trivial) zeros in the critical strip $0 < \sigma < 1$. The number of non-trivial zeros with $0 < t < T$ is $\frac{T}{2\pi} \log \frac{T}{2\pi e} + O(\log T)$, as $T \to \infty$. For such basic knowledge about the Riemann zeta-function and Dirichlet $L$-functions we refer the reader to [4] or [9].

Let $\rho_{\zeta,k}$ denote a non-real zero of the $k^{\text{th}}$ derivative $\zeta^{(k)}(s)$ of $\zeta(s)$ and $\gamma_{\zeta,k} := \Im\rho_{\zeta,k}$. Berndt [2] showed that the number of $\rho_{\zeta,k}$ with $\gamma_{\zeta,k} \in (0,T)$ is $\frac{T}{2\pi} \log \frac{T}{4\pi e} + O_k(\log T)$, as $T \to \infty$.

Similarly a Dirichlet $L$-function is defined as

$$L(s,\psi) := \sum_{n=1}^{\infty} \frac{\psi(n)}{n^s}, \quad (\sigma > 1),$$

where $\psi$ is a Dirichlet character modulo $q$. We will take $q$ fixed (i.e. not depending on $T$) and for the results below we will consider only fixed odd prime values of $q$. In case $\psi$ is a primitive character, $L(s,\psi)$ satisfies the functional equation

$$L(s,\psi) = \chi_\psi(s) L(1-s,\overline{\psi}),$$

where

$$\chi_\psi(s) := \begin{cases} \frac{\tau(\psi)}{q^{\frac{1}{2}}} \left(\frac{\pi}{q}\right)^{s-\frac{1}{2}} \frac{\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)} & \text{if } \psi(-1) = 1, \\ \frac{\tau(\psi)}{iq^{\frac{1}{2}}} \left(\frac{\pi}{q}\right)^{s-\frac{1}{2}} \frac{\Gamma\left(1-\frac{s}{2}\right)}{\Gamma\left(\frac{1+s}{2}\right)} & \text{if } \psi(-1) = -1, \end{cases}$$

and

$$\tau(\psi) = \sum_{m=1}^{q} \psi(m) e^{\frac{2\pi i m}{q}}$$

is the Gaussian sum associated with $\psi$. Apart from the trivial zeros on the non-positive real axis, $L(s,\psi)$ has non-trivial zeros in the critical strip. The number of non-trivial zeros of $L(s,\psi)$ in $0 < \sigma < 1$, $|t| < T$ is $\frac{T}{\pi} \log \frac{qT}{2\pi e} + O(\log qT)$, as $T \to \infty$.

Let $\rho_{\psi,k}$ denote a non-trivial zero of $L^{(k)}(s,\psi)$ and $\gamma_{\psi,k} := \Im\rho_{\psi,k}$. Yıldırım [10] showed that the number of zeros of $L^{(k)}(s,\psi)$ with $|\gamma_{\psi,k}| < T$ in a wide enough strip outside of which there are no zeros is $\frac{T}{\pi} \log \frac{qT}{2\pi e m} + O_q(\log T)$, where $m$ is the smallest prime not dividing $q$ (so, for an odd prime $q$, $m = 2$).

It is known after the works of Conrey and Ghosh [3] (assuming RH, the Riemann Hypothesis) and Karabulut and Yıldırım [8] (unconditionally) that, for $k \in \mathbb{Z}^+$,

$$\sum_{0<\gamma_{\zeta,k}<T} \chi_\zeta(\rho_{\zeta,k}) = \mathscr{A}_k \frac{T}{2\pi} + O_k\left(\frac{T}{\log T}\right), \quad (T\to\infty), \tag{1}$$

where

$$\begin{aligned}\mathscr{A}_k &:= -\sum_{u=0}^{\infty}(-1)^u \sum_{v=1}^{k}\binom{k}{v} \sum_{\substack{i_1+\ldots+i_k=u\\ i_1,\ldots,i_k\geqslant 0}} \binom{u}{i_1,\ldots i_k} \\ &\quad \times \prod_{w=1}^{k}(-1)^w w!\binom{k}{w}^{i_w} \frac{(-1)^v(v+1)!}{(i_1+2i_2+\ldots+ki_k+v)!} \\ &= \sum_{r=1}^{k} e^{-z_r} - k - 1, \qquad (k\geq 1),\end{aligned} \tag{2}$$

with the $z_r\,(r=1,\ldots,k)$ being the zeros of $P_k(z) := \sum_{j=0}^{k}\frac{z^j}{j!}$.

One of the aims of the paper of Conrey and Ghosh was to prove that for any $\varepsilon>0$ there are $\gg_\varepsilon T$ zeros of $\zeta^{(k)}(s)$ in the region $\frac{1}{2}\leq\sigma<\frac{1}{2}+\frac{(1+\varepsilon)\log\log T}{\log T}$, $0<t<T$, and they used (1) for this purpose. That Karabulut and Yıldırım made (1) unconditional did not change the fact that this result of Conrey and Ghosh is dependent on RH because along the way one needs to know that $\zeta^{(k)}(s)$ has at most a finite number of non-real zeros in $\sigma<\frac{1}{2}$ and that depends on RH essentially as the work of Levinson and Montgomery [7] showed.

## 2 Statement of the Results

In this paper, assuming the Generalized Riemann Hypothesis (GRH) for the Riemann zeta-function and Dirichlet $L$-functions, we give results of the calculation of some generalizations of the sum in (1).

Let $k\geq 0$ be a fixed integer, $\mathscr{A}_0 := -1$, $q_1,q_2$ be fixed odd prime numbers. Let $\psi_1(\bmod q_1)$ and $\psi_2(\bmod q_2)$ be non-principal Dirichlet characters.

If $q_1=q_2$, then we have

$$\sum_{0<\gamma_{\psi_1,k}\leq T} \chi_{\psi_2}(\rho_{\psi_1,k}) = \frac{\overline{\tau(\overline{\psi_2})\tau(\psi_1)}}{\varphi(q_1)}\mathscr{A}_k\frac{T}{2\pi} + O_{k,q_1}\left(\frac{T}{\log T}\right), \quad (T\to\infty). \tag{3}$$

On the other hand, if $q_1\neq q_2$, then

$$\sum_{0<\gamma_{\psi_1,k}\leq T} \chi_{\psi_2}(\rho_{\psi_1,k}) \ll_{k,q_1,q_2} T^{1-\frac{1}{\log\log T}}\log T\log\log T, \quad (T\to\infty). \tag{4}$$

Furthermore, for a fixed odd prime $q$, let $\psi$ be a non-principal Dirichlet character $(\mathrm{mod}\ q)$. We have

$$\sum_{0<\gamma_{\psi,k}\leq T} \chi_{\zeta}(\rho_{\psi,k}) \ll_{k,q} T^{1-\frac{1}{\log\log qT}} \log T \log\log T, \quad (T\to\infty). \tag{5}$$

and

$$\sum_{0<\gamma_{\zeta,k}\leq T} \chi_{\psi}(\rho_{\zeta,k}) = -\frac{\overline{\tau(\overline{\psi})}}{\varphi(q)} \mathscr{A}_k \frac{T}{2\pi} + O_{k,q}\left(\frac{T}{\log T}\right), \quad (T\to\infty). \tag{6}$$

## 3 Remarks about the Proofs

For the purpose of gaining some insight about the interaction between two Dirichlet $L$-functions, we were interested in the value of the sums (3)-(6) above. We assumed the GRH to keep the calculations shorter, but as in [8], the results may be obtained without this assumption. The restriction of the Dirichlet characters to fixed prime moduli (in which case all non-principal characters are primitive) was made in order to avoid some minor complications in the calculations.

The proofs begin by expressing the sum under consideration as a contour integral of the form

$$\sum_{A<\gamma\leq B} f(\rho) = \frac{1}{2\pi i}\int_R f(s)\frac{g^{(k+1)}(s)}{g^{(k)}(s)}\,ds,$$

where $\rho$ is a zero of $g^{(k)}(s)$ with imaginary part $\gamma$, and $R$ is an appropriate rectangular contour with vertices at $\sigma_k+iA$, $\sigma_k+iB$, $-\delta+iB$ and $-\delta+iA$, with $\sigma_k$ and $\delta>0$ suitably chosen so as to avoid having any poles of the integrand on the contour. Existence of such suitable contours is given by well-known results on the zeros of derivatives of $\zeta(s)$ and $L(s,\psi)$. For such a contour $R$, the integrals along the horizontal parts and the right side of the contour $R$ can be bounded easily. The results are obtained by careful consideration of the integral along the left vertical segment of the rectangle $R$, from $-\delta+iB$ to $-\delta+iA$. As an example, here we give a sketch of the proof of (3) for $q:=q_1=q_2$. Upon bounding the integrals over the three sides of $R$ (for which we take $A=\frac{T}{2}$, $B=T$), except for the left vertical side, we have

$$\sum_{\frac{T}{2}<\gamma_{\psi_1,k}\leq T} \chi_{\psi_2}(\rho_{\psi_1,k}) = \frac{1}{2\pi i}\int_{-\delta+iT}^{-\delta+i\frac{T}{2}} \chi_{\psi_2}(s)\frac{L^{(k+1)}}{L^{(k)}}(s,\psi_1)\,ds + O_{k,q}(T^{\frac{1}{2}+\delta}\log^2 T). \tag{7}$$

The integral here can be re-expressed as

$$-\frac{1}{2\pi i}\int\limits_{1+\delta+i\frac{T}{2}}^{1+\delta+iT}\chi_{\overline{\psi_2}}(1-s)\frac{L^{(k+1)}}{L^{(k)}}(1-s,\overline{\psi_1})\,ds. \tag{8}$$

Now, using

$$\frac{\chi_\psi^{(m)}}{\chi_\psi}(s)=\left(-\ell+O\left(\frac{1}{|t|}\right)\right)^m,\ (\ell=\log\frac{q|t|}{2\pi},\ m\in\mathbb{N},\ |t|\geq 1),$$

for $s$ lying in a fixed vertical strip, $k$-fold differentiation of the functional equation of $L(s,\psi_1)$ gives

$$L^{(k)}(s,\psi_1)=\chi_{\psi_1}(s)\left(1+O\left(\frac{1}{|t|}\right)\right)\left(-\ell+\frac{d}{ds}\right)^k L(1-s,\overline{\psi_1}).$$

From this we have

$$\frac{L^{(k+1)}}{L^{(k)}}(1-s,\overline{\psi_1})=-\left(\ell+\frac{G_k'}{G_k}(s,\ell,\overline{\psi_1})\right)\left(1+O\left(\frac{1}{|t|}\right)\right), \tag{9}$$

for $\sigma'\leq\sigma\leq\sigma''$ (with $\sigma',\sigma''$ fixed real numbers), where

$$G_k(s,z,\psi_1):=\left(z+\frac{d}{ds}\right)^k L(s,\psi_1)=z^kL(s,\psi_1)+kz^kL'(s,\psi_1)+\ldots+L^{(k)}(s,\psi_1) \tag{10}$$

and the differentiation in $G_k'$ is with respect to $s$. We will substitute (8) and (9) in (7), and use the following generalization of Lemma 5 of [6] and Lemma 2.2 of [8]:

For $m\in\mathbb{Z}$ with $|m|=o(\log T)$ as $T\to\infty$, we have

$$\frac{1}{2\pi}\int\limits_{T/2}^{T}\chi_\psi(1-(a+it))\left(\log\frac{qt}{2\pi}\right)^m\sum_{n=1}^{\infty}\frac{b_n}{n^{a+it}}\,dt=\frac{\tau(\psi)}{q}\sum_{\frac{qT}{4\pi}<n\leqslant\frac{qT}{2\pi}}b_n\,(\log n)^m\,e^{-\frac{2\pi in}{q}}+O_q(K_0^{|m|}T^{a-\frac{1}{2}}(\log T)^m). \tag{11}$$

Here $a>1$ is fixed, $\psi$ is a primitive Dirichlet character modulo $q\geq 3$, $(b_n)_{n\geq 1}$ is a sequence of complex numbers such that $b_n\ll n^{\varepsilon}$ for any $\varepsilon>0$, and $K_0$ is any fixed number $>1$.

But in order to apply this result to our calculation, we have to approximate $\frac{G_k'}{G_k}(s,z,\overline{\psi_1})$, which is not a Dirichlet series for $k\geq 1$, by a Dirichlet series (for $k=0$ we just take the Dirichlet series and the calculation is simpler). From (10) we have

$$\frac{G_k'}{G_k}(s,z,\overline{\psi_1}) = \frac{\sum_{v=0}^{k}\binom{k}{v}\frac{1}{z^v}\frac{L^{(v+1)}}{L}(s,\overline{\psi_1})}{1+\sum_{w=1}^{k}\binom{k}{w}\frac{1}{z^w}\frac{L^{(w)}}{L}(s,\overline{\psi_1})}. \tag{12}$$

Since $\frac{L^{(w)}}{L}(s,\overline{\psi_1}) \ll_w 1$ for $\sigma \geq 1+\delta$, we see

$$\sum_{w=1}^{k}\binom{k}{w}\frac{1}{\ell^w}\frac{L^{(w)}}{L}(s,\overline{\psi_1}) \ll_k \frac{1}{\log T},$$

so that we can expand the denominator of the right-hand side of (12) in a geometric series and write

$$\left(1+\sum_{w=1}^{k}\binom{k}{w}\frac{1}{z^w}\frac{L^{(w)}}{L}(s,\overline{\psi_1})\right)^{-1} \tag{13}$$
$$= \sum_{u\leq\frac{\log T}{\log\log T}}(-1)^u\left(\sum_{w=1}^{k}\binom{k}{w}\frac{1}{\ell^w}\frac{L^{(w)}}{L}(s,\overline{\psi_1})\right)^u + O\left(\frac{1}{T}\right).$$

Thus we obtain

$$\overline{\sum_{\frac{T}{2}<\gamma_{\psi_1,k}\leq T}\chi_{\psi_2}(\rho_{\psi_1,k})} = \sum_{u\leq\frac{\log T}{\log\log T}}\sum_{v=0}^{k}(-1)^u\binom{k}{v}\sum_{\substack{i_1+\ldots+i_k=u\\ i_1,\ldots,i_k\geq 0}}\binom{u}{i_1,\ldots,i_k}\prod_{w=1}^{k}\binom{k}{w}^{i_w}$$
$$\times\frac{\tau(\overline{\psi_2})}{q}\sum_{\frac{qT}{4\pi}<n\leq\frac{qT}{2\pi}}\frac{b_n(i_1,\ldots,i_k;v;\overline{\psi_1})e^{-\frac{2\pi i n}{q}}}{(\log n)^K} + O_{k,q}(T^{\frac{1}{2}+\delta+\varepsilon}), \tag{14}$$

where $K := i_1+2i_2+\ldots+ki_k+v$, and

$$\sum_{n=1}^{\infty}\frac{b_n(i_1,\ldots,i_k;v;\overline{\psi_1})}{n^s} := \frac{L^{(v+1)}}{L}(s,\overline{\psi_1})\prod_{w=1}^{k}\left(\frac{L^{(w)}}{L}(s,\overline{\psi_1})\right)^{i_w}. \tag{15}$$

The sum over $n$ in (14) can be re-expressed as

$$\frac{1}{\varphi(q)}\sum_{\substack{a\,(\mathrm{mod}\ q)\\(a,q)=1}}e^{-\frac{2\pi i a}{q}}\sum_{\psi\,(\mathrm{mod}\ q)}\overline{\psi}(a)\sum_{\frac{qT}{4\pi}<n\leq\frac{qT}{2\pi}}\frac{b_n(i_1,\ldots,i_k;v;\overline{\psi_1})\psi(n)}{(\log n)^K}.$$

Here the sum over $\psi$ is split into two parts. The term $\psi=\psi_1$ (which will lead to the main term), and the terms $\psi\neq\psi_1$ the contribution of which will be denoted by $E_{\psi\neq\psi_1}$ (which turns out to be an error term). So now we have

$$\overline{\sum_{\frac{T}{2}<\gamma_{\psi_1,k}\le T} \chi_{\psi_2}(\rho_{\psi_1,k})} = \tag{16}$$
$$\frac{\tau(\overline{\psi_2})\,\overline{\tau(\psi_1)}}{q\varphi(q)} \sum_{u\le \frac{\log T}{\log\log T}} \sum_{v=0}^{k} (-1)^u \binom{k}{v} \sum_{\substack{i_1+\ldots+i_k=u\\ i_1,\ldots,i_k\ge 0}} \binom{u}{i_1,\ldots,i_k} \prod_{w=1}^{k} \binom{k}{w}^{i_w}$$
$$\times \sum_{\frac{qT}{4\pi}<n\le\frac{qT}{2\pi}} \frac{b_n(i_1,\ldots,i_k;v;\overline{\psi_1})\psi_1(n)}{(\log n)^K} + E_{\psi\ne\psi_1} + O_{k,q}(T^{\frac{1}{2}+\delta+\varepsilon}).$$

The proof is then completed upon using (2) and the following result :

Assume GRH. Let $q$ be a fixed odd prime number and $\psi_1$ be a primitive Dirichlet character modulo $q$; $k,i_1,i_2,\ldots,i_k\in\mathbb{N}$, $v\in\{0,1,\ldots,k\}$. If $\psi$ is a Dirichlet character modulo $q$ such that $\psi\ne\psi_1$, then we have

$$\sum_{n\le x} b_n(i_1,\ldots,i_k;v;\overline{\psi_1})\psi(n) = O(x^{1-\frac{1}{\log\log q(x+4)}}(\log x)(A(k)\log\log q(x+4))^{K+1}). \tag{17}$$

If also $i_1+\ldots+i_k\leqslant\frac{\log x}{\log\log x}$, then we have

$$\sum_{n\le x} b_n(i_1,\ldots,i_k;v;\overline{\psi_1})\psi_1(n) = S(i_1,\ldots,i_k;v)x(\log x)^K + E_b(i_1,\ldots,i_k;v), \tag{18}$$

where

$$S(i_1,\ldots,i_k;v) := \frac{(-1)^{K+1}(v+1)!\prod_{w=1}^{k}(w!)^{i_w}}{K!}, \tag{19}$$

and

$$E_b(i_1,\ldots,i_k;v) := O_q\left((A(k)^K)\left((\log x)^{K+2} + \frac{x(\log x)^{K-1}}{(K-1)!} + \frac{x(\log x)^{(\frac{2}{3}+\varepsilon)(K+3)}}{e^{\delta_1(k)(\log x)^{\frac{1}{3}-\varepsilon}}}\right)\right). \tag{20}$$

The other results, (4)-(6), are proved along similar lines. The detailed proofs of the above results can be found in [1] and in [5].

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## References

1. K. Benli, *The mean values of the functional equation $\chi_\psi$ at the zeros of $\zeta^{(k)}(s)$ and $L^{(k)}(s,\psi_2)$*, M.S. Thesis, Boğaziçi University, 2015.

2. B. C. Berndt, *The number of zeros for* $\zeta^{(k)}(s)$, J. London Math. Soc. (2), Vol. 2, (1970), 577–580.
3. J. B. Conrey, A. Ghosh, *Zeros of derivatives of the Riemann zeta-function near the critical line* in *Analytic Number Theory (Allerton Park, IL 1989)*, Progress in Mathematics, Vol 85, 95–110, Birkhäuser Boston, Boston, 1990.
4. H. Davenport, *Multiplicative Number Theory*, Third edition. Revised by H. L. Montgomery, Springer-Verlag, New York, 2000.
5. E. Elma, *Average values of the functional equation factors at the zeros of derivatives of Dirichlet L-functions*, M.S. Thesis, Boğaziçi University, 2015.
6. S. M. Gonek, *Mean values of the Riemann zeta-function and its derivatives*, Inventiones Mathematicae, Vol. 75, No. 1, (1984), 123–141.
7. N. Levinson, H. L. Montgomery, *Zeros of the derivatives of Riemann's zeta-function*, Acta Math., Vol. 133, (1974), 49–65.
8. Y. Karabulut, C. Y. Yıldırım, *On some averages at the zeros of derivatives of the Riemann zeta-function*, J. Number Theory, Vol. 131, No. 10, (2011), 1939–1961.
9. H. L. Montgomery, R. Vaughan, *Multiplicative Number Theory I. Classical Theory*, Cambridge University Press, Cambridge, 2006.
10. C. Y. Yıldırım, *Zeros of derivatives of Dirichlet L-functions*, Turk. J. Math., Vol. 20, No.4, (1996), 521–534.