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A NOTE ON $\zeta''(s)$ AND $\zeta'''(s)$

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Abstract. There is only one pair of non-real zeros of $\zeta''(s)$, and of $\zeta'''(s)$, in the left half-plane. The Riemann Hypothesis implies that $\zeta''(s)$ and $\zeta'''(s)$ have no zeros in the strip $0 \le \Re s < \frac{1}{2}$.

It was shown by Speiser [3] that the Riemann Hypothesis is equivalent to $\zeta'(s)$ having no zeros in $0 < \sigma < \frac{1}{2}$ (as usual we write $s = \sigma + it$). Levinson and Montgomery [2] gave a different proof; moreover, they showed that $\zeta^{(k)}(s)$ has at most a finite number of non-real zeros in $\sigma < \frac{1}{2}$ for $k \ge 1$ as a consequence of RH. Another result in [2] was that ζ' vanishes exactly once in the interval $(-2n-2, -2n)$ for all $n \ge 1$, these being the only zeros of ζ' in the left half-plane. Spira [4] calculated the zeros of ζ' and ζ'' in the rectangle $-1 \le \sigma \le 5, |t| \le 100$, from which it is seen that $\zeta''(s) \ne 0$ in $0 \le \sigma \le \frac{1}{2}, |t| \le 100$. However, ζ'' has a zero near $-0.36 \pm 3.59i$ (which will be called b_0 and $\overline{b_0}$ below).

In this paper, we shall be concerned with the zeros of $\zeta''(s)$ and $\zeta'''(s)$ lying to the left of the critical line. Only for Theorem 1 will full details of the proof be given here (cf. [6] for Theorems 2, 3 and 4).

Theorem 1. *The Riemann Hypothesis implies that $\zeta''(s)$ has no zeros in the strip $0 \le \sigma < \frac{1}{2}$.*

Proof. Let us denote the real zeros of ζ' as $-a_n, n \ge 1$, where $a_n \in (2n, 2n+2)$. A non-real zero of ζ' will be represented as $\rho_1 = \beta_1 + i\gamma_1$. By what was recounted above, $\beta_1 \ge \frac{1}{2}$ for all ρ_1 (on RH), while Titchmarsh [5, Theorem 11.5(c)] showed that $\beta_1 < 3$. (Since $\Re\frac{\zeta'}{\zeta}(s) < 0$ on $\sigma = \frac{1}{2}$ except when $\zeta(s) = 0$, one has $\beta_1 = \frac{1}{2}$ only at a possible multiple zero of $\zeta(s)$; see [2].) The starting point is the partial fraction representation

$$(1) \qquad \frac{\zeta''}{\zeta'}(s) = \frac{\zeta''}{\zeta'}(0) - 2 - \frac{2}{s-1} + \sum_n \Big(\frac{1}{s+a_n} - \frac{1}{a_n}\Big) + \sum_{\rho_1} \Big(\frac{1}{s-\rho_1} + \frac{1}{\rho_1}\Big),$$

which follows from the Hadamard theory. Taking real parts in (1), we have

$$(2) \qquad \Re\frac{\zeta''}{\zeta'}(s) = \frac{\zeta''}{\zeta'}(0) - 2 + \frac{2(1-\sigma)}{|s-1|^2} + \sum_n \Big(\frac{\sigma+a_n}{|s+a_n|^2} - \frac{1}{a_n}\Big) + \sum_{\rho_1} \Re\frac{1}{s-\rho_1} + \sum_{\rho_1} \frac{1}{\rho_1},$$

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since $\zeta'(\overline{\rho_1}) = 0$ too. We should first like to put a bound on $\sum \frac{1}{\rho_1}$ (in this series it is understood that the terms from ρ_1 and $\overline{\rho_1}$ are grouped together). At $s = 6$, (2) reads

$$(3)\qquad \frac{\zeta''}{\zeta'}(6) = \frac{\zeta''}{\zeta'}(0) - \frac{12}{5} - \sum_n \frac{6}{a_n(a_n+6)} + \sum_{\rho_1} \frac{6-\beta_1}{(6-\beta_1)^2+\gamma_1^2} + \sum_{\rho_1} \frac{\beta_1}{\beta_1^2+\gamma_1^2}.$$

It is known that $\frac{\zeta''}{\zeta'}(0) = 2.183\ldots$ (see [1]), and $\frac{\zeta''}{\zeta'}(6) = -0.773\ldots$. Also

$$\sum_n \frac{6}{a_n(a_n+6)} < \sum_{n=1}^{\infty} \frac{6}{2n(2n+6)} = \frac{11}{12}.$$

Since $6-\beta_1 > \beta_1$ and the least $|\gamma_1|$ is $23.3\ldots$ (see [4]), we have $\frac{6-\beta_1}{(6-\beta_1)^2+\gamma_1^2} > \frac{\beta_1}{\beta_1^2+\gamma_1^2}$ for all ρ_1. Plugging all these in (3), it follows that

$$(4)\qquad \sum_{\rho_1} \frac{1}{\rho_1} < 0.185$$

(from Spira's list of ρ_1 with $|\gamma_1| < 100$ one calculates $\sum \frac{1}{\rho_1} > 0.0249$).

We now examine the value of $\Re\frac{\zeta''}{\zeta'}(s)$ in the region $0 \le \sigma \le \frac{1}{2}, |t| \ge 100$. If ever a zero of ζ' exists on the critical line, this region is to be modified by deleting an arbitrarily small neighbourhood around such a zero. For any s in our region $\frac{2(1-\sigma)}{|s-1|^2} < \frac{1}{5000}, \Re\frac{1}{s-\rho_1} < 0$ for all ρ_1 (on RH), and

$$\begin{aligned}
\sum_n \left(\frac{\sigma+a_n}{|s+a_n|^2} - \frac{1}{a_n}\right) &\le \sum_n \left(\frac{\sigma+a_n}{(\sigma+a_n)^2+10^4} - \frac{1}{a_n}\right) \\
&\le \sum_n \frac{-10^4}{a_n((a_n+\frac{1}{2})^2+10^4)} \\
&< \sum_{n=2}^{\infty} \frac{-10^4}{2n((2n+\frac{1}{2})^2+10^4)} \\
&< -1.74.
\end{aligned}$$

Together with (4), these estimates used in (2) give $\Re\frac{\zeta''}{\zeta'}(s) < -1.37$ at all points of our region. □

Notice that $\zeta''(s)$ can be zero on the critical line only at a multiple (of at least third order) zero of $\zeta(s)$ if ever this exists.

Theorem 2 (unconditional)**.** *There is only one pair of non-real zeros of $\zeta''(s)$ in the left half-plane; viz. b_0 and $\overline{b_0}$.*

Proof. We can choose arbitrarily large N and $\sigma_N = -a_N - \epsilon$, with ϵ small enough so that there is no zero of $\zeta''(s)$ in the interval $[\sigma_N, -a_N]$ and $\frac{\zeta''}{\zeta'}(\sigma_N) < 0$. Consider the rectangle with corners at $\pm iN, \sigma_N \pm iN$. Inside this rectangle there are exactly N zeros of ζ' (all real), so by Rolle's theorem there must be at least $N-1$ real zeros of ζ''. It is known that a pair of zeros of ζ'', viz. b_0 and $\overline{b_0}$, exist within the rectangle. It is now sufficient to check that $arg\frac{\zeta''}{\zeta'}(s)$ changes by 2π as s makes one counterclockwise tour of the rectangle. Some calculation reveals that, on the imaginary axis, $\Re\frac{\zeta''}{\zeta'}(it) \ge 0$ only for $|t| \le |t_0| < 3.8$. As s moves from $-4i$ to $4i$, the image curve $\frac{\zeta''}{\zeta'}(it)$ contains one counterclockwise loop around the origin. On the other three edges of the contour, one verifies that $\Re\frac{\zeta''}{\zeta'}(s)$ is essentially less than $-\frac{1}{2}\log N$. This proves Theorem 2. $\square$

Notice that if one starts from a point on the negative real axis where $\Re\frac{\zeta''}{\zeta'}(s) > 0$ and moves vertically away from the real axis, soon one hits a point where $\Re\frac{\zeta''}{\zeta'}(s) = 0$, and then further away from the axis $\Re\frac{\zeta''}{\zeta'}(s) < 0$.

We may now proceed to examine $\zeta'''(s)$. Similar to (1) we have

$$\begin{aligned} \frac{\zeta'''}{\zeta''}(s) &= \frac{\zeta'''}{\zeta''}(0) - 3 - \frac{3}{s-1} + \sum_{n=1}^{\infty}\left(\frac{1}{s+b_n} - \frac{1}{b_n}\right) + \sum_{\rho_2}\left(\frac{1}{s-\rho_2} + \frac{1}{\rho_2}\right) \\ &\quad + \left(\frac{1}{s-b_0} + \frac{1}{b_0} + \frac{1}{s-\overline{b}_0} + \frac{1}{\overline{b}_0}\right), \end{aligned} \tag{5}$$

where $-b_n, n \geq 1$, are the zeros of $\zeta''(s)$ on the negative real axis, and $\rho_2 = \beta_2 + i\gamma_2$ runs through the zeros of $\zeta''(s)$ in the right half-plane. Some information on the small negative zeros of $\zeta''(s)$ is needed in order to bound the sums over b_n. The intervals where these zeros lie can be found from the functional equation of $\zeta(s)$ differentiated twice. From Spira [4] we know that $\beta_2 < 5$ for all ρ_2, and analogous to (4), by using (5) at $s = 10$, we find

$$\sum_{\rho_2} \frac{1}{\rho_2} < 0.165. \tag{6}$$

Take a contour just as in the proof of Theorem 2 (with a_N replaced by b_N). On the imaginary axis, $\Re\frac{\zeta'''}{\zeta''}(s) \geq 0$ only for $|t| \leq |t_0| < 5$. As s moves up on the imaginary axis from $-5i$ to $5i$, the image curve $\frac{\zeta'''}{\zeta''}(it)$ winds clockwise once around the origin (the presence of b_0 and $\overline{b_0}$ makes the picture different from that of $\frac{\zeta''}{\zeta'}(it)$ not just in the sense of winding, but also the curve crosses the positive real axis three times between $t = -4$ and $t = 4$). On the remaining three edges of the contour the situation is the same as before. Hence, as s makes one counterclockwise tour of the rectangle, $arg\frac{\zeta'''}{\zeta''}(s)$ changes by -2π, allowing us to state

Theorem 3 (unconditional)**.** *There is only one pair of non-real zeros of $\zeta'''(s)$ in the left half-plane.*

If we assume RH, so that $\Re\frac{1}{s-\rho_2} < 0$ for $\sigma < \frac{1}{2}$ by Theorem 1, then a calculation similar to the proof of Theorem 1, and making use of (6), shows that $\Re\frac{\zeta'''}{\zeta''}(s) < 0$, in the region $0 \leq \sigma < \frac{1}{2}, |t| \geq 10$. The region $0 \leq \sigma \leq \frac{1}{2}, |t| \leq 10$ may be swept by integrating $\frac{\zeta^{(iv)}}{\zeta'''}(s)$ around the rectangle, using the expressions from the Euler-Maclaurin formula for the derivatives of $\zeta(s)$ (cf. [4]), and no zeros of $\zeta'''(s)$ are found. Hence we have

Theorem 4. *The Riemann Hypothesis implies that $\zeta'''(s)$ has no zeros in the strip $0 \leq \sigma < \frac{1}{2}$.*

If one determines the 'small' zeros of $\zeta'''(s)$, including the pair in the left half-plane, then one may proceed to the investigation of $\zeta^{(iv)}(s)$.

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References

1. T. M. Apostol, *Formulas for higher derivatives of the Riemann zeta function*, Math. Comp. **44** (1985), 223-232. MR **86c:**11063
2. N. Levinson and H. L. Montgomery, *Zeros of derivatives of the Riemann zeta-function*, Acta Math. **133** (1974), 49-65. MR **54:**5135
3. A. Speiser, *Geometrisches zur Riemannschen zetafunktion*, Math. Ann. **110** (1934), 514-521.
4. R. Spira, *Zero-free regions of* $\zeta^{(k)}(s)$, J. London Math. Soc. **40** (1965), 677-682. MR **31:**5849
5. E. C. Titchmarsh, *The theory of the Riemann zeta-function*, 2nd ed., Oxford, 1986. MR **88c:**11049
6. C. Y. Yıldırım, *A note on* $\zeta''(s)$ *and* $\zeta'''(s)$*, detailed version*, manuscript available by e-mail.

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