1. Find the work done by a particle which follows a trajectory $y = 3x$ under a force field $\vec{F}(x, y) = (\cos x \cos y - \sin x \sin y)(\vec{i} + \vec{j})$ from $(0, 0)$ to $(\pi/6, \pi/2)$.

Solution:

$$ W = \int_C \vec{F}.d\vec{r} , \text{ where } C \text{ is given by } y = 3x. $$

$$ \vec{F}(x, y) = (\cos x \cos y - \sin x \sin y)(\vec{i} + \vec{j}) = M(x, y)\vec{i} + N(x, y)\vec{j}. $$

So $M(x, y) = N(x, y) = (\cos x \cos y - \sin x \sin y) = \cos(x + y)$. 

So, necessarily $M_y(x, y) = N_x(x, y) \Rightarrow \vec{F}(x, y)$ is conservative.

So there’s a potential function $\phi$ defined as follows:

$$ \vec{F}(x, y) = \nabla \phi(x, y), \text{ where } \phi_x = \phi_y = \cos(x + y). \text{ From here, } $$

$$ \phi_x = \cos(x + y) \Rightarrow \phi(x, y) = \sin(x + y) + f(y) $$

$$ \phi_y = \cos(x + y) \Rightarrow \phi(x, y) = \sin(x + y) + g(x) $$

Equating the two solutions,

$$ \phi(x, y) = \sin(x + y) + f(y) = \sin(x + y) + g(x) \Rightarrow f(y) = g(x) = c, \text{ a constant.} $$

$$ \phi(x, y) = \sin(x + y) + c $$

and since $\vec{F}$ is conservative,

$$ W = \int_C \vec{F}.d\vec{r} = \int_{(x_1,y_1)}^{(x_2,y_2)} \vec{F}(x, y).d\vec{r} = \phi(x_2, y_2) - \phi(x_1, y_1) $$

$$ = \phi(\pi/6, \pi/2) + c - \phi(0, 0) - c = \sin(\pi/6 + \pi/2) - \sin(0 + 0) $$

$$ = \sin(\frac{2\pi}{3}) = \frac{\sqrt{3}}{2} $$

2. Let $\nabla f(x(r, \theta), y(r, \theta)) = \frac{\partial f}{\partial r} \vec{e}_1(\theta) + \frac{1}{r} \frac{\partial f}{\partial \theta} \vec{e}_2(\theta)$, where $r$ and $\theta$ are polar coordinates. What are $\vec{e}_1(\theta)$ and $\vec{e}_2(\theta)$ in terms of $\vec{i}$ and $\vec{j}$? Are they perpendicular? Give an interpretation for $\vec{e}_1(\theta)$ and $\vec{e}_2(\theta)$. [Hint : $\arctan x) = \frac{1}{1+x^2}$$]

Solution:

$$ \nabla f(x, y) = \frac{\partial f}{\partial x} \vec{i} + \frac{\partial f}{\partial y} \vec{j} : x = r \cos \theta, y = r \sin \theta \Rightarrow r = \sqrt{x^2 + y^2}, \theta = \arctan(\frac{y}{x}) $$

$$ \frac{\partial f}{\partial x} = \frac{\partial r}{\partial x} \frac{\partial f}{\partial r} + \frac{\partial \theta}{\partial x} \frac{\partial f}{\partial \theta} = \frac{x}{\sqrt{x^2 + y^2}} \frac{\partial f}{\partial r} + \frac{(-y/x^2)}{1 + (y/x)^2} \frac{\partial f}{\partial \theta} = \cos \theta \frac{\partial f}{\partial r} - \frac{\sin \theta \frac{\partial f}{\partial \theta}}{r} $$
\[
\frac{\partial f}{\partial y} = \frac{\partial f}{\partial y} \frac{\partial f}{\partial r} + \frac{\partial \theta}{\partial r} \frac{\partial f}{\partial \theta} = \frac{y}{\sqrt{x^2 + y^2}} \frac{\partial f}{\partial r} + \frac{(1/x)}{1 + (y/x)^2} \frac{\partial f}{\partial \theta} = \sin \theta \frac{\partial f}{\partial r} - \frac{\cos \theta \partial f}{r} \frac{\partial \theta}{\partial \theta}
\]

\[
\Rightarrow \vec{\nabla} f(x(r, \theta), y(r, \theta)) = \left[ \cos \theta \frac{\partial f}{\partial r} - \frac{\sin \theta \partial f}{r} \frac{\partial \theta}{\partial \theta} \right] \vec{i} + \left[ \sin \theta \frac{\partial f}{\partial r} - \frac{\cos \theta \partial f}{r} \frac{\partial \theta}{\partial \theta} \right] \vec{j}
\]

\[
\vec{\nabla} f(x(r, \theta), y(r, \theta)) = \frac{\partial f}{\partial r} (\cos \theta \vec{i} + \sin \theta \vec{j}) + \frac{1}{r} \frac{\partial f}{\partial \theta} (-\sin \theta \vec{i} + \cos \theta \vec{j})
\]

Hence \(e_1(\theta) = \cos \theta \vec{i} + \sin \theta \vec{j}\) and \(e_2(\theta) = -\sin \theta \vec{i} + \cos \theta \vec{j}\)

\(e_1(\theta).e_2(\theta) = 0 \Rightarrow \) They are perpendicular.

These are the basis unit vectors for a system of polar coordinates \(\frac{\partial f}{\partial r} \equiv \) rate of change of \(f\) with respect to \(r\) where \(\theta\) is kept constant

\(\Rightarrow e_1(\theta)\) is the radial unit vector along which the change in \(f\) occurs with respect to \(r\) when \(\theta\) is kept constant. \(\frac{1}{r} \frac{\partial f}{\partial \theta} \equiv \) rate of change of \(f\) with respect to \(\theta\) where \(r\) is kept constant

\(\Rightarrow e_2(\theta)\) is the angular unit vector along which the change in \(f\) occurs with respect to \(\theta\) when \(r\) is kept constant.

3. By drawing relevant graphs, write but not evaluate, triple integrals for the volume of the region bounded outside by \(z = 2 - x^2 - y^2\) and above by \(z = 1\) in the first octant.

(a) in Cartesian coordinates only WITH \(dV = dxdydz\). (b) in cylindrical coordinates.

**Solution:**

(a)

\[
Q = \{(x, z) | 0 \leq x \leq \sqrt{2 - z}, 0 \leq z \leq 1\}
\]

\[
S = \{(x, y, z) | 0 \leq x \leq \sqrt{2 - z}, 0 \leq y \leq \sqrt{2 - x^2 - z}, 0 \leq z \leq 1\}
\]

\[
V = \int_0^1 \int_0^{\sqrt{2-z}} \int_0^{\sqrt{2-y^2-z}} dxdydz
\]
The intersection of $z = 1$ and $z = 2 - r^2$ is found by setting $z = 2 - r^2 = 1$, that is, on $z = 1$ plane. When one draws straight lines along $z$ axis, the region $S$ is seen to be the union of two regions $S_1$ and $S_2$ along $z$ having different boundaries. $Q_1 = \{(r, \theta)|0 \leq \theta \leq \pi/2, 0 \leq r \leq 1\}$, $Q_2 = \{(r, \theta)|0 \leq \theta \leq \pi/2, 1 \leq r \leq \sqrt{2}\}$ and therefore

\[
S = S_1 \cup S_2 = \{(r, \theta,z)|0 \leq \theta \leq \pi/2, 0 \leq r \leq 1, 0 \leq z \leq 1\} \cup \{(r, \theta,z)|0 \leq \theta \leq \pi/2, 1 \leq r \leq \sqrt{2}, 0 \leq z \leq 2 - r^2\}
\]

so that $V = \iiint_{S_1} rdzdrd\theta + \iiint_{S_2} rdzdrd\theta$. Finally,

\[
V = \int_{0}^{\pi/2} \int_{0}^{1} \int_{0}^{1} rdzdrd\theta + \int_{0}^{\pi/2} \int_{1}^{\sqrt{2}} \int_{0}^{2-r^2} rdzdrd\theta.
\]

4. Consider the ellipsoid $\frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{16} = 1$

(a) Find the equation of the plane tangent to the ellipsoid at $P(\frac{2}{\sqrt{3}}, \frac{3}{\sqrt{3}}, \frac{4}{\sqrt{3}})$ by using the gradient.

(b) Find a normal vector at $P$ without using the gradient. (Do not derive the equation of the tangent plane)

Solution:

(a) A normal $\vec{N}(x, y, z)$ to the ellipsoid is:

\[
\vec{N}(x, y, z) = \vec{\nabla}(\frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{16}) = \frac{x}{2}i + \frac{2y}{9}j + \frac{z}{8}k
\]

\[
\vec{N}(\frac{2}{\sqrt{3}}, \frac{3}{\sqrt{3}}, \frac{4}{\sqrt{3}}) = \frac{1}{\sqrt{3}}(i + \frac{2}{3}j - \frac{1}{2}k).
\]

Hence the tangent plane $\sigma$ is given by:

\[
\sigma : \frac{1}{\sqrt{3}}(\vec{v} + \frac{2}{3}\vec{j} - \frac{1}{2}\vec{k}). \left[ (x - \frac{2}{\sqrt{3}})\vec{i} + (y - \frac{3}{\sqrt{3}})\vec{j} + (z + \frac{4}{\sqrt{3}})\vec{k} \right] = 0
\]

\[
(x - \frac{2}{\sqrt{3}}) + \frac{2}{3}(y - \frac{3}{\sqrt{3}}) - \frac{1}{2}(z + \frac{4}{\sqrt{3}}) = 0
\]

\[
\sigma : x + \frac{2}{3}y - \frac{1}{2}z = 2\sqrt{3}.
\]
\[ \frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{16} = 1 \Rightarrow z = \mp 4 \sqrt{1 - \frac{x^2}{4} - \frac{y^2}{9}} \]

Since \( P(\frac{2}{\sqrt{3}}, \frac{3}{\sqrt{3}}, \frac{-4}{3}) \), \( z < 0 \). So one considers the region below the \( xy \) plane

\[ z = -4 \sqrt{1 - \frac{x^2}{4} - \frac{y^2}{9}} \]

So from \( \vec{N}_{up}(x, y) = \frac{\partial f}{\partial x}(x, y)i - \frac{\partial f}{\partial y}(x, y)j + \vec{k} = -\vec{N}_{down}(x, y) \)

\[ \vec{N}_{up}(x, y) = -\frac{x}{\sqrt{1 - \frac{x^2}{4} - \frac{y^2}{9}}} i - \frac{4y}{9 \sqrt{1 - \frac{x^2}{4} - \frac{y^2}{9}}} j + \vec{k} \]

\[ \Rightarrow \vec{N}_{up}(\frac{2}{\sqrt{3}}, \frac{-4}{3}) = -2i - \frac{4}{3} j + \vec{k} \]

The normal vector in part a (\( x = -2\sqrt{3} \)) is found.

5. Consider the function \( f(x, y) = -\sqrt{2x + 1} - \sqrt{8y - 3} \).

(a) What are the domain and range of \( f \)? Find all critical points.

Can one use the second derivative test in this case?

(b) What is the absolute maximum of the function? By studying infinitesimal translations of it,

verify that it is really a maximum.

(Note: Positivity or negativity of the infinitesimals should be discussed).

Solution:

(a)

\( \text{Domain} \{ f \} = \{(x, y)| \frac{-1}{2} \leq x, \frac{3}{8} \leq y \} \Rightarrow \text{Range} \{ f \} = \{ f(x, y)| -\infty < f(x, y) \leq 0 \} \)

\[ \frac{\partial f}{\partial x} = -\frac{1}{\sqrt{2x + 1}} \neq 0, \quad \frac{\partial f}{\partial x} \text{ is undefined at } x = -\frac{1}{2} \]

\[ \frac{\partial f}{\partial y} = -\frac{4}{\sqrt{8y - 3}} \neq 0, \quad \frac{\partial f}{\partial y} \text{ is undefined at } y = \frac{3}{8} \]

So, \( \{(x, y)|x = -\frac{1}{2}, y \geq \frac{3}{8}\} \) and \( \{(x, y)|x \geq -\frac{1}{2}, y = \frac{3}{8}\} \) are the sets of critical points.

Second derivative set cannot be used since the first partial derivatives are undefined.

(b) The value of the function \( f(-\frac{1}{2}, \frac{3}{8}) = 0 \) is the absolute maximum since \( f(x, y) \leq 0 \).

Consider translation at \((-\frac{1}{2}, \frac{3}{8})\) as \( x = -\frac{1}{2} + k, y = \frac{3}{8} + h \). Then,

\[ f(-\frac{1}{2} + k, \frac{3}{8} + h) = -\sqrt{2(-\frac{1}{2} + k) + 1} - \sqrt{8(\frac{3}{8} + h) - 3} = -\sqrt{2k} - \sqrt{8h} \]

But since \( x \geq -\frac{1}{2} \) and \( y \geq \frac{3}{8} \),

\[ -\frac{1}{2} + k \geq -\frac{1}{2} \quad \text{and} \quad \frac{3}{8} + h \geq \frac{3}{8} \Rightarrow h, k \geq 0. \]

So when \( k \geq 0 \) and \( h \geq 0 \), \( f(-\frac{1}{2} + k, \frac{3}{8} + h) \) is defined.
\[ f(-\frac{1}{2} + k, \frac{3}{8} + h) - f(-\frac{1}{2}, \frac{3}{8}) = -\sqrt{2k} - \sqrt{8h} \leq 0 \]

Which means that every translation from \((-\frac{1}{2}, \frac{3}{8})\) gives rise to lower values of \(f\) \(\Rightarrow f(-\frac{1}{2}, \frac{3}{8}) = 0\) is really a maximum for \(f\).

6. Find the distance from the point \(P(1, -1, 2)\) to the surface \(z = 2 - x - 2y\)
(a) By geometrical considerations (without using Lagrange’s method)
(b) Construct explicitly, but do NOT solve, the Lagrange equations to find the distance.
(State clearly the constraint, the function to be extremized etc.)

Solution:

Take an arbitrary point on the plane, e.g. \(Q(0, 0, 2)\) Then, the distance, say \(d\), is given by: \(d = \frac{|\vec{N}.\vec{PQ}|}{\vec{N}}\), where \(\vec{N}\) is the normal vector for the plane.

\[
\vec{N} = \vec{i} + 2\vec{j} + \vec{k}, \quad \vec{PQ} = (0 - 1)\vec{i} + (0 + 1)\vec{j} + (2 - 2)\vec{k} = -\vec{i} + \vec{j}
\]

\[
d = \frac{|\vec{N}.\vec{PQ}|}{\vec{N}} = \frac{1}{\sqrt{6}}.
\]

(b) Let an arbitrary point on the plane be given by \((x, y, z)\). Then the distance between any point of the plane and \(P\) is given by:

\[d(x, y, z) = \sqrt{(x - 1)^2 + (y + 1)^2 + (z - 2)^2}\]

which is the function to be extremized. Define also \(g(x, y, z) = x + 2y - z - 2 = 0\) which is the constraint.

The Lagrange equations are given by:

\[
\vec{\nabla}d(x, y, z) = \lambda \vec{\nabla}g(x, y, z)
\]

\[
\vec{\nabla}(\sqrt{(x - 1)^2 + (y + 1)^2 + (z - 2)^2}) = \lambda \vec{\nabla}(x + 2y - z - 2)
\]

\[
\frac{x - 1}{d}\vec{i} + \frac{y + 1}{d}\vec{j} + \frac{z - 2}{d}\vec{k} = \lambda (\vec{i} + 2\vec{j} + \vec{k}) \Rightarrow
\]

Lagrange equations:

\[
\frac{x - 1}{\sqrt{(x - 1)^2 + (y + 1)^2 + (z - 2)^2}} = \lambda
\]

\[
\frac{y + 1}{\sqrt{(x - 1)^2 + (y + 1)^2 + (z - 2)^2}} = 2\lambda
\]

\[
\frac{z - 2}{\sqrt{(x - 1)^2 + (y + 1)^2 + (z - 2)^2}} = \lambda
\]

With the constraint:

\[x + 2y + z - 2 = 0.\]