B U Department of Mathematics  
Math 102 Calculus II  

Fall 2001 Second Midterm

1. a) Let \( f(x, y) = \begin{cases} \frac{4xy}{\sqrt{x^2 + y^2}} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases} \). Is \( f(x, y) \) continuous at \((0, 0)\)? Justify your answer!

Solution:

First notice that \( \lim_{(x,y) \to (0,0)} f(x, y) \) has a \( \frac{0}{0} \) type indeterminacy. Use polar coordinates:

\( x = r \cos \theta \) and \( y = r \sin \theta \) so that \( r^2 = x^2 + y^2, \ r \geq 0 \). We now have:

\[
\lim_{(x,y) \to (0,0)} f(x, y) = \lim_{r \to 0} \frac{4r^2 \sin \theta \cos \theta}{\sqrt{r^2}} = \lim_{r \to 0} 4r \sin \theta \cos \theta = 0
\]

Hence, \( \lim_{(x,y) \to (0,0)} f(x, y) = 0 = f(0,0) \). Thus, \( f(x, y) \) is continuous at \( (0,0) \).

b) Prove or disprove that the directional derivative of \( f(x, y) \) in the direction of \( u + v \) at the point \( P(x_0, y_0) \) is equal to \( D_u f(x_0, y_0) + D_v f(x_0, y_0) \) where \( u \) and \( v \) are vectors.

Solution:

The equality holds if \( f(x, y) \) is differentiable at \( P \). Otherwise the equality does not necessarily hold. For example, the function \( f(x, y) \) of part (a) is constant along \( x- \) and \( y- \) axes so that \( D_i f(0,0) = D_j f(0,0) = 0 \) but \( D_{i+j} f(0,0) = 2\sqrt{2} \).

2. Find the point(s) on the graph of \( x^2 + 4y^2 + z^2 = 12 \) where the tangent plane is perpendicular to the line \( \frac{1-x}{2} = 1 - y = \frac{z-3}{-2} \).

Solution:

\( (2x, 8y, 2z) \parallel (2, 1, 2) \Rightarrow (2x, 8y, 2z) = \alpha(2, 1, 2) \). Solving for \( x, y \) and \( z \) we get \( x = \alpha, \ y = \alpha/8, \ z = \alpha \) which should determine a point on the surface so that:

\[
\alpha^2 + \frac{\alpha^2}{16} + \alpha^2 = 12 \Rightarrow \alpha^2 = \frac{192}{33} \Rightarrow \alpha = \pm \frac{8}{\sqrt{11}}.
\]

So we get the points \( \left( \frac{8}{\sqrt{11}}, \frac{1}{\sqrt{11}}, \frac{8}{\sqrt{11}} \right) \) and \( \left( -\frac{8}{\sqrt{11}}, -\frac{1}{\sqrt{11}}, -\frac{8}{\sqrt{11}} \right) \).

3. a) Let \( f(x, y) \) and \( g(x, y) \) be differentiable functions. Show that \( \nabla(fg) = f\nabla g + g\nabla f \).

Solution:

\[
\nabla(fg) = (fg_x + gf_x)i + (fg_y + gf_y)j = f(g_x i + g_y j) + g(f_x i + f_y j) = f\nabla g + g\nabla f
\]
b) The plane \( 2y - 3z = 8 \) intersects the cone \( z^2 = 4x^2 + 4y^2 \) in an ellipse. Find the highest and lowest points of intersection.

Solution:

\[
2y - 3z = 8 \Rightarrow z = \frac{2y - 8}{3}. \]

We wish to find the maximum and minimum values for the function
\[
z = f(x, y) = \frac{2y - 8}{3}
\]

subject to the constraint \( z^2 = 4x^2 + 4y^2 \). Hence

\[
\left( \frac{2y - 8}{3} \right)^2 = 4x^2 + 4y^2 \Rightarrow y^2 - 8y + 16 = 9x^2 + 9y^2.
\]

\[
g(x, y) = 9x^2 + 8y^2 + 8y - 16 = 0
\]

\[
\nabla f(x, y) = \lambda \nabla g(x, y)
\]

\[
\frac{2}{3}j = \lambda (18x\mathbf{i} + (16y + 8)\mathbf{j}),
\]

which means

\[
18\lambda x = 0 \quad \text{and} \quad (16y + 8)\lambda = \frac{2}{3}.
\]

So, either \( \lambda = 0 \) or \( x = 0 \). Since \( \lambda = 0 \) is inconsistent with the second equation it follows that \( x = 0 \). Substituting in \( g(x, y) \), we get

\[
y^2 + 8y - 16 = 0 \Rightarrow y^2 + y - 2 = 0 \Rightarrow y = 1, y = -2. \Rightarrow z = f(0, 1) = -2 \text{ and } z = f(0, -2) = -4.
\]

The highest point of intersection is \((0, 1, -2)\) and the lowest point of intersection is \((0, -2, -4)\).

4. Find \( \iint_R \frac{\sin x}{x} \, dA \), where \( R \) is the triangle with vertices \((0, 0), (2, 0)\) and \((2, 2)\). (Sketch the region \( R \).)

Solution:

\[
\iint_R \frac{\sin x}{x} \, dA = \int_0^2 \int_0^x \frac{\sin x}{x} \, dy \, dx = \int_0^2 y \frac{\sin x}{x} \bigg|_{y=0}^{y=x} \, dx = \int_0^2 \sin x \, dx
\]

\[
= -\cos x \bigg|_0^2 = 1 - \cos 2.
\]