1. a) Sketch the region $R$ inside the graph of $r = 3 + \sin \theta$ and outside the graph of $r = 4\sin \theta$.

b) Find the area of the region $R$.

Solution:

![Graph of the region R]

b) $A =$ area of the limaçon $-$ area of the circle

Area of the limaçon $= \frac{1}{2} \int_{0}^{2\pi} (3 + \sin \theta)^2 d\theta = \frac{1}{2} \int_{0}^{2\pi} (9 + 6\sin \theta + \sin^2 \theta) d\theta$

$= \frac{9}{2} \int_{0}^{2\pi} d\theta + 3 \int_{0}^{2\pi} \sin \theta d\theta + \frac{1}{2} \int_{0}^{2\pi} \sin^2 \theta d\theta$

$= \left( \frac{9}{2} \theta - 3 \cos \theta + \frac{1}{4} \theta - \frac{1}{8} \sin 2\theta \right)|_{0}^{2\pi}$

$= 9\pi + \frac{\pi}{2} = 19.5\pi$

Area of the circle $= \frac{1}{2} \int_{0}^{\pi} (4\sin \theta)^2 d\theta$

$= \frac{1}{2} \int_{0}^{\pi} 16 \sin^2 \theta d\theta$

$= 8 \int_{0}^{\pi} \frac{1}{2} (1 - \cos 2\theta) d\theta$

$= (4\theta - 2\sin 2\theta)|_{0}^{\pi} = 4\pi$

Hence we find $A = 19.5\pi - 4\pi$. 
2. a) Find parametric equations for the intersection of the planes \(2x + y - z = 3\) and \(x + 2y + z = 3\).
   b) Find the acute angle between the two planes.

Solution:

a) The normal vectors to the two planes are \(\mathbf{n}_1 = \langle 2, 1, -1 \rangle\) and \(\mathbf{n}_2 = \langle 1, 2, 1 \rangle\) respectively. The line \(L\) of intersection of two planes is parallel to \(\mathbf{n}_1 \times \mathbf{n}_2\).

\[
\mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & -1 \\ 1 & 2 & 1 \end{vmatrix} = 3\mathbf{i} - 3\mathbf{j} + 3\mathbf{k},
\]

or equivalently a vector in the same direction can be taken as: \(\mathbf{i} - \mathbf{j} + \mathbf{k}\).

By inspection the point \((0, 2, -1)\) lies on both planes, so the line has an equation in vector form: \(\mathbf{r}(t) = 2\mathbf{j} - \mathbf{k} + t(\mathbf{i} - \mathbf{j} + \mathbf{k})\). Parametric equations are: \(x = t, y = 2 - t, z = t - 1, \ (t \in \mathbb{R})\).

b) \(\cos \theta = \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{||\mathbf{n}_1|| \cdot ||\mathbf{n}_2||} = \frac{2 + 2 - 1}{\sqrt{6}\sqrt{6}} = \frac{1}{2}\). So, \(\theta = \frac{\pi}{3}\).

3. Let \(C\) be the plane curve parametrized by \(\mathbf{r}(t) = ti + (2t^2 + 3)j\) and let \(l\) be the line parametrized by \(\mathbf{L}(t) = (t + 3)i + (t/4 - 10)j\). Find the point on \(C\) where the tangent line is perpendicular to \(l\).

Solution:

The direction vector for \(\mathbf{L}(t)\) is \(\mathbf{v} = \mathbf{i} + \frac{1}{4}j\). The point sought is the point where \(\mathbf{r}'(t)\) and \(\mathbf{v}\) are orthogonal. \(\mathbf{r}'(t) = \mathbf{i} + 4tj\), we use \(\mathbf{v} \cdot \mathbf{r}'(t) = 0 \Rightarrow \mathbf{v} \cdot \mathbf{r}'(t) = 1 + t = 0 \Rightarrow t = -1\). The point occurs at \(t = -1\) and is therefore the terminal point of \(\mathbf{r}(-1) = -\mathbf{i} + 5\mathbf{j}\) or \((-1, 5)\).

4. a) The curve whose vector equation is \(\mathbf{r}(t) = 2\sqrt{t}\cos t + 3\sqrt{t}\sin t\mathbf{j} + \sqrt{1 - t}\mathbf{k}\) lies on a quadric surface. Find an equation for this surface and identify it.
   b) Find an equation for the sphere with center \((0, 0, 2)\) and \(r = 2\) in spherical coordinates.

Solution:

a) \(x = 2\sqrt{t}\cos t, y = 3\sqrt{t}\sin t, z = \sqrt{1 - t}\).

\[
\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 + z^2 = t + 1 - t
\]

\[
\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 + z^2 = 1,
\]

so the surface is an ellipsoid.

b) Cartesian coordinates: \(x^2 + y^2 + (z - 2)^2 = 4\) or \(x^2 + y^2 + z^2 - 4z = 0\). Using the coordinates: \(\rho^2 = x^2 + y^2 + z^2, z = \rho \cos \phi\)

\[
\rho^2 - 4\rho \cos \phi = 0 \Rightarrow \rho = 4 \cos \phi.
\]