1. Let $\vec{a}$ and $\vec{b}$ be vectors. Show that

(i) $|\vec{a} \cdot \vec{b}| \leq |\vec{a}||\vec{b}|$.

If we have equality, what can you say about $\vec{a}$ and $\vec{b}$?

Solution:

By the definition of dot product we know that

$$|\vec{a} \cdot \vec{b}| = |\vec{a}||\vec{b}| \cos \theta = |\vec{a}||\vec{b}| \cos \theta$$

Since $0 \leq |\cos \theta| \leq 1$

$$|\vec{a} \cdot \vec{b}| \leq |\vec{a}||\vec{b}|$$

If we have equality, $|\cos \theta| = 1$. Thus, $\theta = 0$ or $\theta = \pi$.

When $\theta = 0$, $\vec{a}$ and $\vec{b}$ are parallel and are in the same direction. On the other hand when $\theta = \pi$, $\vec{a}$ and $\vec{b}$ are parallel and are in the opposite direction.

(ii) $|\vec{a} \times \vec{b}| \leq |\vec{a}||\vec{b}|$.

If we have equality, what can you say about $\vec{a}$ and $\vec{b}$?

Solution:

By the definition of cross product we know that

$$|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin \theta$$

Since $-1 \leq \sin \theta \leq 1$

$$|\vec{a} \times \vec{b}| \leq |\vec{a}||\vec{b}|$$

If we have equality, $\sin \theta = 1$. Thus, $\theta = \pi/2$.

Hence $\vec{a}$ and $\vec{b}$ are perpendicular to each other.
2. Find the parametric equations of the line of intersection of the planes,

\[\begin{align*}
3x - 6y - 2z - 15 &= 0 \\
2x + y - 2z - 5 &= 0
\end{align*}\]

Solution:

Let \( P_1 \) be the plane \( 3x - 6y - 2z = 15 \). Then the normal vector for this plane is \( n_1 = \langle 3, -6, -2 \rangle \).

Let \( P_2 \) be the plane \( 2x + y - 2z = 5 \). Then the normal vector for this plane is \( n_2 = \langle 2, 1, -2 \rangle \).

The line of the intersection of two planes is perpendicular to the planes’ normal vectors \( n_1 \) and \( n_2 \), and therefore parallel to \( n_1 \times n_2 \). In other words, the line of intersection is a nonzero scalar multiple of \( n_1 \times n_2 \). In our case,

\[ n_1 \times n_2 = \begin{vmatrix}
\vec{i} & \vec{j} & \vec{k} \\
3 & -6 & -2 \\
2 & 1 & -2
\end{vmatrix} = 14\vec{i} + 2\vec{j} + 15\vec{k} \]

Now, we need to find a point on the line to write the equation of the line. To find a point on the line, we should take a point common to the two planes. So we substitute \( z = 0 \) in the plane equations and solve the equations for \( x \) and \( y \) simultaneously.

\[\begin{align*}
3x - 6y &= 15 \\
2x + y &= 5
\end{align*}\]

So we found the point \( (3, -1, 0) \).

Hence the line is \( x(t) = 3 + 14t, \quad y(t) = -1 + 2t, \quad z(t) = 15t \)

Note that \( (3, -1, 0) \) is not the only intersection point. You may find another intersection point and write the equation of the line according to this point.

3. If \( \vec{r}(t) \) is a differentiable vector-valued function of \( t \) of constant length, then show that \( \vec{r}(t) \perp \vec{r}'(t) \).

Solution:

First consider that \( \frac{d}{dt}(\vec{r}(t) \cdot \vec{r}(t)) = \vec{r}(t) \cdot \frac{d\vec{r}}{dt} + \frac{d\vec{r}}{dt} \cdot \vec{r}(t) = 2\vec{r}(t) \frac{d\vec{r}}{dt} \)

But we know that \( \frac{d}{dt}(\vec{r}(t) \cdot \vec{r}(t)) = \frac{d}{dt} [\|\vec{r}(t)\|^2] \) where \( \|\vec{r}(t)\|^2 \) is a scalar.

So \( \frac{d}{dt} [\|\vec{r}(t)\|^2] = 0 \)

Hence \( \vec{r}(t) \cdot \frac{d\vec{r}}{dt} = 0 \). In other words \( \vec{r}(t) \perp \vec{r}'(t) \).
4. The vector-valued function \( \vec{r}(t) \) is given by \( \vec{r}(t) = \langle \cos t, \sin t, t \rangle \).

(i) Find the arc length from \( t = 0 \) to \( t = 2\pi \).

Solution:

\[
\vec{r}(t) = \langle \cos t, \sin t, t \rangle \quad \text{then} \\
\vec{r}'(t) = \langle -\sin t, \cos t, 1 \rangle \quad \text{and} \\
\|\vec{r}'(t)\| = \sqrt{(-\sin t)^2 + (\cos t)^2 + 1} = \sqrt{2}
\]

Let \( L \) be the arc length of \( \vec{r}(t) \) from \( t = 0 \) to \( t = 2\pi \)

\[
L = \int_0^{2\pi} \|\frac{dr}{dt}\| \, dt = \int_0^{2\pi} \sqrt{2} \, dt = \sqrt{2} \int_0^{2\pi} dt = \sqrt{2} \cdot t \bigg|_{t=0}^{t=2\pi} = 2\sqrt{2}\pi
\]

(ii) Find the principle unit tangent vector \( \vec{T} \).

Solution:

The principle unit tangent vector to the graph of \( \vec{r}(t) \) at \( t \) is \( \vec{T}(t) = \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|} \).

Since \( \vec{r}'(t) = \langle -\sin t, \cos t, 1 \rangle \) and \( \|\vec{r}'(t)\| = \sqrt{2} \) we obtain

\[
\vec{T}(t) = \left\langle -\frac{\sin t}{\sqrt{2}}, \frac{\cos t}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle
\]

(iii) Find the principle unit normal vector \( \vec{N} \).

Solution:

The principle unit normal vector to the graph of \( \vec{r}(t) \) at \( t \) is \( \vec{N}(t) = \frac{\vec{T}'(t)}{\|\vec{T}'(t)\|} \).

Since \( \vec{T}'(t) = \left\langle -\frac{\cos t}{\sqrt{2}}, -\frac{\sin t}{\sqrt{2}}, 0 \right\rangle \) and \( \|\vec{T}'(t)\| = \sqrt{\frac{\cos^2 t}{2} + \frac{\sin^2 t}{2}} = \sqrt{\frac{1}{2}} \)

we obtain

\[
\vec{N}(t) = \langle -\cos t, -\sin t, 0 \rangle = -\cos t \vec{i} - \sin t \vec{j}
\]