1. Use Lagrange multipliers method to find all relative extrema of \( x^2y^2 \) on the ellipse \( 4x^2 + y^2 = 8 \).

Solution:

Let \( f(x, y) = x^2y^2 \) and \( g(x, y) = 4x^2 + y^2 = 8 \). We solve \( \nabla f = \lambda \nabla g \) for \( x \) and \( y \):

\[
2xy^2 = \lambda 8x; \quad 2x^2y = \lambda 2y.
\]

First equation gives either \( x = 0 \) or \( y^2 = 4\lambda \). Second gives \( y = 0 \) or \( x^2 = \lambda \). Either \( (x, y) = (0, 0) \) which is not on the ellipse or \( \lambda = 1, x = \pm 1, y = \pm 2 \). Hence we obtain four points: \((-1, -2), (-1, 2), (1, -2) \) and \((1, 2)\). To show that these points are relative extrema (not saddle points), one can look at the level curves of \( f \), which are hyperbola. At the four points we found, these curves stay at the same side of the ellipse.

2. Let \( \Omega \) be the solid bounded by the half cone \( z = -\sqrt{x^2 + y^2} \) and the circular paraboloid \( z = 2 - x^2 - y^2 \) and suppose \( f : \Omega \to \mathbb{R} \). Express \( \int \int \int _{\Omega} f \, dV \) as an iterated triple integral

(a) in rectangular coordinates in the order \( dz \, dy \, dx \);
(b) in spherical coordinates in the order \( d\rho \, d\phi \, d\theta \);
(c) [Bonus. ] in cylindrical coordinates in the order \( dr \, dz \, d\theta \).

Solution:

The half cone and the paraboloid intersect at points with \( x, y \) satisfying \(-\sqrt{x^2 + y^2} = 2 - x^2 - y^2 \). Set \( u = \sqrt{x^2 + y^2} \) to get \(-u = 2 - u^2 \) so that either \( u = 2 \) or \( u = -1 \). Hence the two surfaces intersect at the circle \( z = -2, x^2 + y^2 = 4 \). (Not at \( z = 1 \!)

See the figure.)

(a) \( \int _{-2}^{2} \int _{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int _{-\sqrt{2-x^2-y^2}}^{\sqrt{2-x^2-y^2}} f \, dz \, dy \, dx \).

(b) Express \( \rho \) on the paraboloid as a function of \( \theta \) and \( \phi \):

\[
z = 2 - x^2 - y^2 \Rightarrow \rho \cos \phi = 2 - \rho^2 \sin^2 \phi \Rightarrow \rho(\phi) = -\frac{\cos \phi + \sqrt{1 + 7 \sin^2 \phi}}{2 \sin^2 \phi}.
\]
Hence, the iterated integral is

\[ \int_0^{2\pi} \int_0^{\frac{3\pi}{4}} \int_0^{\rho(\phi)} \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta \]

where \( \rho(\phi) \) is as above.

(c) \[ \int_0^{2\pi} \int_0^{\frac{3\pi}{4}} f(r) \, dr \, d\phi \, d\theta + \int_0^{2\pi} \int_0^{2} f(r) \, dr \, dz \, d\theta. \]

3. Use the transformation \( u = \frac{1}{2}(x + y), v = \frac{1}{2}(x - y) \) to find

\[ \int \int_T \sin \frac{1}{2}(x + y) \cos \frac{1}{2}(x - y) \, dA \]

over the triangular region \( T \) with vertices \((0,0), (2\pi, 0), (\pi, \pi)\).

Solution:

The region \( T \) is mapped to the triangle \( T' \) in the \( u-v \) world as shown in the figure above. Observing that \( x = u + v \) and \( y = u - v \), the Jacobian \( J = \frac{\partial (x, y)}{\partial (u, v)} = \)
\[
\begin{vmatrix}
1 & 1 \\
1 & -1
\end{vmatrix} = -2. \text{ Hence,}
\]
\[
\int \int_T = \int_0^\pi \int_0^u \sin u \cos v |J| \, dv \, du
\]
\[
= 2 \int_0^\pi \sin u (\sin v)_0^u \, du
\]
\[
= 2 \int_0^\pi \sin^2 u \, du
\]
\[
= \left( u - \frac{1}{2} \sin 2u \right)_0^\pi = \pi
\]

4. Consider the vector field \( \mathbf{F}(x, y) = e^x \cos y \mathbf{i} - e^x \sin y \mathbf{j} \).

(a) Is \( \mathbf{F} \) conservative on \( \mathbb{R}^2 \)?

If yes, find a potential \( \phi \) for \( \mathbf{F} \); check that \( \phi \) is really a potential for \( \mathbf{F} \).

If no, explain the reason in detail, using related theorems.

(b) Find curl \( \mathbf{F} \).

Solution:

(a) \( \mathbf{F} \) is conservative on \( \mathbb{R}^2 \). The potential for \( \mathbf{F} \) is \( \phi(x, y) = e^x \cos y \).

(b) curl \( \mathbf{F} = \text{curl grad } \phi = 0 \).

5. Find the work done by the force field
\[
\mathbf{F}(x, y) = 2xy \mathbf{i} + (x^2 + \cos y) \mathbf{j}
\]
along the curve \( C : r(t) = t \mathbf{i} + t \cos \frac{t}{3} \mathbf{j}, 0 \leq t \leq \frac{\pi}{2} \).

Solution:

Note that \( \mathbf{F} \) is conservative with a potential \( \phi = x^2 y + \sin y \). Hence the work done is
\[
W = \phi\left( \frac{\pi}{2}, \frac{\pi}{2} \cos \frac{\pi}{6} \right) - \phi(0, 0) = \phi\left( \frac{\pi}{2}, \frac{\sqrt{3}\pi}{4} \right) - \phi(0, 0) = \frac{3\pi^3}{64} + \sin \frac{\sqrt{3}\pi}{4}.
\]

6. Consider the vector field
\[
\mathbf{F}(x, y) = \frac{1 - y}{x^2 + (y - 1)^2} \mathbf{i} + \frac{x}{x^2 + (y - 1)^2} \mathbf{j}
\]
and the curves \( C, C_1 \) and \( C_2 \).
(a) Is it always true that \( \int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C} \mathbf{F} \cdot d\mathbf{r} \)? Why?

Solution:

Let \( \mathbf{F} = \langle f, g \rangle \). Observe that \( f_y = g_x \). By theorem, \( \mathbf{F} \) is conservative on any simply connected domain. Since \( \mathbf{F} \) is not defined at \( P = (0, 1) \), if a domain contains \( P \), it must be deleted. One can find a simply-connected domain, as \( \Omega_1 \) in the figure below, which contains \( C \) and \( C_1 \) and does not contain \( P \). On this domain \( \mathbf{F} \) is conservative by theorem, therefore \( \int_{C_1} = \int_{C} \).

(b) Is it always true that \( \int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_{C} \mathbf{F} \cdot d\mathbf{r} \)? Why?

Solution:

Not necessarily. Since every domain containing \( C \) and \( C_2 \) contains \( P \) as well, we have no chance to find a simply connected domain containing the curves but not \( P \). Therefore we cannot apply the theorem. Hence we cannot be sure about the equality. Indeed, the equality is false in general.

7. Let \( C \) be the boundary of the region enclosed between \( y = x^2 \) and \( y = 2x \). Assuming that \( C \) is oriented counterclockwise, use Green's theorem to evaluate \( \oint_C (6xy - y^2)dy \).

Solution:

The region \( \Omega \) is simply connected with a connected, smooth, simple boundary. Set \( \mathbf{F} = 0 \mathbf{i} + (6xy - y^2) \mathbf{j} = \langle f, g \rangle \). Then

\[
\oint_C (6xy - y^2)dy = \oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_{\Omega} \frac{\partial g}{\partial x} dA = \int_0^2 \int_x^{2x} 6y \, dy \, dx = \frac{64}{5}.
\]