1. Consider the polar curves $C_1 : r = f(\theta) = \sqrt{2}\sin \theta$ and $C_2 : r = g(\theta) = \sqrt{\sin 2\theta}$.

5 pts a. Find the $\theta_0 \in (0, \frac{\pi}{2})$ where $C_1$ and $C_2$ intersect.

\[ f(\theta) = g(\theta) \] gives intersections:

\[ \sqrt{2} \sin \theta = \sqrt{\sin 2\theta} = \sqrt{2 \sin \theta \cos \theta} = \sqrt{2} \sqrt{\sin \theta \cos \theta} \]

\[ \sin \theta > 0 \text{ on } (0, \frac{\pi}{2}) \Rightarrow \sqrt{\sin \theta} = \sqrt{2} \cos \theta \]

\[ \Rightarrow \theta = \frac{\pi}{4} \text{ is the only } \theta \in (0, \frac{\pi}{2}) \]

5 pts b. Show that $g(\theta) > f(\theta)$ for all $\theta \in (0, \theta_0)$, for $\theta_0$ found above.

\[ \text{On } (0, \frac{\pi}{4}), \text{ obviously } \cos \theta > \sin \theta \Rightarrow \sqrt{\cos \theta} > \sqrt{\sin \theta} \]

then

\[ g(\theta) = \sqrt{2} \sqrt{\sin \theta \cos \theta} > \sqrt{2} \sqrt{\sin \theta \sin \theta} = \sqrt{2} \sin \theta = f(\theta) \]
15 pts c. Sketch the graphs of $C_1$ and $C_2$ on the same $xy$-plane. Hint: Don't hesitate to use the information obtained in part a and b.

$C_1: \quad r = \frac{1}{\sin \theta} \sin \theta = f$

$\theta = 0, \theta = 2\pi \Rightarrow r = 0$

$\theta = \frac{\pi}{2} \Rightarrow r = \frac{1}{2}$

$\theta = \frac{3\pi}{2} \Rightarrow r = -\frac{1}{2}$

$\pi < \theta < 2\pi \Rightarrow r < 0$

$r \rightarrow (0, \frac{1}{2}) \cup (\frac{3\pi}{2}, 2\pi)$

$r \rightarrow (\frac{\pi}{2}, \frac{3\pi}{2})$

$C_2: \quad r = \sqrt{2} \sin \theta = g$, $\sin 2\theta > 0$ must hold

$0 \leq 2\theta \leq \pi$ and $2\pi \leq 2\theta \leq 3\pi$

$0 \leq \theta \leq \frac{\pi}{2}$ and $\pi \leq \theta \leq \frac{3\pi}{2}$

no graph in 2nd and 4th quadrants.

$r = 0 \iff \theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$ minim

$r = 1 \iff \theta = \frac{\pi}{4}, \frac{3\pi}{4}$ maxim

$r \rightarrow (0, \frac{\pi}{4}) \cup (\frac{\pi}{2}, \frac{3\pi}{4})$, else r =

10 pts d. Find the area of the region inside both $C_1$ and $C_2$.

Desired region zoomed:

Thus,$$
A = \frac{1}{2} \int_0^{\frac{\pi}{4}} 2 \sin^2 \theta \, d\theta + \frac{1}{2} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin 2\theta \, d\theta = \frac{1}{2} \int_0^{\frac{\pi}{4}} \frac{1 - \cos 2\theta}{2} \, d\theta + \frac{1}{2} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin 2\theta \, d\theta
$$

$$
= \frac{1}{2} \left[ \theta - \frac{1}{2} \sin 2\theta \right]_0^{\frac{\pi}{4}} + \frac{1}{2} \left[ -\frac{1}{4} \cos 2\theta \right]_{\frac{\pi}{4}}^{\frac{\pi}{2}} = \frac{1}{2} \left[ \frac{\pi}{4} - \frac{1}{2} \right] - \frac{1}{4} \left[ -1 \right] = \frac{\pi}{8}
$$
2. Let \( l_1 \) be the line \( x = 1 + t, y = -1 + t, z = t \), and \( D \) be the plane through the points \( P(0,0,0), Q(3,2,0) \) and \( R(0,1,-2) \).

10 pts a. Write down an equation for the plane \( D \).

We need a normal vector \( \vec{n} \) for \( D \). Construct two vectors on \( D \):
\[
\vec{PQ} = (3,2,0) \quad \text{and} \quad \vec{PR} = (0,1,-2).
\]
Then \( \vec{PQ} \times \vec{PR} \) (or \( \vec{PR} \times \vec{PQ} \)) serves as \( \vec{n} \).

\[
\begin{vmatrix}
3 & 2 & 0 \\
0 & 1 & -2 \\
-4 & -6 & 3
\end{vmatrix} = 2(-4) - 2(-6) + 1(3) = -8 + 12 + 3 = 7.
\]

Any vector on \( D \) is \( (x-0, y-0, z-0) \) using \( P \) (or can use \( Q \) or \( R \) well) \( D \) is described by \( (x, y, z) \cdot (-4, 6, 3) = 0 \)
\[
\Rightarrow -4x + 6y + 3z = 0.
\]

10 pts b. Find the intersection of the plane \( D \) and the line \( l_1 \).

\( l_1 \) hits \( D \) at a pt if \( x = 1 + t, y = -1 + t, z = t \) satisfy the plane eqn found above:
\[
-4(1+t) + 6(-1+t) + 3t = 0
\]
\[
\Rightarrow -10 + 5t = 0 \Rightarrow t = 2 \quad \text{yielding the pt } (1+t, 1+t, 16) = (3, 3, 2).
\]

15 pts c. Write down an equation for the line \( l \) satisfying both of the following:

(i) \( l \) lies in the plane \( D \); (ii) \( l \) and \( l_1 \) intersect perpendicularly.

Since \( l \) lies in \( D \) and \( l \cap l_1 \neq \emptyset \) and \( l \perp l_1 \), we necessarily have \( l \cap l_1 = \{(3,1,2)\} \), \( \text{i.e. } (3,1,2) \in l \).

\( l \subset D \Rightarrow \vec{n} \perp l \), \( l \perp l_1 \Rightarrow \langle 1,1,1 \rangle \perp l \)

Then \( \vec{n} \times \langle 1,1,1 \rangle \) can serve as the direction of \( l \)
\[
\begin{vmatrix}
7 & 3 & 0 \\
0 & 1 & -2 \\
-4 & 6 & 3
\end{vmatrix} = 7(-4-3) - 3(-46) + 0 = 3\frac{z}{z} + 7\frac{y}{3} - 10\frac{x}{3} = \langle 3, 7, -10 \rangle
\]

Therefore \( l : \langle x(t), y(t), z(t) \rangle = \langle 3,1,2 \rangle + t \langle 3, 7, -10 \rangle \)
\[
\Rightarrow l: \quad x = 3 + 3t \quad y = 1 + 7t \quad z = 2 - 10t \quad \text{is the derived line}
\]
3. Let \( \vec{r}(t) = (\cos t, \sin t, t) \) and \( \vec{q}(t) = (\sin t, 0, \cos t) \) be parametrisations of two curves in \( \mathbb{R}^3 \).

15 pts a. Find all common points of these curves. Be sure that there is no more!

\( \vec{r}(t) \) and \( \vec{q}(t) \) intersect at pt if there exist two numbers \( t_1 \) and \( t_2 \) at \( \vec{r}(t_1) = \vec{q}(t_2) \). Componentwise we have:

\[
\begin{align*}
\cos t_1 &= \sin t_2 & \Rightarrow & \cos k\pi = \sin t_2 \\
\sin t_1 &= 0 & \Rightarrow & t_1 = k\pi, k \in \mathbb{Z} \\
t_1 &= \cos t_2 & \Rightarrow & k\pi = \cos t_2 \Rightarrow \cos t_2 = 0 \Rightarrow t_2 = \frac{\pi}{2} + k\pi
\end{align*}
\]

But \( |\cos t_2| \leq 1 \) and \( |k\pi| \leq 1 \) only when \( k = 0 \Rightarrow t_1 = 0 \)

Now lastly we use the 1st eqn:

\[
\cos 0 = \sin \left( \frac{\pi}{2} + k\pi \right), k \in \mathbb{Z}
\]

\[
1 = \sin \left( \frac{\pi}{2} + k\pi \right) \Rightarrow k \text{ is even}
\]

\( \Rightarrow \vec{r}(t) \) and \( \vec{q}(t) \) intersect when \( t_1 = 0 \), \( t_2 = \frac{\pi}{2} + k\pi \) and the pts of intersection are:

\( \vec{r}(t_1) = (1, 0, 0) \) giving only one pt since \( \vec{r} \) is periodic with period \( 2\pi \).

Only one intersection pt: \( (1, 0, 0) \).

15 pts b. At each common point, find the angle between the tangent lines to these curves.

The angle between tangent lines is the angle between directions of tangent lines, and directions are given by velocity vectors \( \frac{d\vec{r}}{dt} \) and \( \frac{d\vec{q}}{dt} \).

\[
\frac{d\vec{r}}{dt} = \langle -\sin t, \cos t, 1 \rangle \Rightarrow \frac{d\vec{r}}{dt} \bigg|_{(1,0,0)} = \langle 0, 1, 1 \rangle = \vec{d}_r
\]

\( (1,0,0) \) or \( t = 0 \)

\[
\frac{d\vec{q}}{dt} = \langle \cos t, 0, -\sin t \rangle \Rightarrow \frac{d\vec{q}}{dt} \bigg|_{(1,0,0)} = \langle 0, 0, -1 \rangle = \vec{d}_q
\]

\( (1,0,0) \) or \( t = \frac{\pi}{2} + 2k\pi \)

Using \( \vec{u} \cdot \vec{v} = ||\vec{u}|| ||\vec{v}|| \cos \theta \) we get:

\[
\vec{d}_r \cdot \vec{d}_q = -1 = \sqrt{2} \cdot 1 \cdot \cos \theta \Rightarrow \theta = \frac{3\pi}{4}
\]

Note: since we can take \( -\vec{d}_r \) or \( -\vec{d}_q \) as direction vectors, obviously \( \theta = \frac{\pi}{4} \) is also a correct answer.