1. Find a unit normal vector and an equation for the tangent plane for the surface $x^2 + y^2 = 3z$ at the point $(1, 3, 10/3)$

Solution:

Let $F(x, y, z) = x^2 + y^2 - 3z$

We know that $\nabla F$ is normal to the level surface $F = 0$

$\nabla F = 2x \hat{i} + 2y \hat{j} - 3 \hat{k}$

At $(1, 3, 10/3)$ we get $\nabla F = 2 \hat{i} + 6 \hat{j} - 3 \hat{k}$, $|\nabla F| = 7$

$\hat{N} = \frac{1}{7}(2 \hat{i} + 6 \hat{j} - 3 \hat{k})$

Eqn. for the tangent plane $\nabla F(\vec{a}_0) \cdot (\vec{a} - \vec{a}_0) = 0$

$\vec{a}_0 = (1, 3, 10/3)$, $\vec{a} = (x, y, z)$

$2(x - 1) + 6(y - 3) - 3(z - 10/3) = 0$

$2x + 6y - 3z = 10$

2. Let $f(x, y) = (x^2 + y - 2)^4 + (x - y + 2)^3$

(a) Find the total differential $df$ at the point $(1, -2)$

Solution:

$$df(\vec{a}_0) = f_x(\vec{a}_0)dx + f_y(\vec{a}_0)dy, \quad \vec{a}_0 = (1, -2)$$

$$f_x = 8(x^2 + y - 2)^3x + 3(x - y + 2)^2$$

$$f_y = 4(x^2 + y - 2)^3 - 3(x - y + 2)^2$$

$$f_x(1, -2) = 8(-27) + 3(25) = -141$$

$$f_y(1, -2) = 4(-27) - 3(25) = -183$$

$$df(1, -2) = -141dx - 183dy$$

(b) Let $x = u - 2v + 1, y = 2u + v - 2$. Using the chain rule find $\frac{\partial f}{\partial v}$ when $u = 0, v = 0$
Solution:

\[
\frac{\partial f}{\partial x} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial v}; \quad f_v = f_x x_v + f_y y_v
\]

\[
u = 0, \quad v = 0 \Rightarrow x = 1, \quad y = -2
\]

\[
x_v = -2, \quad y_v = 1
\]

\[
\frac{\partial f}{\partial v}(u = 0, \quad v = 0) = -2f_x(1, -2) + f_y(1, -2)
\]

\[
= -2(-141) - 183
\]

\[
= 99
\]

3. A flat circular plate has the shape of the region \(x^2 + y^2 \leq 1\). The plate, including the boundary where \(x^2 + y^2 = 1\), is heated so that the temperature at any point \((x, y)\) is \(T(x, y) = x^2 + 2y^2 - x\).

(a) In which direction \(T(x, y)\) decreases fastest at the origin?

Solution:

\[
\nabla T = (2x - 1) \hat{i} + 4y \hat{j}, \quad \nabla T(0, 0) = -\hat{i}
\]

\(T\) decreases fastest in the \(-\nabla T\) direction. So \(-\nabla T = \hat{i} \Rightarrow \text{along the } x - \text{axis}\)

(b) Find the hottest and the coldest points on the plate and the temperature at each of these points.

Solution:

Critical points: \(T_x = 0, \quad T_y = 0 \Rightarrow x = 1/2, \quad y = 0\)

\[
T_x = 2x - 1, \quad T_{xx} = 2, \quad T_y = 4y, \quad T_{yy} = 4, \quad T_{xy} = 0
\]

\(\Rightarrow A = 2, \quad B = 0, \quad C = 4, \quad D = B^2 - AC = -8\)

\(D < 0, \quad A > 0 \Rightarrow T(1/2, 0) = -1/4, \text{ relative minimum}\)

Must check the behavior at the boundary:

\[
T(x) = T(x, y(x)) = 2 - x^2 - x, \quad y(x) = \pm \sqrt{1 - x^2}
\]

\[
T'(x) = -2x - 1 = 0 \Rightarrow x = -1/2 \Rightarrow y = \pm \frac{\sqrt{3}}{2}
\]

\[
T''(x) = -2 \quad \Rightarrow \text{maxima}
\]

\[
T(-1/2) = 9/4
\]

Hottest points: \((-1/2, \pm \frac{\sqrt{3}}{2}) \Rightarrow T = 9/4\)

Coldest points: \((1/2, 0) \Rightarrow T = -1/4\)

(Boundary would be handled by Lagrange multiplier method which gives \((\pm 1, 0)\) as minima at boundary)

4. (a) Find the volume of the solid bounded by the surface \(z = 6 - xy\) and the planes \(x = 2, \quad x = -2, \quad y = 0, \quad y = 3, \quad \text{and } z = 0\)

Solution:

\[
V = \int_{-2}^{2} \int_{0}^{3} (6 - xy)dy \, dx = \int_{-2}^{2} (18 - \frac{9}{2}x)dx
\]

\[
= \left[ 18x - \frac{9}{4}x^2 \right]_{-2}^{2} = 72
\]
(b) Evaluate the double integral: \( I = \int_0^1 \int_{3y}^3 e^{x^2} \, dx \, dy \)

Solution:

So \( dx \, dy \) can be converted by the help of figure above, to \( dy \, dx \)

\[
I = \int_0^1 \int_{3y}^3 e^{x^2} \, dy \, dx
\]

\[
I = \frac{1}{3} \int_0^3 x e^{x^2} \, dx = \frac{1}{6} \left[ e^{x^2} \right]_0^3
\]

\[
I = \frac{1}{6} (e^9 - 1)
\]