1. Change the order of integration in the following integrals.

(a) $\int_{-1}^{1} \int_{-x}^{1-x} f(x, y) \, dy \, dx$.

**Solution:**

From the sketch we see that to consider $dx \, dy$ order of integration we divide the region into three parts.

$$R_1 : \int_1^2 \int_{-1}^{1-y} f(x, y) \, dx \, dy; \quad R_2 : \int_0^1 \int_{-y}^{1-y} f(x, y) \, dx \, dy; \quad R_3 : \int_{-1}^0 \int_{-y}^{1-y} f(x, y) \, dx \, dy.$$\[\Rightarrow \int_{-1}^{1} \int_{-x}^{1-x} f(x, y) \, dy \, dx = \int_1^2 \int_{-1}^{1-y} f(x, y) \, dx \, dy + \int_0^1 \int_{-y}^{1-y} f(x, y) \, dx \, dy + \int_{-1}^0 \int_{-y}^{1-y} f(x, y) \, dx \, dy.\]

(b) $\int_0^\pi \int_0^{2\sin \theta} r \, dr \, d\theta$.

**Solution:**

One can solve the problem either in $xy$-plane or in $\theta r$-plane. Let us consider the curve $r = 2 \sin \theta$ in $xy$-plane. Using the relations $r^2 = x^2 + y^2$, $y = r \sin \theta$, we get $x^2 + (y - 1)^2 = 1$. From this equation and the given limits of integration, we have a
region as shown above. Now, \(2 \sin \alpha_1 = r\) and \(2 \sin \alpha_2 = r\). As we see \(\alpha_2 > \pi/2 > \alpha_1\) and \(\alpha_2 = \pi - \alpha_3\). Therefore, \(\alpha_1 = \sin^{-1}(r/2)\) and \(\alpha_2 = \pi - \sin^{-1}(r/2)\). (Note that \(\sin^{-1}\) is defined for \(-\pi/2 \leq \theta \leq \pi/2\). It follows that

\[
\int_0^\pi \int_0^{2\sin \theta} dr d\theta = \int_0^2 \int_{\sin^{-1}(r/2)}^{\pi} d\theta dr.
\]

2. Analyze the behavior of \(f(x, y) = x^5 y + xy^5 + xy\) at its critical points.

**Solution:**

The first partial derivatives are

\[
\frac{\partial z}{\partial x} = 5x^4 y + y^5 + y = y(5x^4 + y^4 + 1)
\]

and

\[
\frac{\partial z}{\partial y} = x(5y^4 + x^4 + 1).
\]

The terms \(5x^4 + y^4 + 1\) and \(5y^4 + x^4 + 1\) are always greater than or equal to 1, and so it follows that the only critical point is \((0, 0)\).

The second partial derivatives are

\[
\frac{\partial^2 z}{\partial x^2} = 20x^3 y, \quad \frac{\partial^2 z}{\partial y^2} = 20xy^3
\]

and

\[
\frac{\partial^2 z}{\partial x \partial y} = 5x^4 + 5y^4 + 1.
\]

Thus at \((0, 0)\), \(D = -1\) and so \((0, 0)\) is a saddle point.

3. Find the maximum and minimum values of the function \(f(x, y) = x^2 + y^2 - x - y + 1\) in the disk \(D\) defined by \(x^2 + y^2 \leq 1\).

**Solution:**

(i) To find the critical points we set \(\partial f/\partial x = \partial f/\partial y = 0\). Thus, \(2x - 1 = 0, 2y - 1 = 0\), and hence \((x, y) = (\frac{1}{2}, \frac{1}{2})\) is the only critical point in the open disk \(U = \{(x, y) : x^2 + y^2 < 1\}\).

(ii) The boundary \(\partial U\) can be parametrized by \(c(t) = (\sin t, \cos t), 0 \leq t \leq 2\pi\). Thus

\[
f(c(t)) = \sin^2 t + \cos^2 t - \sin t - \cos t + 1 = 2 - \sin t - \cos t = g(t).
\]

To find the maximum and minimum of \(f\) on \(\partial U\) it suffices to locate the maximum and minimum of \(g\). Now \(g'(t) = 0\) only when

\[
\sin t = \cos t, \text{that is, } t = \frac{\pi}{4}, \frac{5\pi}{4}.
\]
Thus the candidates for the maximum and minimum for $f$ and $\partial U$ are the points $c(\frac{\pi}{4}), c(\frac{3\pi}{4})$ and the endpoints $c(0) = c(2\pi)$.

(iii) The values of $f$ at the critical points are: $f(\frac{1}{2}, \frac{1}{2}) = \frac{1}{2}$ from step (i) and, from step (ii),

$$f(c(\frac{\pi}{4})) = f(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}) = 2 - \sqrt{2},$$

$$f(c(\frac{5\pi}{4})) = f(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}) = 2 + \sqrt{2},$$

and

$$f(c(0)) = f(c(2\pi)) = f(0, 1) = 1.$$  

(iv) Comparing all the values $\frac{1}{2}, 2 - \sqrt{2}, 2 + \sqrt{2}, 1$, it is clear that the absolute minimum occurs at $(\frac{1}{2}, \frac{1}{2})$ and the absolute maximum occurs at $(-\sqrt{2}/2, -\sqrt{2}/2)$.

4. Let $C$ be an oriented simple curve connecting $(1, 1, 1)$ and $(1, 2, 4)$. Evaluate $\int_C 2xyz \, dx + x^2z \, dy + x^2y \, dz$. Justify your evaluation.

Solution:

Observe that $\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k} = 2xyz\vec{i} + x^2z\vec{j} + x^2y\vec{k}$ is conservative with potential $\phi(x, y, z) = x^2yz$, i.e. $\frac{\partial \phi}{\partial x} = F_1, \frac{\partial \phi}{\partial y} = F_2$ and $\frac{\partial \phi}{\partial z} = F_3$. By Fundamental Theorem of Work Integrals, it follows that

$$\int_C 2xyz \, dx + x^2z \, dy + x^2y \, dz = \phi(1, 2, 4) - \phi(1, 1, 1) = 1 \cdot 2 \cdot 4 - 1 \cdot 1 \cdot 1 = 7$$

5. Let $R$ be the region the the first quadrant of $xy$-plane bounded by the curves $y = x^2$, $y = 3x^2$, $y = x$ and $y = 2x$. Evaluate $\iint_R x^2 \, dx \, dy$. Hint: You might try an appropriate change of variables.

Solution:
Consider the transformation

\[ T : \mathbb{R}^2 \to \mathbb{R}^2, \quad (u, v) \to (x = \frac{v}{u}, y = \frac{v^2}{u}) \]

so that \( v = \frac{u}{x} \) and \( u = \frac{y}{x^2} \). With this transformation the rectangle enclosed by the lines \( u = 1, u = 3, v = 1, v = 2 \) is mapped onto the shaded region in \( xy \)-plane. The corresponding Jacobian is:

\[
\frac{\partial (x, y)}{\partial (u, v)} = \begin{vmatrix} \frac{-v}{u^2} & \frac{1}{u} \\ \frac{-v^2}{u^3} & \frac{2v}{u} \end{vmatrix} = -2 \frac{v^2}{u^3} + \frac{v^2}{u^3} = -\frac{v^2}{u^3}.
\]

Note that there is a minus sign since the transformation \( T \) reverses orientation. Then,

\[
\int \int_R x^2 \, dx \, dy = \int_1^3 \int_1^{v_2} \frac{v^2}{u^2} \left( -\frac{v^2}{u^3} \right) dv \, du = -\left[ \frac{1}{4} u^{-4} \right]_1^3 \cdot \left[ \frac{1}{5} v^5 \right]_1^{v_2} = \frac{124}{81}.
\]

6. Let \( C \) be the boundary of the square with corners \((1, 1), (2, 1), (2, 2), (1, 2)\), oriented counterclockwise. Compute

\[
\int_C (x^2 + y) \, dx + (2x + y^2) \, dy.
\]

Solution:

Since (a) the enclosed square \( S \) is simply connected, (b) its boundary is a simple and closed curve and (c) \( f = x^2 + y \) and \( g = 2x + y^2 \) are differentiable over \( S \), one can use Green’s Theorem to change this line integral into a double integral over \( S \):

\[
\int_C = \int_1^2 \int_1^{v_2} \left( \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) \, dy \, dx = \int_1^2 \int_1^{v_2} (2 - 1) \, dy \, dx = 1.
\]

7. Let \( \vec{A}, \vec{B}, \vec{C} \) be the vectors in 3-space. Assume that \( \vec{A} \) is orthogonal to neither \( \vec{B} \) nor \( \vec{C} \).

(a) What is the angle between \( \vec{A} \times (\vec{B} \times \vec{C}) \) and \( \text{proj}_{\vec{B} \times \vec{C}} \vec{A} \)?

Solution:

Either (i) \( \vec{A} \) is normal to \( \vec{B} \times \vec{C} \) such that \( \text{proj}_{\vec{B} \times \vec{C}} \vec{A} \) is the zero vector and the angle is undefined;

or (ii) the nonzero projection \( \text{proj}_{\vec{B} \times \vec{C}} \vec{A} = \frac{\vec{A} \cdot (\vec{B} \times \vec{C})}{||\vec{B} \times \vec{C}||^2} \vec{B} \times \vec{C} \) is parallel to \( \vec{B} \times \vec{C} \), hence orthogonal to \( \vec{A} \times (\vec{B} \times \vec{C}) \) by the definition of cross product. In this case, the angle sought is \( \pi/2 \) radians.

(b) If \( \vec{A} \neq \vec{0} \), then does \( \vec{A} \times \vec{B} = \vec{A} \times \vec{C} \) imply \( \vec{B} = \vec{C} \)? Justify your answer.

Solution:

No, for example, let \( \vec{A} = \vec{i}, \vec{B} = \vec{i} + \vec{j} \) and \( \vec{C} = 2\vec{i} + \vec{j} \). Then \( \vec{A} \) is not orthogonal to \( \vec{B} \), nor to \( \vec{C} \), and \( \vec{A} \times \vec{B} = \vec{i} \times \vec{i} + \vec{i} \times \vec{j} = \vec{0} + \vec{k} = \vec{k} = 2\vec{0} + \vec{k} = \vec{i} \times 2\vec{i} + \vec{i} \times \vec{j} = \vec{A} \times \vec{C} \), but \( \vec{B} = \vec{i} + \vec{j} \neq 2\vec{i} + \vec{j} = \vec{C} \).
8. Evaluate \( \lim_{(x,y) \to (2,1)} \frac{\sin^{-1}(xy - 2)}{\tan^{-1}(3xy - 6)} \).

Solution:

Let \( a = xy - 2 \). Then \( (x, y) \to (2, 1) \) implies \( a \to 0 \). So

\[
\lim_{(x,y) \to (2,1)} \frac{\sin^{-1}(xy - 2)}{\tan^{-1}(3xy - 6)} = \lim_{a \to 0} \frac{\sin^{-1}(a)}{\tan^{-1}(3a)}
\]

which leads to indeterminacy. After application of L’Hôpital’s rule we get:

\[
\lim_{a \to 0} \frac{\sin^{-1}(a)}{\tan^{-1}(3a)} = \lim_{a \to 0} \frac{1 + 9a^2}{3 \sqrt{1 - a^2}} = \frac{1}{3}.
\]

9. **Bonus question.** (a) Let \( \mathbf{F} = 4x \mathbf{i} + 4y \mathbf{j} + 2 \mathbf{k} \). Let \( S \) be the surface that is the bottom of the paraboloid \( z = x^2 + y^2 \) with \( z \leq 1 \). Find the outward flux away from the \( z \)-axis of \( \mathbf{F} \) through \( S \).

(b) Let \( \Omega \) be the solid inside \( z = x^2 + y^2 \) and below \( z = 1 \). Let \( \mathbf{F} \) be as above. Evaluate \( \iiint_{\Omega} \nabla \cdot \mathbf{F} \, dV \).

Are the results in part (a) and (b) equal? Why? [You may answer this question without solving (a) and (b).]

Solution:

(a) The surface \( S \) is parametrized as \( x = u, \ y = v, \ z = u^2 + v^2 \). A normal to \( S \) is at \( (u, v, u^2 + v^2) \) is \( \mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 2u \\ 0 & 1 & 2v \end{vmatrix} = -2u \mathbf{i} - 2v \mathbf{j} + \mathbf{k} \). This vector points towards the \( z \)-axis, not away from the \( z \)-axis. A normal vector away from the \( z \)-axis is \( 2u \mathbf{i} + 2v \mathbf{j} - \mathbf{k} \). The flux is given by the surface integral of \( \mathbf{F} \), namely it is equal to

\[
\iint_S \mathbf{F} \cdot \mathbf{n} \, dS = \iint_{y^2 + z^2 \leq 1} (4u \mathbf{i} + 4v \mathbf{j} + 2 \mathbf{k}) \cdot (2u \mathbf{i} + 2v \mathbf{j} - \mathbf{k}) \, dA
\]

\[
= \iint_{y^2 + z^2 \leq 1} (8u^2 + 8v^2 - 2) \, dA
\]

\[
= \int_0^{2\pi} \int_0^1 (8r^2 - 2) r \, dr \, d\theta
\]

\[
= 2\pi [2r^4 - r^2]_0^1 = 2\pi.
\]

(b) We have \( \nabla \cdot \mathbf{F} = (\frac{\partial}{\partial x} 4x \mathbf{i} + \frac{\partial}{\partial y} 4y \mathbf{j} + \frac{\partial}{\partial z} 2 \mathbf{k}) \cdot (4x \mathbf{i} + 4y \mathbf{j} + 2 \mathbf{k}) = \frac{\partial}{\partial x} 4x + \frac{\partial}{\partial y} 4y + \frac{\partial}{\partial z} 2 = 4 + 4 + 0 = 8 \), so

\[
\iiint_{\Omega} \nabla \cdot \mathbf{F} \, dV = \iiint_{\Omega} 8 \, dV = 8 \iiint_{\Omega} \, dV
\]

\[
= 8 \int_{x^2 + y^2 \leq 1} \int_{y^2 + z^2 \leq 1} \, dz \, dA
\]

\[
= 8 \int_{x^2 + y^2 \leq 1} \int_{y^2 + z^2 \leq 1} [1 - (x^2 + y^2)] \, dA
\]

\[
= 8 \int_0^{2\pi} \int_0^1 [1 - r^2] r \, dr \, d\theta
\]

\[
= 8 \int_0^{2\pi} \left[ \frac{r^2}{2} - \frac{r^4}{4} \right]_0^1 \, d\theta
\]

\[
= 8 \cdot 2\pi \cdot \frac{1}{4} = 4\pi.
\]
(c) The results in (a) and (b) are not equal. In fact, by the divergence theorem, the integral over \( \Omega \) is equal to the surface integral of \( \vec{F} \) over the whole surface enclosing \( \Omega \), and this surface consists of \( S \) and the disk \( D = \{(x, y, z) : x^2 + y^2 \leq 1, \ z = 1\} \). Since
\[
\iiint_{\Omega} \vec{\nabla} \cdot \vec{F} \, dV = \iint_{S} \vec{F} \cdot \vec{n} \, dS + \iint_{D} \vec{F} \cdot \vec{n} \, dS
\]
and since \( \vec{F} \) has a positive component in the direction of \( \vec{k} \), which is \( \vec{n} \) on \( D \), the surface integral over \( D \) is positive, not zero, and the results cannot be equal. In more detail, we have
\[
\iint_{D} \vec{F} \cdot \vec{n} \, dS = \iint_{u^2 + u^2 \leq 1} (4u\vec{i} + 4v\vec{j} + 2\vec{k}) \cdot \vec{k} \, dA
\]
\[
= 2 \iint_{u^2 + u^2 \leq 1} dA
\]
\[
= 2 \text{(area of the unit disk)} = 2 \cdot \pi 1^2 = 2\pi \neq 0.
\]