1. Describe or sketch the surface in $\mathbb{R}^3$ whose equation in spherical coordinates is $\rho = 4 \cos \varphi$.

Solution:

Multiply each side by $\rho$; in rectangular coordinates, we have:

$$x^2 + y^2 + z^2 = 4z$$
$$\Rightarrow x^2 + y^2 + (z - 2)^2 = 4$$

which describes a sphere in $\mathbb{R}^3$ with center at $(0,0,2)$ and radius 2.

2. Answer the questions below. In each case, explain your reasoning.

(a) The curves $r = \theta$, $r = -\theta$, $r = \sqrt{\theta}$, $r = -\sqrt{\theta}$ (in a polar coordinate system whose polar axis coincides with the positive $x$-axis in the usual way) are depicted below in some order. Find out which curve has which equation.

![Curves](image)

Solution:

Set $f(\theta) = \theta$. Then $f(\theta + 2\pi) = f(\theta) + 2\pi$, i.e. after each $2\pi$ rotation around the origin, $f(\theta)$ increases by $2\pi$. Since in the second figure, for $\theta$ close to 0, $r$ is positive, the second figure is the graph of $r = f(\theta) = \theta$ while the third figure is that of $r = -\theta$. First and last figures correspond to $r = \sqrt{\theta}$ and $r = -\sqrt{\theta}$ respectively because $2\pi$ rotations around the origin yield non-uniform increase in $r$.

(b) Are $\vec{u} \cdot \vec{v}$ and $\vec{u} \times \vec{v}$ orthogonal to each other? Why?

Solution:

Orthogonality of a real number and a vector is not defined. This question does not make sense.

(c) Find the vector component of $\vec{u} = \langle 3, -4, 4 \rangle$ orthogonal to $\vec{a} = \langle 2, 2, 1 \rangle$.

Solution:
Express \( \mathbf{u} \) as a sum of two vectors: \( \mathbf{u} = \mathbf{u}_a + \mathbf{u}_\perp \) where \( \mathbf{u}_a \) is the component of \( \mathbf{u} \) along \( \mathbf{a} \) and \( \mathbf{u}_\perp \) the component perpendicular to \( \mathbf{a} \). Then,

\[
\mathbf{u}_\perp = \mathbf{u} - \mathbf{u}_a = \mathbf{u} - \text{proj}_a \mathbf{u} = \langle 3, -4, 4 \rangle - \frac{\mathbf{u} \cdot \mathbf{a}}{||\mathbf{a}||^2} \mathbf{a} = \langle 3, -4, 4 \rangle - \frac{2}{9} \langle 2, 2, 1 \rangle = \left\langle \frac{23}{9}, \frac{-40}{9}, \frac{34}{9} \right\rangle
\]

Observe that \( \mathbf{u}_\perp \cdot \mathbf{a} = 0 \).

(d) Let \( l \) be the line with equation \( \mathbf{r} = \langle -1, 1, 0 \rangle + \langle 0, 1, -1 \rangle t \) and \( P \) be the plane with equation \( x - y - 2z = 3 \). Consider the vectors parallel to \( P \). One of these vectors makes the smallest positive angle with \( l \). Find this smallest positive angle.

Solution:

Let \( \mathbf{w} \) be the vector parallel to \( P \) that gives the smallest angle with the direction vector \( \mathbf{v} \) of \( l \). Then the three vectors \( \mathbf{w}, \mathbf{v} \) and the normal \( \mathbf{n} \) to \( P \) are coplanar. Since

\[
\cos(\theta_{\mathbf{v}, \mathbf{w}}) = \frac{\mathbf{v} \cdot \mathbf{n}}{||\mathbf{v}|| \cdot ||\mathbf{n}||} = \frac{1}{\sqrt{12}}.
\]

we have:

\[
\theta_{\mathbf{v}, \mathbf{n}} = \sin^{-1} \frac{1}{\sqrt{12}}.
\]

Bonus question. Draw some curves around \( (0, 0) \) which can be level curves of a function of two variables that is not continuous at \( (0, 0) \).

Solution:

Examples:

- All level curves pass through the origin where function is not defined.
- Function has a jump along \( y \)-axis; the level curves break.
3. A curve \( C \) is given in polar coordinates by the equation \( r = \frac{1}{2} + \cos 2\theta \). Sketch the curve.

Find the area of the region in the \( xy \)-plane in the first quadrant bounded by \( C \).

Find a surface in \( \mathbb{R}^3 \) that cuts the \( xy \)-plane in the curve \( C \).

**Solution:**

\[
\theta = \frac{2\pi}{3} \quad \theta = \frac{\pi}{3}
\]

\[
\theta = \frac{4\pi}{3} \quad \theta = \frac{5\pi}{3}
\]

A surface in \( \mathbb{R}^3 \) that cuts the \( xy \)-plane in the above curve \( C \) may be a cylinder over \( C \), given in cylindrical coordinates by \( r = \frac{1}{2} + \cos 2\theta \).

The shaded region has area \( A \) equal to:

\[
A = \frac{1}{2} \int_{0}^{\frac{\pi}{3}} \left( \frac{1}{2} + \cos 2\theta \right)^2 d\theta + \frac{1}{2} \int_{\frac{2\pi}{3}}^{\pi} \left( \frac{1}{2} + \cos 2\theta \right)^2 d\theta
\]

\[
= \frac{1}{2} \int_{0}^{\frac{\pi}{3}} \left( \frac{1}{2} + \cos 2\theta \right)^2 d\theta + \frac{1}{2} \int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} \left( \frac{1}{2} + \cos 2\theta \right)^2 d\theta, \text{ (using symmetry with respect to origin)}
\]

\[
= \frac{1}{2} \left\{ \left( \frac{\theta}{4} + \frac{\sin 2\theta}{2} + \frac{\sin 4\theta}{8} + \frac{\theta}{4} \right)^{\pi/3}_{0} + \left( \frac{\theta}{4} + \frac{\sin 2\theta}{2} + \frac{\sin 4\theta}{8} + \frac{\theta}{2} \right)^{\pi/3}_{\pi/3} \right\}
\]

\[
= \frac{1}{2} \left( \frac{\pi}{8} + \frac{\pi}{4} \right) = \frac{3\pi}{16}.
\]

4. A point moves on a curve according to the vector equation

\[
\mathbf{r}(t) = 4\cos t\mathbf{i} + 4\sin t\mathbf{j} + 4\cos t\mathbf{k} \quad (0 \leq t \leq 2\pi)
\]

(a) This curve lies on a plane. Find an equation for that plane.

**Solution:**

Since \( x(t) = z(t) \) for all \( t \), the curve lies on the plane \( x = z \).

(b) Find the point(s) at which the curvature of this curve is maximum.

**Solution:**
\[
\begin{align*}
\vec{r}'(t) &= \langle -4 \sin t, 4 \cos t, -4 \sin t \rangle; \\
\vec{r}''(t) &= \langle -4 \cos t, -4 \sin t, -4 \cos t \rangle; \\
\kappa(t) &= \frac{||\vec{r}'(t) \times \vec{r}''(t)||}{||\vec{r}'(t)||^3} \\
&= \frac{16 \cdot ||\langle -1, 0, 1 \rangle||}{(1 + \sin^2 t)^{3/2}} \\
&= \frac{16 \sqrt{2}}{(1 + \sin^2 t)^{3/2}}.
\end{align*}
\]

Extrema for \( \kappa(t) \) occurs when

\[
\kappa'(t) = 16 \sqrt{2} \cdot \frac{-3/2 \cdot 2 \sin t \cos t}{(1 + \sin^2 t)^{5/2}} = 0
\]

\[\Rightarrow 2 \sin t \cos t = \sin 2t = 0 \quad \text{and} \quad 1 + \sin^2 t \neq 0\]

\[\Rightarrow t \in \{0, \frac{\pi}{2}, \frac{3\pi}{2}, 2\pi \} \quad \text{in} \quad [0, 2\pi].\]

Now, \( \kappa''(t) = -24 \sqrt{2} \cdot \left[ \frac{2 \cos 2t}{(1 + \sin^2 t)^{5/2}} - \frac{5/2 \cdot \sin^2 2t}{(1 + \sin^2 t)^{7/2}} \right]. \)

Since

\[\kappa''(0), \kappa''(\pi) < 0 \quad \text{and} \quad \kappa''(\pi/2), \kappa''(3\pi/2) > 0,\]

maximum for curvature occurs when \( t = 0 \) or \( t = \pi \), i.e. at points \((4, 0, 4)\) and \((-4, 0, -4)\) of the curve.

5. Let \( f(x, y) = \frac{x + y}{x^2 + y} \) when \( x^2 + y \neq 0 \).

Is it possible to define \( f(1, -1) \) in such a way that \( f(x, y) \to f(1, -1) \) as \( (x, y) \to (1, -1) \) along the line \( x = 1 \)?

Is it possible to define \( f(1, -1) \) in such a way that \( f(x, y) \to f(1, -1) \) as \( (x, y) \to (1, -1) \) along the line \( y = -1 \)?

Is it possible to define \( f(1, -1) \) in such a way that \( f \) is continuous at \((1, -1)\)? Give reason for your answers.

Solution:

\[
\lim_{(x, y) \to (1, -1)} f(x, y) = \lim_{y \to -1} \frac{1 + y}{1 + y} = 1. \text{ Define } f(1, -1) \text{ as } 1.
\]

\[
\lim_{(x, y) \to (1, -1)} f(x, y) = \lim_{x \to 1} \frac{x - 1}{x^2 - 1} = \lim_{x \to 1} \frac{1}{x + 1} = \frac{1}{2}. \text{ Hence define } f(1, -1) \text{ as } \frac{1}{2}.
\]

Finally, since the two limits above are not equal to each other, \( f(x, y) \) is by no means continuous at \((1, -1)\) and therefore there is no value for \( f(1, -1) \) to make \( f \) continuous there.