1. What horizontal plane is tangent to the surface \( z = x^2 - 4xy - 2y^2 + 12x - 12y - 1 \) and what is the point of the tangency?

Solution:

Begin by rewriting the surface equation as

\[
F(x, y, z) = x^2 - 4xy - 2y^2 + 12x - 12y - 1 - z.
\]

Obviously, the gradient of this function at the point \((x_0, y_0, z_0)\) is a normal vector to the surface at \((x_0, y_0, z_0)\), where

\[
\vec{\nabla}F(x_0, y_0, z_0) = (2x_0 - 4y_0 + 12)\hat{i} + (-4x_0 - 4y_0 - 12)\hat{j} + (-1)\hat{k}
\]

For \(\vec{\nabla}F(x_0, y_0, z_0)\) to be the normal vector to the horizontal plane, the first two components of \(\vec{\nabla}F(x_0, y_0, z_0)\) should be zero. i.e.

\[
2x_0 - 4y_0 + 12 = 0
\]
\[
-4x_0 - 4y_0 - 12 = 0
\]

From these equations and the equation of surface, the point of tangency is \((-4, 1, -31)\). Hence the equation of the tangent plane is

\[
\vec{\nabla}F(x_0, y_0, z_0) \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0
\]
\[
< 0, 0, -1 > \cdot < x + 4, y - 1, z + 31 > = 0
\]

\[\Rightarrow z = -31\]
2. Evaluate \( \int_{\sqrt{3}/2}^{1} \int_{\sqrt{1-x^2}}^{x} \frac{1}{\sqrt{x^2 + y^2}} \, dy \, dx \).

Solution:

We will use polar coordinates to evaluate this integral. So, first we will identify the region of integration.

Consider that for fixed \( x \), \( y \) runs from \( \sqrt{1-x^2} \) to \( x \), and for fixed \( y \), \( x \) runs from \( \sqrt{\frac{2}{2}} \) to 1. So, the region is bounded below by a semicircle of radius 1 centered at the origin, and is bounded above by the line \( x = y \). The region of the integration is the shaded area in the figure:

Now let us describe this area in polar coordinates. Consider that \( r \) runs from the circle with radius 1 to the line \( x = 1 \). The equation of the circle is \( r = 1 \), and the equation for \( x = 1 \) is \( r \cos \theta = 1 \) \( \Rightarrow r = \frac{1}{\cos \theta} \). So, \( r \) varies from 1 to \( \frac{1}{\cos \theta} \) and \( \theta \) varies from 0 to \( \frac{\pi}{4} \). Thus,

\[
\int_{\sqrt{3}/2}^{1} \int_{\sqrt{1-x^2}}^{x} \frac{1}{\sqrt{x^2 + y^2}} \, dy \, dx = \int_{0}^{\pi/4} \int_{1}^{1/\cos \theta} \frac{1}{r} \, r \, dr \, d\theta
\]

\[
= \int_{0}^{\pi/4} \left( \frac{1}{\cos \theta} - 1 \right) d\theta
\]

\[
= \int_{0}^{\pi/4} \frac{d\theta}{\cos \theta} - \int_{0}^{\pi/4} d\theta
\]

\[
= \ln |\sec \theta + \tan \theta| - \theta \bigg|_{0}^{\pi/4}
\]

\[
= \ln |\sqrt{2} + 2| - \frac{\pi}{4}
\]
3. Find the volume of the solid in the first octant bounded by the coordinate planes and the surface $z = 4 - x^2 - y$.

Solution:

The projection of the solid G on the xy-plane is shown in the figure:

![Diagram](image)

The volume of the solid can be found as follows:

$$V = \iiint dV = \iint_R \left[ \int_0^{4-x^2-y} dz \right] dA$$

$$= \int_0^2 \int_0^{4-x^2} \int_0^{4-x^2-y} dz \, dy \, dx$$

$$= \int_0^2 \int_0^{4-x^2} (4 - x^2 - y) \, dy \, dx$$

$$= \int_0^2 \left[ 4y - x^2y - \frac{y^2}{2} \right]_{y=0}^{4-x^2} \, dx$$

$$= \int_0^2 \left[ 8 - 4x^2 + \frac{x^4}{2} \right] dx$$

$$= 8x - \frac{4x^3}{3} + \frac{x^5}{10} \bigg|_0^2$$

$$= \frac{128}{15}$$
4. Let C be the triangle whose edges are \( x=0, x+y = 1, y=0 \). Apply Green’s theorem to evaluate the line integral

\[
\int_C y^2 \, dx + x^2 \, dy
\]

Solution:

Since \( f(x, y) = y^2 \) and \( g(x, y) = x^2 \), it follows from Green’s theorem that

\[
\int_C y^2 \, dx + x^2 \, dy = \iint_R \left[ \frac{\partial}{\partial x} (x^2) - \frac{\partial}{\partial y} (y^2) \right] \, dA
\]

\[
= \iint_R (2x - 2y) \, dA
\]

As given in the question, the region of integration is

Thus,

\[
\int_C y^2 \, dx + x^2 \, dy = \int_0^1 \int_0^{1-x} (2x - 2y) \, dy \, dx
\]

\[
= \int_0^1 \left[ 2xy - y^2 \right]_{y=0}^{1-x} \, dx
\]

\[
= \int_0^1 [2x - 2x^2 - 1 + 2x - x^2] \, dx
\]

\[
= x^2 - \frac{2x^3}{3} - x + x^2 - \frac{x^3}{3} \bigg|_0^1
\]

\[
= 0
\]
5. Let $C$ be the curve $x = \frac{\pi}{2} t^5 + \frac{\pi}{2} t^4$, $y = t^8 + t^9 + 1$, $0 \leq t \leq 1$.

Find the line integral

$$\int_C \left( y \cos xy + \frac{1}{y} e^{x/y} \right) \, dx + \left( x \cos xy - \frac{x}{y^2} e^{x/y} \right) \, dy$$

Solution:

When $t = 0$, we have $x = 0$, $y = 1$ and when $t = 1$, we have $x = \pi$, $y = 3$. The curve $C$ is from $(x_0, y_0) = (0, 1)$ to $(x_1, y_1) = (\pi, 3)$.

Put $P(x, y) = y \cos xy + \frac{1}{y} e^{x/y}, Q(x, y) = x \cos xy - \frac{x}{y^2} e^{x/y}$. Then

$$\frac{\partial P}{\partial y} = y(- \sin xy)x + \cos xy - \frac{1}{y^2} e^{x/y} + \frac{1}{y} \left( -\frac{x}{y^2} \right) e^{x/y}$$

$$= -xy \sin xy + \cos xy - \frac{1}{y^2} e^{x/y} - \frac{x}{y^2} e^{x/y}.$$ 

$$\frac{\partial Q}{\partial x} = x(- \sin xy)y + \cos xy - \frac{1}{y^2} e^{x/y} - \frac{x}{y^2} \left( -\frac{1}{y} \right) e^{x/y}$$

$$= -xy \sin xy + \cos xy - \frac{1}{y^2} e^{x/y} - \frac{x}{y^3} e^{x/y}.$$ 

So, $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ and $F = P \hat{i} + Q \hat{j}$ is conservative.

Let’s find a potential function $\phi$ for $F$. We have

$$\frac{\partial \phi}{\partial x} = y \cos xy + \frac{1}{y} e^{x/y}$$

$$\phi = \int \left( y \cos xy + \frac{1}{y} e^{x/y} \right) \, dx$$

$$\phi(x, y) = \sin xy + e^{x/y} + f(y)$$

We also have

$$Q(x, y) = x \cos xy - \frac{x}{y^2} e^{x/y} = \frac{\partial \phi}{\partial y}$$

$$\frac{\partial \phi}{\partial y} = (\cos xy)x + \frac{x}{y^2} e^{x/y} + f'(y)$$

$$\Rightarrow f'(y) = 0, f(y) = c,$
So,

\[
\phi(x, y) = \sin xy + e^{x/y} + c
\]

\[
\int_C Pdx + Qdy = \phi(x_1, y_1) - \phi(x_0, y_0)
\]

\[
= \phi(\pi, 3) - \phi(0, 1)
\]

\[
= (\sin 3\pi + e^{\pi/3} + c) - (\sin \pi + e^{0/1} + c)
\]

\[
= e^{\pi/3} - 1
\]