1. (a) (30 points) Sketch the graph of the polar equation \( r = 1 + \cos(2\theta) \) and find the area of the region enclosed by the graph.

\[
\begin{array}{c|c|c|c|c|c|c|c|c|c|c}
\theta & 0 & \frac{\pi}{4} & \frac{\pi}{2} & \frac{3\pi}{4} & \pi & \frac{5\pi}{4} & \frac{3\pi}{2} & \frac{7\pi}{4} & 2\pi \\
-----&-----&-----&-----&-----&-----&-----&-----&-----&-----
\
r & 2 & 1 & 0 & 1 & 2 & 1 & 0 & 1 & 2 \\
\end{array}
\]

Total Area = \( 4A \)

\[
= 4 \int_{0}^{\pi/2} \frac{1}{2} (1 + \cos(2\theta))^2 \, d\theta
\]

\[
= 2 \int_{0}^{\pi/2} \left[ 1 + 2\cos(2\theta) + \cos^2(2\theta) \right] \, d\theta
\]

\[
= 2 \left[ \theta + \sin(2\theta) + 1 - \frac{1}{2} \left( \frac{\sin(4\theta)}{8} + \frac{\sin(2\theta)}{2} \right) \right]_{0}^{\pi/2}
\]

\[
= 2 \left( \frac{\pi}{2} + \frac{\pi}{4} \right)
\]

\[
= \frac{3\pi}{2}
\]
2. (a) (20 points) Find the area of the triangle with vertices $P_1(3, 0, 1)$, $P_2(2, -1, 2)$, $P_3(1, 3, -2)$ using cross product.

We observe
\[ P_1P_2 = (-1, -1, 1) \]
\[ P_1P_3 = (-2, 3, -3) \]

the area of the triangle with vertices $P_1, P_2, P_3$
is the half of the area of the parallelogram generated
by $P_1P_2$ and $P_1P_3$. So

\[
\text{Area} = \frac{1}{2} \| P_1P_2 \times P_1P_3 \|
\]

\[
= \frac{1}{2} \| (-1, -1, 1) \times (-2, 3, -3) \|
\]

\[
= \frac{1}{2} \| (0, -5, -5) \|
\]

\[
= \frac{5\sqrt{2}}{2}
\]
3. Let \( L_1 : x = t + 1, y = t + 2, z = 3 \) and \( L_2 : x = 3t - 2, y = 5t - 3, z = t + 2 \) be the parametric equations of two lines.

(a) (10 points) Find the intersection point of the lines \( L_1 \) and \( L_2 \).

\( L_1 \) and \( L_2 \) intersects if there exist \( t_1, t_2 \in \mathbb{R} \) such that:

\[
\begin{align*}
  t_1 + 1 &= 3t_2 - 2 \\
  t_1 + 2 &= 5t_2 - 3 \\
  3 &= t_2 + 2
\end{align*}
\]

Solving these we get \( t_2 = 1, t_1 = 0 \) and if we put \( t_1 = 1 \) and \( t_2 = 0 \) into the first equation we get:

\[
0 + 1 = 3 \cdot 1 - 2 \quad \text{that is} \quad t_1 = 1
\]

So the system is consistent and lines intersect at \((1, 2, 3)\).

(b) (10 points) What is the equation of the plane \( P \) containing the lines \( L_1 \) and \( L_2 \)?

See that \( \vec{v}_1 = <1, 1, 0> \) and \( \vec{v}_2 = <3, 5, 1> \) are the direction vectors for \( L_1 \) and \( L_2 \) respectively. So \( \vec{v}_1 \) and \( \vec{v}_2 \) should be parallel to \( P \) and therefore \( \vec{v}_1 \times \vec{v}_2 \) is perpendicular to the plane \( P \). We compute \( \vec{v}_1 \times \vec{v}_2 = <-1, -1, 2> \). We know that the point \((1, 2, 3)\) is on the plane \( P \) so we get:

\[
(<x, y, z> - <1, 2, 3>) \cdot <-1, -1, 2> = 0 \iff x - y + 2z - 5 = 0
\]

(c) (10 points) What is the distance between the above plane \( P \) and the origin?

We know that the distance is given by:

\[
D = \frac{|1 \cdot 0 + (-1) \cdot 0 + 2 \cdot 0 - 5|}{\sqrt{1^2 + (-1)^2 + 2^2}} = \frac{5}{\sqrt{6}}
\]
4. (a) (20 points) Find the arc length parametrization of the curve

\[ r(t) = \cos^3 t \mathbf{i} + \sin^3 t \mathbf{j}, \quad \text{for } 0 \leq t \leq \pi/2. \]

We take \( r(0) = <1, 0> \) as a reference point and define

\[ s(t) = \int_0^t \| r'(z) \| \, dz \]

\[ = \int_0^t \| <-3 \cos^2 z \sin z, 3 \sin^2 z \cos z> \| \, dz \]

\[ = \int_0^t 3 \cos z \sin z \, dz \]

\[ = \left[ \frac{3}{2} \sin^2 z \right]_0^t \]

So we get \( s(t) = \frac{3}{2} \sin^2 t \), and \( t = \arcsin \left( \sqrt{\frac{s}{3}} \right) \)

Hence we get

\[ r(s) = \cos^3 \left( \arcsin \sqrt{\frac{s}{3}} \right) \mathbf{i} + \sin^3 \left( \arcsin \sqrt{\frac{s}{3}} \right) \mathbf{j} \]

where \( s \in [0, \frac{\pi}{2}] \).

Or one can continue from (\( \ast \)) as the following:

\[ \int_0^t 3 \sin z \cos z \, dz = \frac{3}{2} \int_0^t 2 \sin z \cos z \, dz = \frac{3}{2} \int_0^t \sin 2z \, dz \]

\[ = \left[ -\frac{3}{4} \cos 2z \right]_0^t \]

\[ = \ldots \]