1. The surface of a mountain is described by the function \( h(x, y) = 3e^{-(3x^4+y^2)} \), where \( h, x, y \) are in km; and the \( x \)-axis points towards east, the \( y \)-axis points towards north. A mountaineer is at the point \( P(1, 1) \). As seen from \( P \):

   (a) In which direction does the height increase the fastest?
   (b) What is the rate of increase of height with respect to horizontal distance for the direction found in (a)?
   (c) What is the rate of increase of height with respect to horizontal distance for the NW direction?

**Solution:**

(a) The direction of the gradient vector gives the fastest increase in height.
\[
\nabla h(x, y) = -36x^3e^{-(3x^4+y^2)}\mathbf{i} - 6ye^{-(3x^4+y^2)}\mathbf{j},
\]
so the vector \( \nabla h(1, 1) = -36e^{-4}\mathbf{i} - 6e^{-4}\mathbf{j} \) gives the direction of the fastest increase.

(b) The equation for the line passing through \( P(1, 1) \) in the direction of \( \nabla h(1, 1) \) is
\[
\frac{x - 1}{-36e^{-4}} = \frac{y - 1}{-6e^{-4}},
\]
from which we obtain \( y = (x + 5)/6 \).

\[
\frac{d}{dx} h(x, y(x)) \]
will give the rate of increase in the given direction with respect to \( x \), where \( y(x) = (x + 5)/6 \) is the equation of the line.

Using the chain rule, we have:
\[
\frac{dh}{dx} = \frac{\partial h}{\partial x} + \frac{\partial h}{\partial y} \frac{dy}{dx} = -36x^3e^{-(3x^4+y^2)} - 6ye^{-(3x^4+y^2)} \frac{1}{6}.
\]

Thus,
\[
\frac{d}{dx} h(x, y(x)) \bigg|_{x=1} = -36e^{-4} - 6e^{-4} \frac{1}{6} = -37e^{-4}
\]

(c) Similar to part (b), we use \( y(x) = 2 - x \), which is the equation of the line passing through \( P(1, 1) \) in the NW direction, and we get
\[
\frac{dh}{dx} = \frac{\partial h}{\partial x} + \frac{\partial h}{\partial y} \frac{dy}{dx} = -36x^3e^{-(3x^4+y^2)} - 6y^2e^{-(3x^4+y^2)}(-1)
\]
\[
\frac{d}{dx} h(x, y(x)) \bigg|_{x=1} = -36e^{-4} - 6e^{-4}(-1) = -30e^{-4}
\]
2. Using the method of Lagrange multipliers, find the greatest and smallest values that the function \( z = f(x, y) = xy \) takes on the ellipse \( \frac{x^2}{8} + \frac{y^2}{2} = 1 \).

Solution:

The constraint curve is \( g(x, y) = \frac{x^2}{8} + \frac{y^2}{2} - 1 = 0 \). We solve \( \nabla f(x, y) = \lambda \nabla g(x, y) \):

\[
y \mathbf{i} + x \mathbf{j} = \lambda \left( \frac{x}{4} \mathbf{i} + y \mathbf{j} \right)
\]

So,

\[
y = \lambda \frac{x}{4}; \quad x = \lambda y.
\]

It follows that \( y = \lambda^2 (y/4) \), which implies \( \lambda^2 = \pm 2 \) or \( y = 0 \). If \( y = 0 \), then \( x = \lambda y = 0 \), thus \( (x, y) = (0, 0) \), so this choice ends up with a point that is not on the curve. If \( \lambda = 2 \), then \( x = 2y \), and putting this in \( g(x, y) = 0 \), we find \( (x, y) = \pm (2, 1) \). Similarly if \( \lambda = -2 \) then \( (x, y) = \pm (2, -1) \).

We also need to check the points on the constraint curve, where \( \nabla g(x, y) = \mathbf{0} \). The only point satisfying \( \nabla g(x, y) = \mathbf{0} \) is \( (x, y) = (0, 0) \), and it is not on the constraint curve \( g(x, y) = 0 \). So there is no such point.

We now evaluate the function \( f(x, y) \) at the points we have found: \( f(2, 1) = 2, f(-2, -1) = 2, f(2, -1) = -2, f(-2, 1) = 2 \). Therefore, the maximum and minimum values of \( f \) are 2 and -2, respectively, on the given ellipse.

3. Find the volume bounded by the surfaces \( x + y = 1, \sqrt{x} + \sqrt{y} = 1 \), \( z = 0, z = 10 \).

Solution:

We solve the equations \( x + y = 1, \sqrt{x} + \sqrt{y} = 1 \) together, to find their intersection points. Squaring the second equation gives \( x + 2\sqrt{xy} + y = 1 \). Plugging the first equation, \( x + y = 1 \), into this result, we get \( 1 + 2\sqrt{xy} = 1 \), or \( \sqrt{xy} = 0 \). So, the points of intersection are \( (x, y) = (0, 1) \) and \( (x, y) = (1, 0) \).

Rearranging the equations, we get that the region in the plane bounded by the curves \( x + y = 1 \) and \( \sqrt{x} + \sqrt{y} = 1 \) is the region bounded by the curves \( y = 1-x \) and \( y = 1 + x - 2\sqrt{x} \).

Thus we obtain the volume by the triple integral:

\[
\int_0^{10} \int_0^1 \int_{x-1}^{1-x+2\sqrt{x}} dz \, dy \, dx = \int_0^{10} \int_0^1 \left( 2 - 2\sqrt{x} \right) dx \, dz \\
= \int_0^{10} \left[ 2x - \frac{4}{3} x^{3/2} \right]_0^1 dz \\
= \int_0^{10} \frac{20}{3} \, dz = \frac{200}{3}.
\]

4. Find the area of the region \( Q \) that lies inside the cardioid \( r = 1 + \cos \theta \) and outside the circle \( r = 1 \).

Solution:
The region corresponds to $1 \leq r \leq 1 + \cos \theta$ and $-\pi/2 \leq \theta \leq \pi/2$ in the $r$-$\theta$ plane. So the area can be found as follows:

\[
\int_{-\pi/2}^{\pi/2} \int_{1}^{1+\cos \theta} r \, dr \, d\theta = \int_{-\pi/2}^{\pi/2} \left[ \frac{r^2}{2} \right]_{1}^{1+\cos \theta} \, d\theta
\]

\[
= \int_{-\pi/2}^{\pi/2} 2 \cos \theta + \cos^2 \theta \, d\theta
\]

\[
= \int_{-\pi/2}^{\pi/2} 2 \cos \theta + \frac{1 + \cos 2\theta}{2} \, d\theta
\]

\[
= \left[ 2 \sin \theta + \frac{\theta}{2} + \frac{\sin 2\theta}{4} \right]_{-\pi/2}^{\pi/2}
\]

\[
= 4 + \frac{\pi}{2}.
\]