1.) Find the volume of the part of the sphere of radius 3 that is left after drilling a cylindrical hole of radius 2 through the center.

Solution:

The sphere has equation: $x^2 + y^2 = 9$

In polar coordinates, $r^2 + z^2 = 9 \Rightarrow z = \pm \sqrt{9 - r^2}$

$$V = \int_0^{2\pi} \int_2^3 \left[ \sqrt{9 - r^2} - (-\sqrt{9 - r^2}) \right] r dr d\theta$$

Let $u = 9 - r^2 \Rightarrow du = -2r dr$

$$\Rightarrow V = -\frac{1}{2} \int_0^{2\pi} \int_5^{0} 2u^{1/2} du d\theta = -\frac{2}{3} \int_0^{2\pi} -5^{3/2} d\theta = \frac{20\sqrt{5}\pi}{3}$$
2.) Let $S$ denote the set of topless and bottomless right circular cylinders of fixed non-zero surface area $A$. Use Lagrange multipliers to prove or disprove that the set $S$ has an element with maximal volume. If your answer is positive then compute the maximal possible volume in terms of only $A$. If your answer is negative tell us precisely the lack of which properties of the solution set of $A = 2\pi rh$ is responsible for the non-existence. (Note that the surface area of the cylinder of radius $r$ and height $h$ with no top and bottom is $2\pi rh$.)

Solution:

$$S(r, h) = 2\pi rh = A \Rightarrow g(r, h) = 2\pi rh - A$$

$$V(r, h) = \pi r^2 h \Rightarrow f(r, h) = \pi r^2 h$$

$$\nabla f = 2\pi rh \mathbf{i} + \pi r^2 h \mathbf{j}$$

$$\nabla g = 2\pi h \mathbf{i} + 2\pi r \mathbf{j}$$

$$\nabla f = \lambda \nabla g \Rightarrow 2\pi rh = \lambda \cdot 2\pi h \quad \text{and} \quad \pi r^2 = \lambda \cdot 2\pi r$$

$$r = \lambda, r = 2\lambda \Rightarrow r = \lambda = 0.$$ 

But $r = 0$ gives $S(r, h) = 2\pi \cdot 0 \cdot h = 0$. But the surface area is nonzero. So using the Lagrange multiplier method we conclude that NO such maximal volume cylinder exists.

Note that the variables $r, h$ are such that $r > 0, h > 0$. Hence, the solution set ($A = 2\pi rh$) is an unbounded set. For the existence of extremum, solution set must be closed and bounded. Also, $V = \frac{Ar}{2}$ and since $r > 0, V \rightarrow \infty$ and so no cylinder with maximal volume!
3.) Transform the following integral into spherical coordinates throughly but do not evaluate.

\[ \int_{-2}^{0} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2-y^2}} \int_{0}^{\sqrt{4-x^2-y^2}} z^3 \sqrt{x^2 + y^2 + z^2} \, dz \, dy \, dx \]

Solution:

Let \( I \) be the value of the given integral.

\[ 0 \leq z \leq \sqrt{4 - x^2 - y^2} \Rightarrow \text{The solid over which we integrate is bounded below by } z = 0 \text{ (xy-plane) and above by the sphere } x^2 + y^2 = 4. \]

\[ -\sqrt{4 - x^2} \leq y \leq 0 \text{ and } -2 \leq x \leq 0 \text{ tell us that the projection of this solid onto xy-plane is the quarter of the circle: } x^2 + y^2 = 4 \text{ in the third quadrant.} \]

In spherical coordinates: \( z = \rho \cdot \cos \phi \) and \( \rho = \sqrt{x^2 + y^2 + z^2} \)

\[ I = \int_{\pi}^{3\pi/2} \int_{\pi/2}^{\pi/2} \int_{0}^{2} (\rho \cdot \cos \phi)^3 \cdot (\rho) \cdot \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta \]
4.) Evaluate $\int \int_R \sin(y^3) \ dA$, where $R$ is the region bounded by $y = \sqrt{x}$, $y = 2$ and $x = 0$.

Solution:

Let $I = \int \int_R \sin(y^3) \ dA = \int_0^2 \int_0^{\sqrt[x]} \sin y^3 \ dy \ dx = \int_0^2 \int_0^y \sin y^3 \ dy \ dx$

This last integral seems to be more easier to take.

So, $I = \int_0^2 [x \sin y^3]_0^y \ dy = \int_0^2 y^2 \sin y^3 \ dy$

Let $u = y^3$, then $du = 3y^2 \ dy$ and therefore,

$I = \int_0^2 \sin u \ du = - \left[ \frac{\cos u}{3} \right]_0^y = - \left[ \frac{\cos y^3}{3} \right]_0^y = 1 - \cos \frac{8}{3}$