1.) A) Find the equation of the plane through the points (1, 1, -1), (2, 0, 2), (0, -2, 1).
B) Find parametric equations of the line perpendicular to this plane at (4, 2, 3).

Solution:

A) \( A(1, 1, -1), \ B(2, 0, 2), \ C(0, -2, 1) \)

\[ \vec{AB} = (1, -1, 3), \quad \vec{AC} = (-1, -3, 2) \]

\[ \vec{AB} \times \vec{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 3 \\ -1 & -3 & 2 \end{vmatrix} = 7\mathbf{i} - 5\mathbf{j} - 4\mathbf{k} \]

So \( \langle 7, -5, -4 \rangle \) is normal to the plane. Hence the equation of the plane through the given points is:

\[
7(x - 1) - 5(y - 1) - 4(z + 1) = 0
\]

\[
7x - 5y - 4z = 6
\]

B) \( x = 4 + 7t, \quad y = 2 - 5t, \quad z = 3 - 4t \)
2.) A) Find a unit vector such that the derivative of $x^3y^2$ along this vector at the point (2, 1) is zero.

B) Is there a unit vector along which the derivative of $x^3y^2$ at the point (2, 1) is 100? Explain.

Solution:

A) \( \nabla f(x, y) = (\frac{\partial}{\partial x}(x^3y^2), \frac{\partial}{\partial y}(x^3y^2)) = (3x^2y^2, 2x^3y) \)

\( \nabla f(2, 1) = (12, 16) \)

The required unit vector, say \( \vec{u} = (x, y) \), must satisfy \( \vec{u} \cdot (12, 16) = 0 \). Let \( x = t \), then

\[ 12t + 16y = 0 \implies y = -\frac{3}{4}t. \]

Also, since \( \vec{u} \) is a unit vector we have \( ||\vec{u}|| = 1 \). So

\[ t^2 + \frac{9}{16}t^2 = 1, \quad t = \pm \frac{4}{5}. \]

Hence we have \( \vec{u} = (-\frac{4}{5}, \frac{3}{5}) \) or \( \vec{u} = (\frac{4}{5}, -\frac{3}{5}) \).

B) The derivative along any unit vector \( \vec{u} \) at (2, 1) is given by

\[ \vec{u} \cdot \nabla f(2, 1) = ||\vec{u}|| ||\nabla f(2, 1)|| \cos \theta = 20 \cos \theta \]

where \( \theta \) is the acute angle between \( \vec{u} \) and \( \nabla f(2, 1) \). Clearly \( 20 \cos \theta \leq 20 \) so there is no such unit vector.
3.) Find point(s) on the surface $z^2 = xy + 4$ closest to the origin.

Solution:

We want to minimize $f = x^2 + y^2 + z^2$ under the constraint $g = xy - z^2 + 4 = 0$.

$$\nabla f(x, y, z) = \langle 2x, 2y, 2z \rangle, \quad \nabla g(x, y, z) = \langle y, x, -2z \rangle.$$

A point $(x_0, y_0, z_0)$ is an extremum of $f$ only if $\nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0)$ for some $\lambda \neq 0$.

$$\nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0) \implies 2x_0 = \lambda y_0 \quad 2y_0 = \lambda x_0 \quad 2z_0 = -2\lambda z_0$$

If $z_0 \neq 0$, we have $\lambda = -1$, and so $x_0 = y_0 = 0$ The points on the surface $g(x, y, z) = 0$ satisfying this are $P_1(0, 0, 2)$ and $P_2(0, 0, -2)$.

Otherwise if $z_0 = 0$, we have

$$x_0 = \frac{-4}{y_0}, \quad x_0 = \frac{-8}{\lambda x_0}, \quad x_0^2 = \frac{-8}{\lambda}.$$

Also, $x_0 y_0 = -4$ and $2x_0 = \lambda y_0$ so $x_0^2 = -2\lambda$. It follows that $x_0^4 = 16$ so $x_0 = \pm 2$, and the points on $g(x, y, z) = 0$ satisfying this are $P_3(2, -2, 0)$ and $P_4(-2, 2, 0)$.

Therefore the points closest to the origin on the surface $z^2 = xy + 4$ are $P_1$ and $P_2$. 
4.) Find the area of the region shared by \( r = 2 + 2 \cos \theta \) and \( r = 2 - 2 \cos \theta \).

Solution:

\[
A = 4 \cdot \frac{1}{2} \int_0^{\pi/2} r^2 d\theta = 2 \int_0^{\pi/2} (2 + 2 \cos \theta)^2 d\theta
\]

\[
= 2 \int_0^{\pi/2} 4d\theta - 2 \int_0^{\pi/2} 8 \cos \theta d\theta + 2 \int_0^{\pi/2} 4 \cos^2 \theta d\theta
\]

We have the identity \( \cos^2 \theta = \frac{\cos 2\theta + 1}{2} \). Hence

\[
A = 4\pi - 16 \sin \theta \bigg|_0^{\pi/2} + 2 \sin 2\theta \bigg|_0^{\pi/2} + 4\theta \bigg|_0^{\pi/2}
\]

\[
= 6\pi - 16
\]