1. Use variation of parameters to find the general solution to

\[ x^2 y'' + 3xy' + y = \frac{1}{x} \quad \text{for } x > 0. \]

**Solution:**

The corresponding homogenous equation \( x^2 y'' + 3xy' + y = 0 \) is Euler’s equation. The indicial equation is

\[ r^2 + (3 - 1)r + 1 = r^2 + 2r + 1 = (r + 1)^2 = 0. \]

So we have equal roots \( r_1 = r_2 = -1 \). Hence,

\[ y_1(x) = x^{-1}, \quad y_2(x) = x^{-1} \ln x \]

form a fundamental set of solutions.

Now by variation of parameters, we assume

\[ y(x) = c_1(x)y_1 + c_2(x)y_2 \]

is a particular solution. We insert \( y \) in the given differential equation. Assuming, as usual, that

\[ c_1'y_1 + c_2'y_2 = 0 \]

we obtain as the second identity:

\[ x^2(c_1'y_1 + c_2'y_2) = \frac{1}{x}; \]

that is, we get:

\[
\begin{bmatrix}
  y_1 & y_2 \\
  y_1' & y_2'
\end{bmatrix}
\begin{bmatrix}
  c_1' \\
  c_2'
\end{bmatrix}
= \begin{bmatrix}
  0 \\
  x^{-3}
\end{bmatrix}.
\]

Therefore,

\[
c_1' = -\frac{y_2/x^3}{W(y_1, y_2)} = -x^{-3}x^{-1}\ln x \cdot \begin{vmatrix}
  x^{-1} & x^{-1} \ln x \\
  -x^{-2} & -x^{-2} \ln x + x^{-2}
\end{vmatrix}^{-1} = -x^{-3}x^{-1} \ln x \cdot x^3 = -\ln x.
\]

\[
c_2' = -\frac{y_1/x^3}{W(y_1, y_2)} = x^{-3}x^{-1}x^3 = x^{-1}.
\]

Finally,

\[
c_1 = -\int \frac{\ln x}{x} dx = -\int \ln x \cdot (\ln x)' dx = -\frac{1}{2} (\ln x)^2 \quad \text{and}
\]

\[
c_2 = \int x^{-1} dx = \ln x.
\]

Hence, \( y_p = -\frac{1}{2}(\ln x)^2 x^{-1} + \ln x (x^{-1} \ln x) = \frac{(\ln x)^2}{2x} \).
2. Consider the piecewise continuous function
\[ f(t) = \begin{cases} t^2, & 0 \leq t < 3 \\ 9, & t \geq 3 \end{cases} \]
(a) Show that \( f \) is of exponential order.
(b) Express \( f \) in terms of the unit step function.
(c) Find Laplace transform of \( f \) and determine the allowed values for \( s \).

Solution:
(a) Since \( f(t) \leq 9e^{0t} = 9 \) for all \( t \), \( f(t) \) is of exponential order.
(b) Think of \( f \) in two parts: \( t \geq 3 \) and \( t < 3 \). In this way we see that
\[ f(t) = t^2(1 - u_3(t)) + 9u_3(t) = t^2 + (9 - t^2)u_3(t). \]
(c) Use the fact that \( \mathcal{L}(u_3(t)g(t - 3)) = e^{-3s}G(s) \). For this, we put the expression for \( f \) in an appropriate form:
\[ f(t) = t^2 + (9 - t^2)u_3(t) = t^2 - ((t - 3)^2 + 6t - 18)u_3(t) = t^2 - (t - 3)^2u_3(t) - 6(t - 3)u_3(t). \]
Then
\[ \mathcal{L}(f)(s) = \frac{2}{s^3} - \frac{2e^{-3s}}{s^3} - \frac{6e^{-3s}}{s^2}. \]
Here \( s > 0 \).

3. Find all singular points of
\[ 2xy'' + y' + y = 0 \]
and determine whether each one is regular or irregular. If possible, find the series solution corresponding to these points by finding the indicial equation, its roots and the recurrence relation.

Solution:
Let \( P(x) = 2x \), \( Q(x) = R(x) = 1 \). Observe \( P(x) = 0 \) only if \( x = 0 \). Since \( \lim_{x \to 0} \frac{x}{2x} = \frac{1}{2} \) and \( \lim_{x \to 0} x^2 \frac{1}{2x} = 0 \) (both finite) then \( x = 0 \) is a regular singular point of the differential equation. Assume a solution of the form \( y(x) = \sum_{n=0}^{\infty} a_n x^{r+n} \). Then
\[ y' = \sum_{n=0}^{\infty} (r + n)a_n x^{r+n-1}; y'' = \sum_{n=0}^{\infty} (r + n)(r + n - 1)a_n x^{r+n-2}. \]
Insert these in the differential equation:
\[ 0 = 2xy'' + y' + y = 2 \sum_{n=0}^{\infty} (r + n)(r + n - 1)a_n x^{r+n-1} + \sum_{n=0}^{\infty} (r + n)a_n x^{r+n-1} + \sum_{n=0}^{\infty} a_n x^{r+n} \]
\[ = 2r(r - 1)a_0 x^{r-1} + 2 \sum_{n=1}^{\infty} (r + n)(r + n - 1)a_n x^{r+n-1} + ra_0 x^{r-1} + \sum_{n=1}^{\infty} (r + n)a_n x^{r+n-1} + \sum_{n=0}^{\infty} a_n x^{r+n} \]
\[ = (2r(r - 1) + r)a_0 x^{r-1} + \sum_{n=0}^{\infty} (2(r + n)(r + n + 1)a_{n+1} + (r + n + 1)a_{n+1} + a_n)x^{r+n} \]
Requiring each coefficient to be 0, we get:

\[ 0 = 2r^2 - r = r(2r - 1) \text{ so that } r_1 = 0, r_2 = \frac{1}{2} \]

\[ a_{n+1} = -\left( \frac{r+n+1}{2r+2n+1} \right)^{-1} a_n. \]

Now first put \( r = r_1 = 0 \). The recurrence relation is

\[ a_{n+1} = -\frac{a_n}{(n+1)(2n+1)} \]

and the general term becomes

\[ a_n = (-1)^n \frac{a_0}{2 \cdot 3 \cdots n \cdot 3 \cdot 5 \cdots (2n-1)} = (-1)^n \frac{2^n n! a_0}{n!(2n)!} = (-1)^n \frac{2^n a_0}{(2n)!}. \]

Similarly, put \( r = r_2 = \frac{1}{2} \). We have the recurrence relation

\[ a_{n+1} = -\frac{a_n}{(n+\frac{3}{2})(2n+2)} = -\frac{a_n}{(2n+3)(n+1)} \]

and the general term becomes

\[ a_n = (-1)^n \frac{a_0}{3 \cdot 5 \cdots (2n+1) \cdot 2 \cdot 3 \cdots n} = (-1)^n \frac{2^n n! a_0}{(2n+1)!n!} = (-1)^n \frac{2^n a_0}{(2n+1)!}. \]

As a result we have a series solution

\[ y(t) = c_1 \sum_{n=0}^{\infty} (-1)^n \frac{2^n}{(2n)!} x^n + c_2 x^\frac{1}{2} \sum_{n=0}^{\infty} (-1)^n \frac{2^n}{(2n+1)!}. \]

4. Suppose that \( f(t) = t^3 \) is a solution of the differential equation

\[ P(t)y'' + Q(t)y' + R(t)y = 0 \]

where \( P(t) \), \( Q(t) \) and \( R(t) \) are continuous functions that are defined everywhere. Show that \( P(0) = 0 \).

(Hint: Note that \( P(0) = 0 \) means the differential equation has a singular point at \( t = 0 \). Now remember the uniqueness and existence theorem for some initial value problems and question 5(b) of Midterm 1.)

Solution:

Observe that \( f(t) = t^3 \) is a solution for the initial value problem

\[ P(t)y'' + Q(t)y' + R(t)y = 0, y(0) = 0, y'(0) = 0 \]

i.e. \( f(0) = 0 \) and \( f'(0) = 0 \). But \( g(t) = 0 \) is also a solution for this initial value problem. By Theorem, the solution is unique for such a problem provided that \( \frac{Q(t)}{P(t)} \) and \( \frac{R(t)}{P(t)} \) are continuous at \( t = 0 \). Since the solution is not unique \((t^3 \text{ and } 0 \text{ are distinct functions})\), \( 0 \) must be a singular point of the differential equation; in other words \( P(0) = 0 \).
5. Find the inverse Laplace transform of \( F(s) = \frac{e^{-2s}}{s^2 + 2s + 2} \) and determine the allowed values for \( s \).

Solution:

\[
\mathcal{L}^{-1}\left( \frac{e^{-2s}}{s^2 + 2s + 2} \right) = u_2(t)f(t - 2)
\]

where

\[
f(t) = \mathcal{L}^{-1}\left( \frac{1}{s^2 + 2s + 2} \right) = \mathcal{L}^{-1}\left( \frac{1}{(s + 1)^2 + 1} \right) = e^{-t}\sin t, \quad s > -1.
\]

Hence we get

\[
\mathcal{L}^{-1}(F(s)) = u_2(t)e^{-(t-2)}\sin(t - 2), \quad s > -1.
\]